Pathak et al. *Fixed Point Theory and Applications* (2015) 2015:101
DOI 10.1186/s13663-015-0343-0

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Some generalized fixed point results in a b-metric space and application to matrix equations

Hemant Kumar Pathak[1], Reny George[2,3*], Hossam A Nabwey[2,4], Mahdy S El-Paoumy[2,5] and Kaivatath Puthalath Reshma[6,7]

*Correspondence: renygeorge02@yahoo.com
[2]Department of Mathematics, College of Science, Salman bin Abdulaziz University, Al-Kharj, Kingdom of Saudi Arabia
[3]Department of Mathematics and Computer Science, St. Thomas College, Bhilai, Chhattisgarh, India
Full list of author information is available at the end of the article

Abstract
We have proved a generalized Presic-Hardy-Rogers contraction principle and Ciric-Presic type contraction principle for two mappings in a b-metric space. As an application, we derive some convergence results for a class of nonlinear matrix equations. Numerical experiments are also presented to illustrate the convergence algorithms.

MSC: coincident point; common fixed point; b-metric space; matrix equation

Keywords: 47H10

1 Introduction

There appears in literature several generalizations of the famous Banach contraction principle. One such generalization was given by Presic [1, 2] as follows.

Theorem 1.1 [2] *Let (X,d) be a metric space, k be a positive integer, $T : X^k \to X$ be a mapping satisfying the following condition*:

$$\begin{aligned} & d\bigl(T(x_1,x_2,\dots,x_k), T(x_2,x_3,\dots,x_{k+1})\bigr) \\ & \quad \le q_1 \cdot d(x_1,x_2) + q_2 \cdot d(x_2,x_3) + \dots + q_k \cdot d(x_k,x_{k+1}), \end{aligned} \tag{1.1}$$

where $x_1, x_2, \dots, x_{k+1}$ are arbitrary elements in X and $q_1, q_2, \dots, q_k$ are nonnegative constants such that $q_1 + q_2 + \dots + q_k < 1$. Then there exists some $x \in X$ such that $x = T(x,x,\dots,x)$. Moreover, if $x_1, x_2, \dots, x_k$ are arbitrary points in X and for $n \in N$, $x_{n+k} = T(x_n, x_{n+1}, \dots, x_{n+k-1})$, then the sequence $\langle x_n \rangle$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \dots, \lim x_n)$.

Note that for $k = 1$ the above theorem reduces to the well-known Banach contraction principle. Ciric and Presic [3] generalizing the above theorem proved the following.

Theorem 1.2 [3] *Let (X,d) be a metric space, k be a positive integer, $T : X^k \to X$ be a mapping satisfying the following condition*:

$$\begin{aligned} & d\bigl(T(x_1,x_2,\dots,x_k), T(x_2,x_3,\dots,x_{k+1})\bigr) \\ & \quad \le \lambda \cdot \max\bigl\{d(x_1,x_2), d(x_2,x_3), \dots, d(x_k,x_{k+1})\bigr\}, \end{aligned} \tag{1.2}$$

where $x_1, x_2, \ldots, x_{k+1}$ are arbitrary elements in X and $\lambda \in (0,1)$. Then there exists some $x \in X$ such that $x = T(x, x, \ldots, x)$. Moreover, if $x_1, x_2, \ldots, x_k$ are arbitrary points in X and for $n \in N$, $x_{n+k} = T(x_n, x_{n+1}, \ldots, x_{n+k-1})$, then the sequence $\langle x_n \rangle$ is convergent and $\lim x_n = T(\lim x_n, \lim x_n, \ldots, \lim x_n)$. If in addition T satisfies $D(T(u, u, \ldots, u), T(v, v, \ldots, v)) < d(u, v)$ for all $u, v \in X$, then x is the unique point satisfying $x = T(x, x, \ldots, x)$.

In [4, 5] Pacurar gave a classic generalization of the above results. Later the above results were further extended and generalized by many authors (see [6–14]). Generalizing the concept of metric space, Bakhtin [15] introduced the concept of b-metric space which is not necessarily Hausdorff and proved the Banach contraction principle in the setting of a b-metric space. Since then several papers have dealt with fixed point theory or the variational principle for single-valued and multi-valued operators in b-metric spaces (see [16–23] and the references therein). In this paper we have proved common fixed point theorems for the generalized Presic-Hardy-Rogers contraction and Ciric-Presic contraction for two mappings in a b-metric space. Our results extend and generalize many well-known results. As an application, we have derived some convergence results for a class of nonlinear matrix equations. Numerical experiments are also presented to illustrate the convergence algorithms.

2 Preliminaries

Definition 2.1 [15] Let X be a nonempty set and $d : X \times X \to [0, \infty)$ satisfy:

(bM1) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(bM2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(bM3) there exists a real number $s \geq 1$ such that $d(x, y) \leq s[d(x, z) + d(z, y)]$ for all $x, y, z \in X$.

Then d is called a b-metric on X and (X, d) is called a b-metric space (in short bMS) with coefficient s.

Convergence, Cauchy sequence and completeness in b-metric space are defined as follows.

Definition 2.2 [15] Let (X, d) be a b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then:

(a) The sequence $\{x_n\}$ is said to be convergent in (X, d), and it converges to x if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$, and this fact is represented by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(b) The sequence $\{x_n\}$ is said to be Cauchy sequence in (X, d) if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+p}) < \varepsilon$ for all $n > n_0$, $p > 0$ or, equivalently, if $\lim_{n\to\infty} d(x_n, x_{n+p}) = 0$ for all $p > 0$.

(c) (X, d) is said to be a complete b-metric space if every Cauchy sequence in X converges to some $x \in X$.

Definition 2.3 [9] Let (X, d) be a metric space, k be a positive integer, $T : X^k \to X$ and $f : X \to X$ be mappings.

(a) An element $x \in X$ is said to be a *coincidence point* of f and T if and only if $f(x) = T(x, x, \ldots, x)$. If $x = f(x) = T(x, x, \ldots, x)$, then we say that x is a *common fixed point* of f and T. If $w = f(x) = T(x, x, \ldots, x)$, then w is called a point of coincidence of f and T.

(b) Mappings f and T are said to be *commuting* if and only if $f(T(x,x,\dots,x)) = T(fx,fx,\dots,fx)$ for all $x \in X$.

(c) Mappings f and T are said to be *weakly compatible* if and only if they commute at their coincidence points.

Remark 2.4 For $k = 1$ the above definitions reduce to the usual definition of commuting and weakly compatible mappings in a metric space.

The set of coincidence points of f and T is denoted by $C(f,T)$.

Lemma 2.5 [24] *Let X be a nonempty set, k be a positive integer and $f : X^k \to X$, $g : X \to X$ be two weakly compatible mappings. If f and g have a unique point of coincidence $y = f(x,x,\dots,x) = g(x)$, then y is the unique common fixed point of f and g.*

Khan *et al.* [8] defined the set function $\theta : [0,\infty)^4 \to [0,\infty)$ as follows:

1. θ is continuous,
2. for all $t_1,t_2,t_3,t_4 \in [0,\infty)$, $\theta(t_1,t_2,t_3,t_4) = 0 \Leftrightarrow t_1t_2t_3t_4 = 0$.

3 Main results

Throughout this paper we assume that the b-metric $d : X \times X \to [0,\infty)$ is continuous on X^2.

Theorem 3.1 *Let (X,d) be a b-metric space with coefficient $s \ge 1$. For any positive integer k, let $f : X^k \to X$ and $g : X \to X$ be mappings satisfying the following conditions:*

$$f\bigl(X^k\bigr) \subseteq g(X), \tag{3.1}$$

$$\begin{aligned} &d\bigl(f(x_1,x_2,\dots,x_k),f(x_2,x_3,\dots,x_{k+1})\bigr) \\ &\quad\le \sum_{i=1}^{k} \alpha_i d(gx_i,gx_{i+1}) + \sum_{i=1}^{k+1}\sum_{j=1}^{k+1} \beta_{i,j} d\bigl(gx_i,f(x_j,x_j,\dots,x_j)\bigr) \\ &\qquad + L\cdot\theta\bigl(d\bigl(gx_1,f(x_{k+1},x_{k+1},x_{k+1},\dots,x_{k+1})\bigr), d\bigl(gx_{k+1},f(x_1,x_1,x_1,\dots,x_1)\bigr), \\ &\qquad d\bigl(gx_1,f(x_1,x_1,\dots,x_1)\bigr), d\bigl(gx_{k+1},f(x_{k+1},x_{k+1},\dots,x_{k+1})\bigr)\bigr), \end{aligned} \tag{3.2}$$

where $x_1,x_2,\dots,x_{k+1}$ are arbitrary elements in X and α_i, β_{ij}, L are nonnegative constants such that $\sum_{n=1}^{k} s^{k+3-n}[\alpha_n + \sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j}] < 1$ and

$$g(X) \text{ is complete.} \tag{3.3}$$

Then f and g have a unique coincidence point, i.e., $C(f,g) \ne \emptyset$. In addition, if f and g are weakly compatible, then f and g have a unique common fixed point. Moreover, for any $x_1 \in X$, the sequence $\{y_n\}$ defined by $y_n = g(x_n) = f(x_{n-1},x_{n-1},\dots,x_{n-1}) = Fx_{n-1}$ converges to the common fixed point of f and g.

Proof Let $x_1 \in X$, then $f(x_1,x_1,\dots,x_1) \in f(X^k) \subset g(X)$. So there exists $x_2 \in X$ such that $f(x_1,x_1,\dots,x_1) = g(x_2)$. Now $f(x_2,x_2,\dots,x_2) \in f(X^k) \subset g(X)$ and so there exists $x_3 \in X$ such that $f(x_2,x_2,\dots,x_2) = g(x_3)$. Continuing this process we define the sequence $\langle y_n\rangle$ in $g(X)$ as

$y_n = g(x_n) = f(x_{n-1}, x_{n-1}, \ldots, x_{n-1}) = Fx_{n-1}$, $n = 1, 2, \ldots, k+1$, where F is the associate operator for f. Let $d_n = d(y_n, y_{n+1}) = d(gx_n, gx_{n+1})$ and $D_{ij} = d(gx_i, f(x_j, x_j, \ldots, x_j))$.

Then we have

$$\begin{aligned}
d_{n+1} &= d\bigl(g(x_{n+1}), g(x_{n+2})\bigr) \\
&= d(Fx_n, Fx_{n+1}) \\
&= d\bigl(f(x_n, x_n, \ldots, x_n), f(x_{n+1}, x_{n+1}, \ldots, x_{n+1})\bigr) \\
&\leq sd\bigl(f(x_n, x_n, \ldots, x_n), f(x_n, x_n, \ldots, x_{n+1})\bigr) \\
&\quad + s^2 d\bigl(f(x_n, x_n, \ldots, x_{n+1}), f(x_n, x_n, \ldots, x_{n+1}, x_{n+1})\bigr) \\
&\quad + s^3 d\bigl(f(x_n, x_n, \ldots, x_{n+1}, x_{n+1}), f(x_n, \ldots, x_{n+1}, x_{n+1}, x_{n+1})\bigr) + \cdots \\
&\quad + s^k d\bigl(f(x_n, x_{n+1}, \ldots, x_{n+1}, x_{n+1}), f(x_{n+1}, \ldots, x_{n+1}, x_{n+1}, x_{n+1})\bigr).
\end{aligned}$$

Using (3.2) we get

$$\begin{aligned}
d_{n+1} &\leq s\Biggl\{\alpha_k d_n + \Biggl[\sum_{j=1}^{k}\beta_{1,j} + \sum_{j=1}^{k}\beta_{2,j} + \cdots + \sum_{j=1}^{k}\beta_{kj}\Biggr] D_{n,n} \\
&\quad + \Biggl[\sum_{i=1}^{k}\beta_{i,k+1}\Biggr] D_{n,n+1} + \Biggl[\sum_{j=1}^{k}\beta_{k+1,j}\Biggr] D_{n+1,n} + \beta_{k+1,k+1} D_{n+1,n+1}\Biggr\} \\
&\quad + s^2\Biggl\{\alpha_{k-1} d_n + \Biggl[\sum_{j=1}^{k-1}\beta_{1,j} + \sum_{j=1}^{k-1}\beta_{2,j} + \cdots + \sum_{j=1}^{k-1}\beta_{k-1,j}\Biggr] D_{n,n} \\
&\quad + \Biggl[\sum_{i=1}^{k-1}\beta_{i,k} + \sum_{i=1}^{k-1}\beta_{i,k+1}\Biggr] D_{n,n+1} + \Biggl[\sum_{j=1}^{k-1}\beta_{k,j} + \sum_{j=1}^{k-1}\beta_{k+1,j}\Biggr] D_{n+1,n} \\
&\quad + \Biggl[\sum_{j=k}^{k+1}\beta_{k,j} + \sum_{j=k}^{k+1}\beta_{k+1,j}\Biggr] D_{n+1,n+1}\Biggr\} \\
&\quad + \cdots + s^k\Biggl\{\alpha_1 d_n + \beta_{1,1} D_{n,n} + \Biggl[\sum_{j=2}^{k+1}\beta_{1,j}\Biggr] D_{n,n+1} + \Biggl[\sum_{i=2}^{k+1}\beta_{i,1}\Biggr] D_{n+1,n} \\
&\quad + \Biggl[\sum_{j=2}^{k+1}\beta_{2,j} + \sum_{j=2}^{k+1}\beta_{3,j} + \cdots + \sum_{j=2}^{k+1}\beta_{k+1,j}\Biggr] D_{n+1,n+1}\Biggr\} \\
&\quad + L\cdot\theta\Bigl(d\bigl(gx_n, (fx_{n+1}, x_{n+1}, x_{n+1}, \ldots, x_{n+1})\bigr), d\bigl(gx_{n+1}, f(x_n, x_n, x_n, \ldots, x_n)\bigr), \\
&\quad d\bigl(gx_n, f(x_n, x_n, \ldots, x_n)\bigr), d\bigl(gx_{n+1}, f(x_{n+1}, x_{n+1}, \ldots, x_{n+1})\bigr)\Bigr),
\end{aligned}$$

i.e.,

$$\begin{aligned}
d_{n+1} &\leq \bigl[s\alpha_k + s^2\alpha_{k-1} + s^3\alpha_{k-2} + \cdots + s^k\alpha_1\bigr] d_n + s\Biggl\{\Biggl[\sum_{i=1}^{k}\sum_{j=1}^{k}\beta_{i,j}\Biggr] D_{n,n} \\
&\quad + \Biggl[\sum_{i=1}^{k}\beta_{i,k+1}\Biggr] D_{n,n+1} + \Biggl[\sum_{j=1}^{k}\beta_{k+1,j}\Biggr] D_{n+1,n} + \beta_{k+1,k+1} D_{n+1,n+1}\Biggr\}
\end{aligned}$$

$$
\begin{aligned}
&+ s^2\Biggl\{\Biggl[\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\beta_{i,j}\Biggr]D_{n,n} + \Biggl[\sum_{i=1}^{k-1}\sum_{j=k}^{k+1}\beta_{i,j}\Biggr]D_{n,n+1} + \Biggl[\sum_{i=k}^{k+1}\sum_{j=1}^{k-1}\beta_{i,j}\Biggr]D_{n+1,n} \\
&+ \Biggl[\sum_{i=k}^{k+1}\sum_{j=k}^{k+1}\beta_{i,j}\Biggr]D_{n+1,n+1}\Biggr\} + \cdots + s^k\Biggl\{\beta_{1,1}D_{n,n} + \Biggl[\sum_{j=2}^{k+1}\beta_{1,j}\Biggr]D_{n,n+1} \\
&+ \Biggl[\sum_{i=2}^{k+1}\beta_{i,1}\Biggr]D_{n+1,n} + \Biggl[\sum_{i=2}^{k+1}\sum_{j=2}^{k+1}\beta_{i,j}\Biggr]D_{n+1,n+1}\Biggr\} + L\cdot 0,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
d_{n+1} \leq{}& \bigl[s\alpha_k + s^2\alpha_{k-1} + s^3\alpha_{k-2} + \cdots + s^k\alpha_1\bigr]d_n \\
&+ \Biggl[s\sum_{i=1}^{k}\sum_{j=1}^{k}\beta_{i,j} + s^2\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=1}^{2}\sum_{j=1}^{2}\beta_{i,j} + s^k\beta_{1,1}\Biggr]D_{n,n} \\
&+ \Biggl[s\sum_{i=1}^{k}\beta_{i,k+1} + s^2\sum_{i=1}^{k-1}\sum_{j=k}^{k+1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=1}^{2}\sum_{j=3}^{k+1}\beta_{i,j} + s^k\sum_{j=2}^{k+1}\beta_{1,j}\Biggr]D_{n,n+1} \\
&+ \Biggl[s\sum_{j=1}^{k}\beta_{k+1,j} + s^2\sum_{i=k}^{k+1}\sum_{j=1}^{k-1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=3}^{k+1}\sum_{j=1}^{2}\beta_{i,j} + s^k\sum_{i=2}^{k+1}\beta_{i,1}\Biggr]D_{n+1,n} \\
&+ \Biggl[s^k\sum_{i=2}^{k+1}\sum_{j=2}^{k+1}\beta_{i,j} + s^{k-1}\sum_{i=3}^{k+1}\sum_{j=3}^{k+1}\beta_{i,j} + \cdots + s^2\sum_{i=k}^{k+1}\sum_{j=k}^{k+1}\beta_{i,j} + s\beta_{k+1,k+1}\Biggr]D_{n+1,n+1} \\
={}& Ad_n + BD_{n,n} + CD_{n,n+1} + ED_{n+1,n} + FD_{n+1,n+1},
\end{aligned}
$$

where A, B, C, E and F are the coefficients of $d_n, D_{n,n}, D_{n,n+1}, D_{n+1,n}$ and $D_{n+1,n+1}$ respectively in the above inequality. By the definition, $D_{n,n} = d(gx_n, f(x_n, x_n, \ldots, x_n)) = d(gx_n, gx_{n+1}) = d_n$, $D_{n,n+1} = d(gx_n, f(x_{n+1}, x_{n+1}, \ldots, x_{n+1})) = d(gx_n, gx_{n+2})$, $D_{n+1,n} = d(gx_{n+1}, f(x_n, x_n, \ldots, x_n)) = d(gx_{n+1}, gx_{n+1}) = 0$, $D_{n+1,n+1} = d(gx_{n+1}, f(x_{n+1}, x_{n+1}, \ldots, x_{n+1})) = d(gx_{n+1}, gx_{n+2}) = d_{n+1}$; therefore,

$$
\begin{aligned}
d_{n+1} &\leq Ad_n + Bd_n + Cd(gx_n, gx_{n+2}) + Fd_{n+1} \\
&\leq Ad_n + Bd_n + Csd(gx_n, gx_{n+1}) + Csd(gx_{n+1}, gx_{n+2}) + Fd_{n+1} \\
&= (A + B + Cs)d_n + (Cs + F)d_{n+1},
\end{aligned}
$$

i.e., $(1 - Cs - F)d_{n+1} \leq (A + B + Cs)d_n$. Again, interchanging the role of x_n and x_{n+1} and repeating the above process, we obtain $(1 - Es - B)d_{n+1} \leq (A + F + Es)d_n$. It follows that

$$
\begin{aligned}
\bigl(2 - (C + E)s - F - B\bigr)d_{n+1} &\leq \bigl(2A + B + F + s(C + E)\bigr)d_n \\
d_{n+1} &\leq \frac{2A + B + F + s(C + E)}{2 - B - F - (C + E)s}d_n \\
d_{n+1} &\leq \lambda d_n,
\end{aligned}
$$

where $\lambda = \frac{2A+B+F+s(C+E)}{2-B-F-(C+E)s}$. Thus we have

$$
d_{n+1} \leq \lambda^{n+1}d_0 \quad \text{for all } n \geq 0. \tag{3.4}
$$

We will show that $\lambda < 1$ and $s\lambda < 1$. We have

$$\begin{aligned}
&A + B + F + s(C + E) \\
&\quad \le s[A + B + C + E + F] \\
&\quad = s\bigl[s\alpha_k + s^2\alpha_{k-1} + s^3\alpha_{k-2} + \cdots + s^k\alpha_1\bigr] \\
&\qquad + s\Biggl[s\sum_{i=1}^{k}\sum_{j=1}^{k}\beta_{i,j} + s^2\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=1}^{2}\sum_{j=1}^{2}\beta_{i,j} + s^k\beta_{1,1}\Biggr] \\
&\qquad + s\Biggl[s\sum_{i=1}^{k}\beta_{i,k+1} + s^2\sum_{i=1}^{k-1}\sum_{j=k}^{k+1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=1}^{2}\sum_{j=3}^{k+1}\beta_{i,j} + s^k\sum_{j=2}^{k+1}\beta_{1,j}\Biggr] \\
&\qquad + s\Biggl[s\sum_{j=1}^{k}\beta_{k+1,j} + s^2\sum_{i=k}^{k+1}\sum_{j=1}^{k-1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=3}^{k+1}\sum_{j=1}^{2}\beta_{i,j} + s^k\sum_{i=2}^{k+1}\beta_{i,1}\Biggr] \\
&\qquad + s\Biggl[s\beta_{k+1,k+1} + s^2\sum_{i=k}^{k+1}\sum_{j=k}^{k+1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=3}^{k+1}\sum_{j=3}^{k+1}\beta_{i,j} + s^k\sum_{i=2}^{k+1}\sum_{j=2}^{k+1}\beta_{i,j}\Biggr] \\
&\quad = \bigl[s^2\alpha_k + s^3\alpha_{k-1} + s^4\alpha_{k-2} + \cdots + s^{k+1}\alpha_1\bigr] + s^2\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} + s^3\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} \\
&\qquad + s^4\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} + \cdots + s^{k+1}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} \\
&\quad = \bigl[s^2\alpha_k + s^3\alpha_{k-1} + s^4\alpha_{k-2} + \cdots + s^{k+1}\alpha_1\bigr] \\
&\qquad + \bigl[s^2 + s^3 + s^4 + \cdots + s^{k+1}\bigr]\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} \\
&\quad \le \bigl[s^3\alpha_k + s^4\alpha_{k-1} + s^5\alpha_{k-2} + \cdots + s^{k+2}\alpha_1\bigr] \\
&\qquad + \bigl[s^3 + s^4 + s^5 + \cdots + s^{k+2}\bigr]\sum_{i=1}^{k+1}\sum_{j=1}^{k+2}\beta_{i,j} \\
&\quad = \sum_{n=1}^{k} s^{k+3-n}\Biggl[\alpha_n + \sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j}\Biggr] < 1,
\end{aligned}$$

and so $\lambda < 1$. We also have $sA + sB + sF + s(C + E) = s(A + B + F + C + E) < 1$ (proved above) and

$$\begin{aligned}
&sA + B + F + s^2(C + E) \\
&\quad \le s^2[A + B + C + E + F] \\
&\quad = s^2\bigl[s\alpha_k + s^2\alpha_{k-1} + s^3\alpha_{k-2} + \cdots + s^k\alpha_1\bigr] \\
&\qquad + s^2\Biggl[s\sum_{i=1}^{k}\sum_{j=1}^{k}\beta_{i,j} + s^2\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=1}^{2}\sum_{j=1}^{2}\beta_{i,j} + s^k\beta_{1,1}\Biggr] \\
&\qquad + s^2\Biggl[s\sum_{i=1}^{k}\beta_{i,k+1} + s^2\sum_{i=1}^{k-1}\sum_{j=k}^{k+1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=1}^{2}\sum_{j=3}^{k+1}\beta_{i,j} + s^k\sum_{j=2}^{k+1}\beta_{1,j}\Biggr]
\end{aligned}$$

$$
\begin{aligned}
&+ s^2\left[s\sum_{j=1}^{k}\beta_{k+1,j} + s^2\sum_{i=k}^{k+1}\sum_{j=1}^{k-1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=3}^{k+1}\sum_{j=1}^{2}\beta_{i,j} + s^k\sum_{i=2}^{k+1}\beta_{i,1}\right] \\
&+ s^2\left[s\beta_{k+1,k+1} + s^2\sum_{i=k}^{k+1}\sum_{j=k}^{k+1}\beta_{i,j} + \cdots + s^{k-1}\sum_{i=3}^{k+1}\sum_{j=3}^{k+1}\beta_{i,j} + s^k\sum_{i=2}^{k+1}\sum_{j=2}^{k+1}\beta_{i,j}\right] \\
= {}& \left[s^3\alpha_k + s^4\alpha_{k-1} + s^5\alpha_{k-2} + \cdots + s^{k+2}\alpha_1\right] + s^3\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} + s^4\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} \\
&+ s^5\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} + \cdots + s^{k+2}\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} \\
= {}& \left[s^3\alpha_k + s^4\alpha_{k-1} + s^5\alpha_{k-2} + \cdots + s^{k+2}\alpha_1\right] \\
&+ \left[s^3 + s^4 + s^5 + \cdots + s^{k+2}\right]\sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j} \\
= {}& \sum_{n=1}^{k} s^{k+3-n}\left[\alpha_n + \sum_{i=1}^{k+1}\sum_{j=1}^{k+1}\beta_{i,j}\right] < 1,
\end{aligned}
$$

and so $s\lambda < 1$.

Thus, for all $n, p \in N$,

$$
\begin{aligned}
d(gx_n, gx_{n+p}) &\leq sd(gx_n, gx_{n+1}) + s^2 d(gx_{n+1}, gx_{n+2}) + \cdots + s^{p-1} d(gx_{n+(p-1)}, gx_{n+p}) \\
&= sd_n + s^2 d_{n+1} + \cdots + s^{p-1} d_{n+(p-1)} \\
&\leq s\lambda^n d_0 + s^2\lambda^{n+1} d_0 + \cdots + s^{p-1}\lambda^{n+(p-1)} d_0 \\
&\leq \frac{s\lambda^n}{1-s\lambda} d_0 \to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

Thus $\{gx_n\}$ is a Cauchy sequence. By completeness of $g(X)$, there exists $u \in X$ such that

$$
\lim_{n\to\infty} gx_n = u \text{ and there exists } p \in X \text{ such that } g(p) = u. \tag{3.5}
$$

We shall show that u is the fixed point of f and g. Using a similar process as the one used in the calculation of d_{n+1}, we obtain

$$
\begin{aligned}
d\big(g(p), f(p,\ldots,p)\big) &\leq s\big[d\big(g(p), y_{n+1}\big) + d\big(y_{n+1}, f(p,p,\ldots,p)\big)\big] \\
&\leq s\big[d\big(g(p), y_{n+1}\big) + d(Fx_n, Fp)\big] \\
&\leq s\big[d\big(g(p), y_{n+1}\big) + Ad(gx_n, gp) + Bd\big(gx_n, f(x_n, x_n, \ldots, x_n)\big) \\
&\quad + Cd\big(gx_n, f(p,p,\ldots,p)\big) \\
&\quad + Ed\big(gp, f(x_n, x_n, \ldots, x_n)\big) + Fd\big(gp, f(p,p,\ldots,p)\big)\big].
\end{aligned}
$$

It follows from (3.5) that

$$
d\big(g(p), f(p,\ldots,p)\big) \leq s(C+F)d\big(gp, f(p,p,\ldots,p)\big). \tag{3.6}
$$

As $s(C+F)<1$, we obtain $F(p)=g(p)=f(p,p,\dots,p)=u$ Thus, u is a point of coincidence of f and g. If u' is another point of coincidence of f and g, then there exists $p'\in X$ such that $F(p')=g(p')=f(p',p',\dots,p')=u'$.

Then we have

$$\begin{aligned} d(u,u') &= d(Fp,Fp') \\ &\le Ad(gp,gp') + Bd(gp,f(p,p,\dots,p)) \\ &\quad + Cd(gp,f(p',p',\dots,p')) \\ &\quad + Ed(gp',f(p,p,\dots,p)) + Fd(gp',f(p',p',\dots,p')) \\ &= Ad(u,u') + Bd(u,u) + Cd(u,u') + Ed(u',u) + Fd(u',u) \\ &= (A+C+E+F)d(u,u'). \end{aligned}$$

As $A+C+E+F<1$, we obtain from the above inequality that $d(u,u')=0$, that is, $u=u'$. Thus the point of coincidence u is unique. Further, if f and g are weakly compatible, then by Lemma 2.5, u is the unique common fixed point of f and g. □

Remark 3.2 Taking $s=1$, $g=I$ and $\theta(t_1,t_2,t_3,t_4)=0$ in Theorem 3.1, we get Theorem 4 of Shukla *et al.* [13].

Remark 3.3 For $s=1$, $g=I$, $i=j$, $\beta_{ij}=\delta_{k+1}$, $\forall i$, $L=1$, we obtain Theorem 2.1 of Khan *et al.* [8].

Remark 3.4 For $s=1$, $g=I$, $\beta_{ij}=0$, $\forall i,j\in\{1,2,\dots,k+1\}$ and $\theta(t_1,t_2,t_3,t_4)=\min\{(t_1,t_2,t_3,t_4)\}$, we obtain the result of Pacurar [5].

Remark 3.5 For $s=1$, $g=I$, $\alpha_i=0$, $i=j$, $\beta_{ij}=a$, $L=0$, we obtain the result of Pacurar [4].

Remark 3.6 For $s=1$, $g=I$, $\beta_{ij}=0$, $\forall i,j\in\{1,2,\dots,k+1\}$, $L=0$, we obtain the result of Presic [2].

Next we prove a generalized Ciric-Presic type fixed point theorem in a b-metric space. Consider a function $\phi:R^k\to R$ such that

1. ϕ is an increasing function, *i.e.*, $x_1<y_1,x_2<y_2,\dots,x_k<y_k$ implies $\phi(x_1,x_2,\dots,x_k)<\phi(y_1,y_2,\dots,y_k)$;
2. $\phi(t,t,t,\dots)\le t$ for all $t\in X$;
3. ϕ is continuous in all variables.

Theorem 3.7 *Let (X,d) be a b-metric space with $s\ge 1$. For any positive integer k, let $f:X^k\to X$ and $g:X\to X$ be mappings satisfying the following conditions*:

$$f(X^k)\subseteq g(X), \tag{3.7}$$

$$\begin{aligned} &d(f(x_1,x_2,\dots,x_k),f(x_2,x_3,\dots,x_{k+1})) \\ &\quad \le \lambda\phi(d(gx_1,gx_2),d(gx_2,gx_3),d(gx_3,gx_4),\dots,d(gx_k,gx_{k+1})), \end{aligned} \tag{3.8}$$

where $x_1, x_2, \ldots, x_{k+1}$ are arbitrary elements in X, $\lambda \in (0, \frac{1}{s^k})$,

$$g(X) \text{ is complete} \tag{3.9}$$

and

$$d\bigl(f(u,u,\ldots,u), f(v,v,\ldots,v)\bigr) < d(gu, gv) \tag{3.10}$$

for all $u, v \in X$. Then f and g have a coincidence point, i.e., $C(f,g) \neq \emptyset$. In addition, if f and g are weakly compatible, then f and g have a unique common fixed point. Moreover, for any $x_1 \in X$, the sequence $\{y_n\}$ defined by $y_n = g(x_n) = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$ converges to the common fixed point of f and g.

Proof For arbitrary $x_1, x_2, \ldots, x_k$ in X, let

$$R = \max\left(\frac{d(gx_1, gx_2)}{\theta}, \frac{d(gx_2, gx_3)}{\theta^2}, \ldots, \frac{d(gx_k, f(x_1, x_2, \ldots, x_k))}{\theta^k}\right), \tag{3.11}$$

where $\theta = \lambda^{\frac{1}{k}}$. By (3.7) we define the sequence $\langle y_n \rangle$ in $g(X)$ as $y_n = gx_n$ for $n = 1, 2, \ldots, k$ and $y_{n+k} = g(x_{n+k}) = f(x_n, x_{n+1}, \ldots, x_{n+k-1})$, $n = 1, 2, \ldots$.

Let $\alpha_n = d(y_n, y_{n+1})$. By the method of mathematical induction, we will prove that

$$\alpha_n \leq R\theta^n \quad \text{for all } n. \tag{3.12}$$

Clearly, by the definition of R, (3.12) is true for $n = 1, 2, \ldots, k$. Let the k inequalities $\alpha_n \leq R\theta^n, \alpha_{n+1} \leq R\theta^{n+1}, \ldots, \alpha_{n+k-1} \leq R\theta^{n+k-1}$ be the induction hypothesis. Then we have

$$\begin{aligned}
\alpha_{n+k} &= d(y_{n+k}, y_{n+k+1}) \\
&= d\bigl(f(x_n, x_{n+1}, \ldots, x_{n+k-1}), f(x_{n+1}, x_{n+2}, \ldots, x_{n+k})\bigr) \\
&\leq \lambda\phi\bigl(d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}), \ldots, d(gx_{n+k-1}, gx_{n+k}), \\
&\quad d\bigl(gx_n, f(x_n, x_n, \ldots, x_n)\bigr), d\bigl(gx_{n+k}, f(x_{n+k}, x_{n+k}, \ldots, x_{n+k})\bigr)\bigr) \\
&= \lambda\phi(\alpha_n, \alpha_{n+1}, \ldots, \alpha_{n+k-1}) \\
&\leq \lambda\phi\bigl(R\theta^n, R\theta^{n+1}, \ldots, R\theta^{n+k-1}\bigr) \\
&\leq \lambda\phi\bigl(R\theta^n, R\theta^n, \ldots, R\theta^n\bigr) \\
&\leq \lambda R\theta^n \\
&= R\theta^{n+k}.
\end{aligned}$$

Thus the inductive proof of (3.12) is complete. Now, for $n, p \in N$, we have

$$\begin{aligned}
d(y_n, y_{n+p}) &\leq sd(y_n, y_{n+1}) + s^2 d(y_{n+1}, y_{n+2}) + \cdots + s^{p-1} d(y_{n+p-1}, y_{n+p}), \\
&\leq sR\theta^n + s^2 R\theta^{n+1} + \cdots + s^{p-1} R\theta^{n+p-1} \\
&\leq sR\theta^n\bigl(1 + s\theta + s^2\theta^2 + \cdots\bigr) \\
&= \frac{sR\theta^n}{1 - s\theta}.
\end{aligned}$$

Hence the sequence $\langle y_n \rangle$ is a Cauchy sequence in $g(X)$ and since $g(X)$ is complete, there exist $v, u \in X$ such that $\lim_{n\to\infty} y_n = v = g(u)$,

$$\begin{aligned}
d\bigl(gu, f(u,u,\ldots,u)\bigr) &\le s\bigl[d(gu, y_{n+k}) + d\bigl(y_{n+k}, f(u,u,\ldots,u)\bigr)\bigr] \\
&= s\bigl[d(gu, y_{n+k}) + d\bigl(f(x_n, x_{n+1},\ldots,x_{n+k-1}), f(u,u,\ldots,u)\bigr)\bigr] \\
&= sd(gu, y_{n+k}) + sd\bigl(f(x_n, x_{n+1},\ldots,x_{n+k-1}), f(u,u,\ldots,u)\bigr) \\
&\le sd(gu, y_{n+k}) + s^2 d\bigl(f(u,u,\ldots,u), f(u,u,\ldots,x_n)\bigr) \\
&\quad + s^3 d\bigl(f(u,u,\ldots,x_n), f(u,u,\ldots,x_n,x_{n+1})\bigr) \\
&\quad + \cdots + s^{k-1} d\bigl(f(u,x_n,\ldots,x_{n+k-2}), f(x_n,x_{n+1},\ldots,x_{n+k-1})\bigr) \\
&\le sd(gu, y_{n+k}) + s^2\lambda\phi\bigl\{d(gu,gu), d(gu,gu),\ldots,d(gu,gx_n)\bigr\} \\
&\quad + s^3\lambda\phi\bigl\{d(gu,gu), d(gu,gu),\ldots,d(gu,gx_n), d(gx_n,gx_{n+1})\bigr\} + \cdots \\
&\quad + s^{k-1}\lambda\phi\bigl\{d(gu,gx_n), d(gx_n,gx_{n+1}),\ldots,d(gx_{n+k-2},gx_{n+k-1})\bigr\} \\
&= sd(gu, y_{n+k}) + s^2\lambda\phi\bigl(0,0,\ldots,d(gu,gx_n)\bigr) \\
&\quad + s^3\lambda\phi\bigl(0,0,\ldots,d(gu,gx_n), d(gx_n,gx_{n+1})\bigr) + \cdots \\
&\quad + s^{k-1}\lambda\phi\bigl(d(gu,gx_n), d(gx_n,gx_{n+1}),\ldots,d(gx_{n+k-2},gx_{n+k-1})\bigr).
\end{aligned}$$

Taking the limit when n tends to infinity, we obtain $d(gu, f(u,u,\ldots,u)) \le 0$. Thus $gu = f(u,u,u,\ldots,u)$, *i.e.*, $C(g,f) \neq \emptyset$. Thus there exist $v, u \in X$ such that $\lim_{n\to\infty} y_n = v = g(u) = f(u,u,u,\ldots,u)$. Since g and f are weakly compatible, $g(f(u,u,\ldots,u)) = f(gu,gu,gu,\ldots,gu)$. By (3.10) we have that

$$\begin{aligned}
d(ggu, gu) &= d\bigl(gf(u,u,\ldots,u), f(u,u,\ldots,u)\bigr) \\
&= d\bigl(f(gu,gu,gu,\ldots,gu), f(u,u,\ldots,u)\bigr) \\
&< d(ggu, gu)
\end{aligned}$$

implies $d(ggu, gu) = 0$ and so $ggu = gu$. Hence we have $gu = ggu = g(f(u,u,\ldots,u)) = f(gu,gu,gu,\ldots,gu)$, *i.e.*, gu is a common fixed point of g and f, and $\lim_{n\to\infty} y_n = g(u)$. Now suppose that x, y are two fixed points of g and f. Then

$$\begin{aligned}
d(x,y) &= d\bigl(f(x,x,x,\ldots,x), f(y,y,y,\ldots,y)\bigr) \\
&< d(gx, gy) \\
&= d(x,y).
\end{aligned}$$

This implies $x = y$. Hence the common fixed point is unique. □

Remark 3.8 Taking $s = 1$, $g = I$ and $\phi(x_1,x_2,\ldots,x_k) = \max\{x_1,x_2,\ldots,x_k\}$ in Theorem 3.7, we obtain Theorem 1.2, *i.e.*, the result of Ciric and Presic [3].

Remark 3.9 For $\lambda \in (0, \frac{1}{s^{k+1}})$, we can drop the condition (3.10) of Theorem 3.7. In fact we have the following.

Theorem 3.10 *Let* (X,d) *be a b-metric space with* $s \geq 2$. *For any positive integer* k, *let* $f : X^k \to X$ *and* $g : X \to X$ *be mappings satisfying conditions* (3.7), (3.8) *and* (3.9) *with* $\lambda \in (0, \frac{1}{s^{k+1}})$. *Then all conclusions of Theorem* 3.7 *hold.*

Proof As proved in Theorem 3.7, there exist $v, u \in X$ such that $\lim_{n\to\infty} y_n = v = g(u) = f(u,u,\dots,u)$, *i.e.*, $C(g,f) \neq \emptyset$. Since g and f are weakly compatible, $g(f(u,u,\dots,u)) = f(gu,gu,gu,\dots,gu)$. By (3.8) we have

$$\begin{aligned}
d(ggu,gu) &= d\bigl(gf(u,u,\dots,u),f(u,u,\dots,u)\bigr) \\
&= d\bigl(f(gu,gu,gu,\dots,gu),f(u,u,\dots,u)\bigr) \\
&\leq sd\bigl(f(gu,gu,gu,\dots,gu),f(gu,gu,\dots,gu,u)\bigr) \\
&\quad + s^2 d\bigl(f(gu,gu,\dots,gu,u),f(gu,gu,\dots,u,u)\bigr) \\
&\quad + \cdots + s^{k-1} d\bigl(f(gu,gu,\dots,u,u),f(u,u,\dots,u)\bigr) \\
&\quad + s^{k-1} d\bigl(f(gu,u,\dots,u,u),f(u,u,\dots,u)\bigr) \\
&\leq s\lambda\phi\bigl(d(ggu,ggu),\dots,d(ggu,ggu),d(ggu,gu)\bigr) \\
&\quad + s^2\lambda\phi\bigl(d(ggu,ggu),\dots,d(ggu,gu),d(gu,gu)\bigr) \\
&\quad + \cdots + s^{k-1}\lambda\phi\bigl(d(ggu,gu),\dots,d(gu,gu),d(gu,gu)\bigr) \\
&= s\lambda\phi\bigl(0,0,0,\dots,d(ggu,gu)\bigr) + s^2\lambda\phi\bigl(0,0,\dots,0,d(ggu,gu),0\bigr) \\
&\quad + \cdots + s^{k-1}\lambda\phi\bigl(d(ggu,gu),0,0,\dots,0\bigr) \\
&= s\lambda\bigl[1+s+s^2+s^3+\cdots+s^{k-2}+s^{k-2}\bigr]d(ggu,gu) \\
&\leq s\lambda\bigl[1+s+s^2+s^3+\cdots+s^{k-2}+s^{k-1}\bigr]d(ggu,gu) \\
&= s\lambda\frac{s^k-1}{s-1}d(ggu,gu).
\end{aligned}$$

$s\lambda\frac{s^k-1}{s-1} < 1$ implies $d(ggu,gu) = 0$ and so $ggu = gu$. Hence we have $gu = ggu = g(f(u,u,\dots,u)) = f(gu,gu,gu,\dots,gu)$, *i.e.*, gu is a common fixed point of g and f, and $\lim_{n\to\infty} y_n = g(u)$. Now suppose that x, y are two fixed points of g and f. Then

$$\begin{aligned}
d(x,y) &= d\bigl(f(x,x,x,\dots,x),f(y,y,y,\dots,y)\bigr) \\
&\leq sd\bigl(f(x,x,\dots,x),f(x,x,\dots,x,y)\bigr) + s^2 d\bigl(f(x,x,\dots,x,y), \\
&\quad f(x,x,x,\dots,x,y,y)\bigr) + \cdots + s^{k-1} d\bigl(f(x,x,y,\dots,y),f(y,y,\dots,y)\bigr) \\
&\quad + s^{k-1} d\bigl(f(x,y,y,\dots,y),f(y,y,\dots,y)\bigr) \\
&\leq s\lambda\phi\bigl\{d(fx,fx),d(fx,fx),\dots,d(fx,fy)\bigr\} + s^2\lambda\phi\bigl\{d(fx,fx), \\
&\quad d(fx,fx),\dots,d(fx,fy),d(fy,fy)\bigr\} \\
&\quad + \cdots + s^{k-1}\lambda\phi\bigl\{d(fx,fy),d(fy,fy),\dots,d(fy,fy)\bigr\} \\
&= s\lambda\phi\bigl(0,0,\dots,d(fx,fy)\bigr) + s^2\lambda\phi\bigl(0,0,\dots,d(fx,fy),0\bigr) + \cdots \\
&\quad + s^{k-1}\lambda\phi\bigl(d(fx,fy),0,0,0,\dots,0\bigr)
\end{aligned}$$

$$\begin{aligned}
&= \lambda\big[s + s^2 + s^3 + \cdots + s^{k-1} + s^{k-1}\big]d(fx,fy) \\
&= s\lambda\big[1 + s + s^2 + s^3 + \cdots + s^{k-2} + s^{k-2}\big]d(fx,fy) \\
&\le s\lambda\big[1 + s + s^2 + s^3 + \cdots + s^{k-2} + s^{k-1}\big]d(fx,fy) \\
&= s\lambda\frac{s^k - 1}{s-1}d(fx,fy). \\
&= s\lambda\frac{s^k - 1}{s-1}d(x,y).
\end{aligned}$$

This implies $x = y$. Hence the common fixed point is unique. □

Example 3.11 Let $X = R$ and $d : X \times X \to X$ such that $d(x,y) = |x-y|^3$. Then d is a b-metric on X with $s = 4$. Let $f : X^2 \to X$ and $g : X \to X$ be defined as follows:

$$\begin{aligned}
&f(x,y) = \frac{x^2 + y^2}{13} + \frac{18}{13} \quad \text{if } (x,y) \in R, \\
&gx = x^2 - 2 \quad \text{if } x \in R.
\end{aligned}$$

We will prove that f and g satisfy condition (3.8):

$$\begin{aligned}
d\big(f(x,y),f(y,z)\big) &= \big|f(x,y) - f(y,z)\big|^3 \\
&= \left|\frac{x^2 - z^2}{13}\right|^3 = \left|\frac{x^2 - y^2 + y^2 - z^2}{13}\right|^3 \\
&\le 4\left(\left|\frac{x^2 - y^2}{13}\right|^3 + \left|\frac{y^2 - z^2}{13}\right|^3\right) \\
&= \frac{4}{13^3}\big[\big|x^2 - y^2\big|^3 + \big|y^2 - z^2\big|^3\big] \\
&= \frac{8}{13^3}\frac{1}{2}\big[\big|x^2 - y^2\big|^3 + \big|y^2 - z^2\big|^3\big] \\
&\le \frac{8}{13^3}\max\big\{\big|x^2 - y^2\big|^3, \big|y^2 - z^2\big|^3\big\} \\
&= \frac{8}{13^3}\max\big\{d(gx,gy), d(gy,gz)\big\}.
\end{aligned}$$

Thus, f and g satisfy condition (3.8) with $\lambda = \frac{8}{13^3} \in (0, \frac{1}{4^3})$. Clearly $C(f,g) = 2$, f and g commute at 2. Finally, 2 is the unique common fixed point of f and g. But f and g do not satisfy condition (3.10) as at $x = -1$ and $y = 1$, $d(f(x,x),f(y,y)) = d(f(-1,-1),f(1,1)) = d(\frac{2}{13} + \frac{18}{13}, \frac{2}{13} + \frac{18}{13}) = 0 = d(-1,-1) = d(g(-1),g(1)) = d(gx,gy)$.

4 Application to matrix equation

In this section we have applied Theorem 3.7 to study the existence of solutions of the nonlinear matrix equation

$$X = Q + \sum_{i=1}^{m} A_i X^{\delta_i} A_i^*, \quad 0 < |\delta_i| < 1, \tag{4.1}$$

where Q is an $n \times n$ positive semidefinite matrix and A_i's are nonsingular $n \times n$ matrices, or Q is an $n \times n$ positive definite matrix and A_i's are arbitrary $n \times n$ matrices, and a positive definite solution X is sought. Here A_i^* denotes the conjugate transpose of the matrix A_i. The existence and uniqueness of positive definite solutions and numerical methods for finding a solution of (4.1) have recently been studied by many authors (see [25–30]). The Thompson metric on the open convex cone $P(N)$ ($N \geq 2$), the set of all $N \times N$ Hermitian positive definite matrices, is defined by

$$d(A,B) = \max\bigl\{\log M(A/B), \log M(B/A)\bigr\}, \tag{4.2}$$

where $M(A/B) = \inf\{\lambda > 0 : A \leq \lambda B\} = \lambda_{\max}(B^{-1/2}AB^{-1/2})$, the maximal eigenvalue of $B^{-1/2}AB^{-1/2}$. Here $X \leq Y$ means that $Y - X$ is positive semidefinite and $X < Y$ means that $Y - X$ is positive definite. Thompson [31] has proved that $P(N)$ is a complete metric space with respect to the Thompson metric d and $d(A,B) = \| \log(A^{-1/2}BA^{-1/2})\|$, where $\| \cdot \|$ stands for the spectral norm. The Thompson metric exists on any open normal convex cone of real Banach spaces [31, 32]; in particular, the open convex cone of positive definite operators of a Hilbert space. It is invariant under the matrix inversion and congruence transformations:

$$d(A,B) = d\bigl(A^{-1}, B^{-1}\bigr) = d\bigl(MAM^*, MBM^*\bigr) \tag{4.3}$$

for any nonsingular matrix M. One remarkable and useful result is the nonpositive curvature property of the Thompson metric:

$$d\bigl(X^r, Y^r\bigr) \leq rd(X,Y), \quad r \in [0,1]. \tag{4.4}$$

By the invariant properties of the metric, we then have

$$d\bigl(MX^rM^*, MY^rM^*\bigr) = |r|d(X,Y), \quad r \in [-1,1] \tag{4.5}$$

for any $X, Y \in P(N)$ and a nonsingular matrix M. Proceeding as in [30] we prove the following lemma.

Lemma 4.1 *For any* $A_1, A_2, \ldots, A_k \in P(N)$, $B_1, B_2, \ldots, B_k \in P(N)$, $d(A_1 + A_2 + \cdots + A_k, B_1 + B_2 + \cdots + B_k) \leq \max\{d(A_1,B_1), d(A_2,B_2), \ldots, d(A_k,B_k)\}$.

Proof Without loss of generality we can assume that $d(A_1,B_1) \leq d(A_2,B_2) \leq \cdots \leq d(A_k,B_k) = \log r$. Then $B_1 \leq rA_1, B_2 \leq rA_2, \ldots, B_k \leq rA_k$ and $A_1 \leq rB_1, A_2 \leq rB_2, \ldots, A_k \leq rB_k$, and thus $B_1 + A_1 \leq r(A_1 + B_1), B_2 + A_2 \leq r(A_2 + B_2), \ldots, B_k + A_k \leq r(A_k + B_k)$. Hence $A_1 + A_2 + \cdots + A_k \leq r[B_1 + B_2 + \cdots + B_k]$ and $B_1 + B_2 + \cdots + B_k \leq r[A_1 + A_2 + \cdots + A_k]$. Hence $d(A_1 + A_2 + \cdots + A_k, B_1 + B_2 + \cdots + B_k) \leq \log r = d(A_k,B_k) = \max\{d(A_1,B_1), d(A_2,B_2), \ldots, d(A_k,B_k)\}$. □

For arbitrarily chosen positive definite matrices $X_{n-r}, X_{n-(r-1)}, \ldots, X_n$, consider the iterative sequence of matrices, given by

$$X_{n+1} = Q + A_1^* X_{n-r}^{\alpha_1} A_1 + A_2^* X_{n-(r-1)}^{\alpha_2} A_2 + \cdots + A_{r+1}^* X_n^{\alpha_{r+1}} A_{r+1}, \tag{4.6}$$

$\alpha_1, \alpha_2, \ldots, \alpha_{r+1}$ are real numbers.

Theorem 4.2 *Suppose that* $\lambda = \max\{|\alpha_1|, |\alpha_2|, \dots, |\alpha_{r+1}|\} \in (0,1)$.

(i) *Equation* (4.6) *has a unique equilibrium point in* $P(N)$, *that is, there exists unique* $U \in P(N)$ *such that*

$$U = Q + A_1^* U^{\alpha_1} A_1 + A_2^* U^{\alpha_2} A_2 + \cdots + A_{r+1}^* U^{\alpha_{r+1}} A_{r+1}. \tag{4.7}$$

(ii) *The iterative sequence* $\{X_n\}$ *defined by* (4.6) *converges to a unique solution of* (4.1).

Proof Define the mapping $f : P(N) \times P(N) \times P(N) \times \cdots \times P(N) \to P(N)$ by

$$f(X_1, X_2, X_{n-(r-2)}, \dots, X_k) = Q + A_1^* X_1^{\alpha_1} A_1 + A_2^* X_2^{\alpha_2} A_2 + \cdots + A_{r+1}^* X_k^{\alpha_{r+1}} A_{r+1}, \tag{4.8}$$

where $X_1, X_2, \dots, X_k \in P(N)$.

For all $X_{n-r}, X_{n-(r-1)}, X_{n-(r-2)}, \dots, X_{n+1} \in P(N)$, we have

$$\begin{aligned}
& d\bigl(f(X_{n-r}, X_{n-(r-1)}, X_{n-(r-2)}, \dots, X_n), f(X_{n-(r-1)}, X_{n-(r-2)}, X_{n-(r-2)}, \dots, X_{n+1})\bigr) \\
&\quad = d\bigl(Q + A_1^* X_{n-r}^{\alpha_1} A_1 + A_2^* X_{n-(r-1)}^{\alpha_2} A_2 + \cdots + A_{r+1}^* X_n^{\alpha_{r+1}} A_{r+1}, \\
&\qquad Q + A_2^* X_{n-(r-1)}^{\alpha_1} A_2 + A_3^* X_{n-(r-2)}^{\alpha_3} A_3 + \cdots + A_{r+2}^* X_{n+1}^{\alpha_{r+2}} A_{r+2}\bigr) \\
&\quad \le d\bigl(A_1^* X_{n-r}^{\alpha_1} A_1 + A_2^* X_{n-(r-1)}^{\alpha_2} A_2 + \cdots + A_{r+1}^* X_n^{\alpha_{r+1}} A_{r+1}, \\
&\qquad A_2^* X_{n-(r-1)}^{\alpha_1} A_2 + A_3^* X_{n-(r-2)}^{\alpha_3} A_3 + \cdots + A_{r+2}^* X_{n+1}^{\alpha_{r+2}} A_{r+2}\bigr) \\
&\quad \le \max\bigl\{d\bigl(A_1^* X_{n-r}^{\alpha_1} A_1, A_2^* X_{n-(r-1)}^{\alpha_1} A_2\bigr), d\bigl(A_2^* X_{n-(r-1)}^{\alpha_2} A_2, A_3^* X_{n-(r-2)}^{\alpha_3} A_3\bigr), \\
&\qquad \dots, d\bigl(A_{r+1}^* X_n^{\alpha_{r+1}} A_{r+1}, A_{r+2}^* X_{n+1}^{\alpha_{r+2}} A_{r+2}\bigr)\bigr\} \\
&\quad \le \max\bigl\{|\alpha_1| d(X_{n-r}, X_{n-(r-1)}), |\alpha_2| d(X_{n-(r-1)}, X_{n-(r-2)}), \\
&\qquad \dots, |\alpha_{r+1}| d(X_n, X_{n+1})\bigr\} \\
&\quad \le \max\bigl\{|\alpha_1|, |\alpha_2|, \dots, |\alpha_{r+1}|\bigr\} \max\bigl\{d(X_{n-r}, X_{n-(r-1)}), d(X_{n-(r-1)}, X_{n-(r-2)}), \\
&\qquad \dots, d(X_n, X_{n+1})\bigr\} \\
&\quad \le \lambda \max\bigl\{d(X_{n-r}, X_{n-(r-1)}), d(X_{n-(r-1)}, X_{n-(r-2)}), \dots, d(X_n, X_{n+1})\bigr\}
\end{aligned} \tag{4.9}$$

for all $X_{n-r}, X_{n-(r-1)}, X_{n-(r-2)}, \dots, X_{n+1} \in P(N)$. $X, Y \in P(N)$, we have

$$\begin{aligned}
& d\bigl(f(X, X, \dots, X), f(Y, Y, \dots, Y)\bigr) \\
&\quad = d\bigl(Q + A_1^* X^{\alpha_1} A_1 + A_2^* X^{\alpha_2} A_2 + \cdots + A_{r+1}^* X^{\alpha_{r+1}} A_{r+1}, \\
&\qquad Q + A_2^* Y^{\alpha_1} A_2 + A_3^* Y^{\alpha_3} A_3 + \cdots + A_{r+2}^* Y^{\alpha_{r+2}} A_{r+2}\bigr) \\
&\quad \le d\bigl(A_1^* X^{\alpha_1} A_1 + A_2^* X^{\alpha_2} A_2 + \cdots + A_{r+1}^* X^{\alpha_{r+1}} A_{r+1}, \\
&\qquad A_2^* Y^{\alpha_1} A_2 + A_3^* Y^{\alpha_3} A_3 + \cdots + A_{r+2}^* Y^{\alpha_{r+2}} A_{r+2}\bigr) \\
&\quad \le \max\bigl\{d\bigl(A_1^* X^{\alpha_1} A_1, A_2^* Y^{\alpha_1} A_2\bigr), d\bigl(A_2^* X^{\alpha_2} A_2, A_3^* Y^{\alpha_3} A_3\bigr), \\
&\qquad \dots, d\bigl(A_{r+1}^* X^{\alpha_{r+1}} A_{r+1}, A_{r+2}^* Y^{\alpha_{r+2}} A_{r+2}\bigr)\bigr\} \\
&\quad \le \max\bigl\{|\alpha_1| d(X, Y), |\alpha_2| d(X, Y), \dots, |\alpha_{r+1}| d(X, Y)\bigr\}
\end{aligned}$$

$$\begin{aligned}
&\le \max\bigl\{|\alpha_1|, |\alpha_2|, \ldots, |\alpha_{r+1}|\bigr\} \max\bigl\{d(X,Y), d(X,Y), \ldots, d(X,Y)\bigr\} \\
&\le \lambda \max\bigl\{d(X,Y), d(X,Y), \ldots, d(X,Y)\bigr\} \\
&< d(X,Y).
\end{aligned}$$

Since $\lambda \in (0,1)$, (i) and (ii) follow immediately from Theorem 3.7 with $s = 1$ and $g = I$. □

Numerical experiment illustrating the above convergence algorithm

Consider the nonlinear matrix equation

$$X = Q + A^* X^{\frac{1}{2}} A + B^* X^{\frac{1}{3}} B + C^* X^{\frac{1}{4}} C, \tag{4.10}$$

where

$$A = \begin{pmatrix} 14/3 & 1/3 & 1/4 \\ 2/15 & 1/12 & 1/23 \\ 3/10 & 9/20 & 11/4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2/5 & 3/2 & 4/6 \\ 10/4 & 6/13 & 7/46 \\ 5/2 & 4/7 & 6/13 \end{pmatrix},$$

$$C = \begin{pmatrix} 1/3 & 19/24 & 22/55 \\ 17/10 & 27/15 & 45/17 \\ 13/8 & 1/3 & 1/4 \end{pmatrix}, \qquad Q = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 6 & 4 \\ 1 & 2 & 7 \end{pmatrix}.$$

We define the iterative sequence $\{X_n\}$ by

$$X_{n+1} = Q + A^* X_{n-2}^{\frac{1}{2}} A + B^* X_{n-1}^{\frac{1}{3}} B + C^* X_n^{\frac{1}{4}} C. \tag{4.11}$$

Let R_m $(m \ge 2)$ be the residual error at the iteration m, that is, $R_m = \|X_{m+1} - (Q + A^* X_{m+1}^{\frac{1}{2}} A + B^* X_{m+1}^{\frac{1}{3}} B + C^* X_{m+1}^{\frac{1}{4}} C)\|$, where $\|\cdot\|$ is the spectral norm. For initial values

$$X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

$$X_2 = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix},$$

we computed the successive iterations and the error R_m using MATLAB and found that after thirty five iterations the sequence given by (4.11) converges to

$$U = X_{35} = \begin{pmatrix} 639.1810 & 54.1681 & 107.3574 \\ 54.1285 & 44.7768 & 44.1469 \\ 104.3977 & 42.1095 & 112.5509 \end{pmatrix},$$

which is clearly a solution of (4.10). The convergence history of algorithm (4.11) is given in Figure 1.

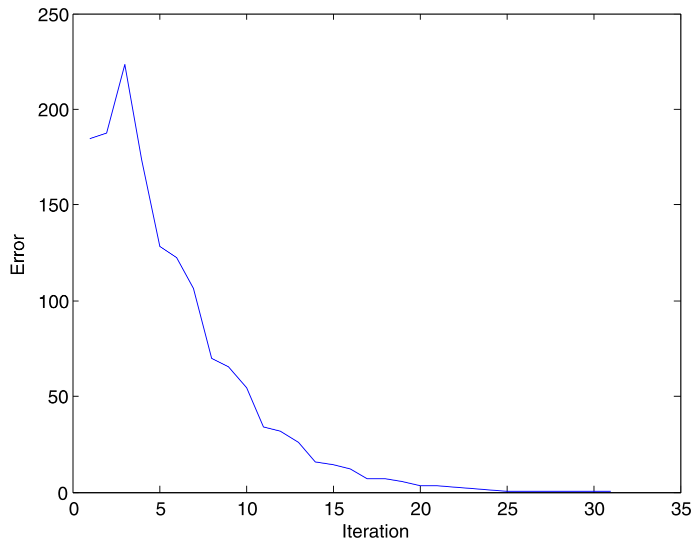

Figure 1 Convergence history for Equation (4.11).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in this work. All authors read and approved the final manuscript.

Author details
[1]School of Studies in Mathematics, Pt. Ravishankar Shukla University, Raipur, Chhattisgarh, India. [2]Department of Mathematics, College of Science, Salman bin Abdulaziz University, Al-Kharj, Kingdom of Saudi Arabia. [3]Department of Mathematics and Computer Science, St. Thomas College, Bhilai, Chhattisgarh, India. [4]Department of Engineering Basic Sciences, Faculty of Engineering, Menofia University, Menofia, Egypt. [5]Department of Mathematics, Faculty of Commerce, Alazhar University, Menofia, Egypt. [6]Department of Mathematics, Government VYT PG Autonomous College, Durg, Chhattisgarh, India. [7]Department of Mathematics, Rungta College of Engineering and Technology, Bhilai, Chhattisgarh, India.

Acknowledgements
The authors are thankful to the learned referees for their valuable comments which helped in bringing this paper to its present form. The second, third and fourth authors would like to thank the Deanship of Scientific Research, Salman bin Abdulaziz University, Al Kharj, Kingdom of Saudi Arabia for the financial assistance provided. The research of second, third and fourth authors is supported by the Deanship of Scientific Research, Salman bin Abdulaziz University, Alkharj, Kingdom of Saudi Arabia, Research Grant No. 2014/01/2046.

Received: 30 December 2014 Accepted: 29 May 2015 Published online: 01 July 2015

References
1. Presic, SB: Sur la convergence des suites. C. R. Acad. Sci. Paris **260**, 3828-3830 (1965)
2. Presic, SB: Sur une classe d'inequations aux differences finite et sur la convergence de certaines suites. Publ. Inst. Math. (Belgr.) **5**(19), 75-78 (1965)
3. Ciric, LB, Presic, SB: On Presic type generalisation of Banach contraction principle. Acta Math. Univ. Comen. **LXXVI**(2), 143-147 (2007)
4. Pacurar, M: Approximating common fixed points of Presic-Kannan type operators by a multi-step iterative method. An. Ştiinţ. Univ. 'Ovidius' Constanţa **17**(1), 153-168 (2009)
5. Pacurar, M: Fixed points of almost Presic operators by a *k*-step iterative method. An. Ştiinţ. Univ. 'Al.I. Cuza' Iaşi, Mat. **LVII**, 199-210 (2011)
6. Rao, KPR, Mustaq Ali, M, Fisher, B: Some Presic type generalizations of the Banach contraction principle. Math. Morav. **15**(1), 41-47 (2011)
7. Pacurar, M: Common fixed points for almost Presic type operators. Carpath. J. Math. **28**(1), 117-126 (2012)
8. Khan, MS, Berzig, M, Samet, B: Some convergence results for iterative sequences of Presic type and applications. Adv. Differ. Equ. (2012). doi:10.1186/1687-1847-2012-38
9. George, R, Reshma, KP, Rajagopalan, R: A generalised fixed point theorem of Presic type in cone metric spaces and application to Markov process. Fixed Point Theory Appl. (2011). doi:10.1186/1687-1812-2011-85
10. Shukla, S: Presic type results in 2-Banach spaces. Afr. Math. (2013). doi:10.1007/s13370-013-0174-2
11. Shukla, S, Fisher, B: A generalization of Presic type mappings in metric-like spaces. J. Oper. **2013**, Article ID 368501 (2013)

12. Shukla, S, Sen, R: Set-valued Presic-Reich type mappings in metric spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. (2012). doi:10.1007/s13398-012-0114-2
13. Shukla, S, Radenovic, S, Pantelic, S: Some fixed point theorems for Presic-Hardy-Rogers type contractions in metric spaces. J. Math. **2013**, Article ID 295093 (2013). doi:10.1155/2013/295093
14. Chen, YZ: A Presic type contractive condition and its applications. Nonlinear Anal. (2009). doi:10.1016/j.na.2009.03.006
15. Bakhtin, IA: The contraction mapping principle in quasimetric spaces. Funct. Anal. Unianowsk Gos. Pedagog. Inst. **30**, 26-37 (1989)
16. Boriceanu, M: Strict fixed point theorems for multivalued operators in *b*-metric spaces. Int. J. Mod. Math. **4**(3), 285-301 (2009)
17. Boriceanu, M, Bota, M, Petrusel, A: Multivalued fractals in *b*-metric spaces. Cent. Eur. J. Math. **8**(2), 367-377 (2010)
18. Bota, M, Molnar, A, Csaba, V: On Ekeland's variational principle in *b*-metric spaces. Fixed Point Theory **12**, 21-28 (2011)
19. Plebaniak, R: New generalized pseudodistance and coincidence point theorem in a *b*-metric space. Fixed Point Theory Appl. **2013**, 270 (2013)
20. Czerwik, S: Contraction mappings in *b*-metric spaces. Acta Math. Inform. Univ. Ostrav. **1**, 5-11 (1993)
21. Czerwik, S: Nonlinear set-valued contraction mappings in *b*-metric spaces. Atti Semin. Mat. Fis. Univ. Modena **46**, 263-276 (1998)
22. Czerwik, S, Dlutek, K, Singh, SL: Round-off stability of iteration procedures for operators in *b*-metric spaces. J. Natur. Phys. Sci. **11**, 87-94 (1997)
23. Czerwik, S, Dlutek, K, Singh, SL: Round-off stability of iteration procedures for set valued operators in *b*-metric spaces. J. Natur. Phys. Sci. **15**, 1-8 (2001)
24. Pacurar, M: A multi-step iterative method for approximating common fixed points of Presic-Rus type operators on metric spaces. Stud. Univ. Babeş-Bolyai, Math. **LV**(1), 149-162 (2010)
25. Ferrante, A, Levy, B: Hermitian solution of the equation $X = Q + N^*X^{-1}N$. Linear Algebra Appl. **247**, 359-373 (1996)
26. Huang, M, Huang, C, Tsai, T: Application of Hilbert's projective metric to a class of positive nonlinear operators. Linear Algebra Appl. **413**, 202-211 (2006)
27. Hasanov, VI: Positive definite solutions of matrix equations $X \pm A^T X^{-q} A = Q$. Linear Algebra Appl. **404**, 166-182 (2005)
28. Duan, X, Liao, A, Tang, B: On the nonlinear matrix equation $X - \sum_{i=1}^m A_i^* X^{\delta_i} A_i^* = Q$. Linear Algebra Appl. **429**, 110-121 (2008)
29. Liu, XG, Gao, H: On the positive definite solutions of the matrix equation $X^s = \pm A^T X^{-t} A = I_n$. Linear Algebra Appl. **368**, 83-97 (2003)
30. Lim, Y: Solving the nonlinear matrix equation $X = Q + \sum_{i=1}^m M_i^* X^{\delta_i} M_i^*$, via a contraction principle. Linear Algebra Appl. **430**, 1380-1383 (2009)
31. Thompson, AC: On certain contraction mappings in a partially ordered vector space. Proc. Am. Math. Soc. **14**, 438-443 (1963)
32. Nussbaum, RD: Hilbert's projective metric and iterated nonlinear maps. Mem. Am. Math. Soc. **14**, 438-443 (1963)