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Strong convergence theorems for Bregman quasi-strict pseudo-contractions in reflexive Banach spaces with applications

Zi-Ming Wang[*]

[*]Correspondence:
wangziming@ymail.com
Department of Foundation,
Shandong Yingcai University, Jinan,
250104, P.R. China

Abstract

In this paper, a simple iterative algorithm is introduced for finding a fixed point of a Bregman quasi-strict pseudo-contraction. Furthermore, strong convergence results are established in a reflexive Banach space. Finally, the solution of equilibrium problem, variational inequality, and zero point problem of maximal monotone operator are considered as applications.

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1 Introduction

Let E denote a real reflexive Banach space with the norm $\|\cdot\|$, E^* stand for the dual space of E. The normalized duality mapping from E to 2^{E^*} denoted by J is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E, \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between E and E^*. Let C be a nonempty, closed, and convex subset of E, $F(T) = \{x \in C : Tx = x\}$ denote the set of fixed points of an operator T. $\mathbb{R}$ and $\mathbb{N}$ stand for the set of real numbers and positive integers, respectively.

Given a nonempty, closed, and convex subset C of a Hilbert space H, an operator $T : C \to C$ is said to be a strict pseudo-contraction [1] if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in C, \tag{1.2}$$

and an operator $T : C \to C$ is said to be a nonexpansive mapping if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

It is obvious that the class of strict pseudo-contractions includes the class of nonexpansive mappings.

Since nonexpansive fixed point theory can be applied to the solution of diverse problems such that solving variational inequality problems, equilibrium problems, and convex feasibility problems, strict pseudo-contractions have more powerful applications than nonexpansive mappings in solving inverse problems [2]. In recent years, construction of an iterative algorithm for seeking fixed points of nonexpansive mappings and strict pseudo-contractions has extensively been investigated; see [1, 3–12] and the references therein. It is well known that, in an infinite-dimensional Hilbert space, the normal Mann iterative algorithm has only weak convergence, in general, even for nonexpansive mappings. So, in order to get strong convergence for nonexpansive mappings and strict pseudo-contractive mappings, one has to modify the normal Mann's iterative algorithm such as the so-called hybrid projection iteration method.

When one tries to extend this theory to general Banach spaces, some difficulties must be encountered because many of the useful examples of nonexpansive mappings in Hilbert spaces are no longer nonexpansive in Banach spaces, for example, the resolvent $R_A := (I + A)^{-1}$ of a maximal monotone mapping $A : H \to 2^H$ and the metric projection P_C. In this connection, Alber [13] introduced a generalized projection operator Π_C in Banach spaces which is an analogue of the metric projection in Hilbert spaces. Recently, applying the generalized projection operator in reflexive, strictly convex and smooth Banach spaces with some property, Zhou and Gao [14] introduced a modified hybrid projection iterative algorithm and proved a strong convergence theorem for a closed quasi-strict pseudo-contraction which is an extension of strict pseudo-contractive mappings and relatively nonexpansive mappings [15–17]. In fact, there are several ways to overcome these difficulties. Another way is to use the Bregman distance instead of the norm, Bregman (quasi-)nonexpansive mappings instead of the (quasi-)nonexpansive mappings and the Bregman projection instead of the metric projection.

In recent years, many authors focused attention on constructing the fixed point of Bregman nonlinear operators by utilizing the Bregman distance and the Bregman projection, see [18–23] and the references therein. In 2014, Zegeye and Shahzad [21] investigated an iterative scheme for a Bregman relatively nonexpansive mapping. Very recently, Ugwunnadi *et al.* [23] introduced the concept of Bregman quasi-strict pseudo-contraction and proved the strong convergence by using hybrid Bregman projection iterative algorithm for Bregman quasi-strict pseudo-contractions.

Motivated and inspired by the above works, in this paper we aim to propose a new simple hybrid Bregman projection iterative algorithm for a Bregman quasi-strict pseudo-contraction and to prove strong convergence results in the framework of reflexive Banach spaces. The results presented in this paper improve the known corresponding results announced in the literature sources listed in this work.

2 Preliminaries

In this section, we collect some preliminaries and lemmas which will be used to prove our main results.

Throughout this paper E is a real reflexive Banach space with the norm $\|\cdot\|$ and E^* is the dual space of E. $f : E \to (-\infty, +\infty]$ is a proper, convex, and lower semi-continuous function. We denote by $\operatorname{dom} f$ the domain of f, that is, $\operatorname{dom} f := \{x \in E : f(x) < +\infty\}$. For any $x \in \operatorname{int}(\operatorname{dom} f)$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by

$$f^{\circ}(x,y) = \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}. \tag{2.1}$$

The function f is said to be Gâteaux differentiable at x if $f^{\circ}(x,y)$ exists for any y. In this case, $f^{\circ}(x,y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f of f at x. The function f is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{int}\operatorname{dom} f$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is called uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

Let E be a smooth, strictly convex, and reflexive Banach space. Let $\phi : E \times E \to [0,\infty)$ denote the Lyapunov functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E. \tag{2.2}$$

It is obvious from the definition of ϕ that

$$\left(\|x\| - \|y\|\right)^2 \le \phi(x,y) \le \left(\|x\| + \|y\|\right)^2, \quad \forall x,y \in E. \tag{2.3}$$

The generalized projection [2] Π_C from E onto C is defined and denoted by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y,x). \tag{2.4}$$

A point $p \in C$ is said to be an asymptotic fixed point of a mapping T if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\widehat{F}(T)$ the set of asymptotic fixed points of T. A point $p \in C$ is said to be a strong asymptotic fixed point of a mapping T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. We denote by $\widetilde{F}(T)$ the set of strong asymptotic fixed points of T.

Let $T : C \to C$ be a mapping, and recall the following definitions:

(a) T is said to be relatively nonexpansive [15–17] if $\widehat{F}(T) = F(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p,x), \quad \forall x \in C, p \in F(T). \tag{2.5}$$

(b) T is said to be relatively weak nonexpansive if $\widetilde{F}(T) = F(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p,x), \quad \forall x \in C, p \in F(T). \tag{2.6}$$

(c) T is said to be hemi-relatively nonexpansive if $F(T) \neq \emptyset$ and

$$\phi(p, Tx) \le \phi(p,x), \quad \forall x \in C, p \in F(T). \tag{2.7}$$

(d) T is said to be quasi-ϕ-strictly pseudo-contractive [14] if $F(T) \neq \emptyset$ and there exists a constant $k \in [0,1)$ such that

$$\phi(p, Tx) \le \phi(p,x) + k\phi(x, Tx), \quad \forall x \in C, p \in F(T). \tag{2.8}$$

Remark 2.1 From the definitions, one has the following facts.

(1) The class of relatively nonexpansive mappings is included by the class of relatively weak nonexpansive mappings. In fact, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is a relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$.

(2) The class of relatively weak nonexpansive mappings is contained by the class of hemi-relatively nonexpansive mappings. Hemi-relatively nonexpansive mappings do not require $F(T) = \widetilde{F}(T)$.

(3) The class of quasi-ϕ-strict pseudo-contractions is more general than the class of hemi-relatively nonexpansive mappings. In fact, a hemi-relatively nonexpansive mapping is a quasi-ϕ-strict pseudo-contraction with $k = 0$.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : \operatorname{dom} f \times \operatorname{int}\operatorname{dom} f \to [0, +\infty)$ defined by

$$D_f(y,x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \tag{2.9}$$

is called the Bregman distance with respect to f, see [24]. The Bregman distance has the following two important properties:

(i) (The three point identity): for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{int}\operatorname{dom} f$,

$$D_f(x,y) + D_f(y,z) - D_f(x,z) = \langle \nabla f(z) - \nabla f(y), x - y \rangle; \tag{2.10}$$

(ii) (The four point identity): for any $y, w \in \operatorname{dom} f$ and $x, z \in \operatorname{int}\operatorname{dom} f$,

$$D_f(y,x) - D_f(y,z) - D_f(w,x) + D_f(w,z) = \langle \nabla f(z) - \nabla f(x), y - w \rangle. \tag{2.11}$$

Recall that the Bregman projection [25] of $x \in \operatorname{int}\operatorname{dom} f$ onto the nonempty closed and convex set $C \subset \operatorname{dom} f$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f\bigl(P_C^f(x), x\bigr) = \inf\bigl\{D_f(y,x) : y \in C\bigr\}. \tag{2.12}$$

It should be observed that if E is a smooth Banach space, setting $f(x) = \|x\|^2$ for all $x \in E$, we have $\nabla f(x) = 2Jx$ for all $x \in E$. Hence $D_f(x,y)$ reduces to the Lyapunov function $\phi(x,y) = \|2\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$ and the Bregman projection $P_C^f(x)$ reduces to the generalized projection Π_C from E onto C. If E is a Hilbert space H, then $D_f(x,y)$ becomes $\phi(x,y) = \|x - y\|^2$ for $x, y \in H$ and the Bregman projection $P_C^f(x)$ becomes the metric projection P_C from E onto C.

Similarly to the metric projection in a Hilbert space, Bregman projections with respect to totally convex and differentiable functions have variational characterizations.

Lemma 2.2 (see Butnariu and Resmerita [26]) *Suppose that f is Gâteaux differentiable and totally convex on* $\operatorname{int}\operatorname{dom} f$. *Let $x \in \operatorname{int}\operatorname{dom} f$ and let $C \subset \operatorname{int}\operatorname{dom} f$ be a nonempty, closed, and convex set. If $\hat{x} \in C$, then the following conditions are equivalent*:

(a) *The vector $\hat{x}$ is the Bregman projection of x onto C with respect to f, i.e., $z = P_C^f(x)$;*
(b) *The vector $\hat{x}$ is the unique solution of the variational inequality*

$$\langle \nabla f(x) - \nabla f(z), z - y\rangle \geq 0, \quad \forall y \in C; \tag{2.13}$$

(c) *The vector $\hat{x}$ is the unique solution of the inequality*

$$D_f(y,z) + D_f(z,x) \leq D_f(y,x), \quad \forall y \in C. \tag{2.14}$$

Let $x \in \operatorname{int}\operatorname{dom} f$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y), \forall y \in E\}. \tag{2.15}$$

The Fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}, \quad x^* \in E^*. \tag{2.16}$$

The function f is said to be essentially smooth if ∂f is both locally bounded and single-valued on its domain. It is called essentially strictly convex if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every convex subset of $\operatorname{dom} \partial f$. f is said to be a Legendre if it is both essentially smooth and essentially strictly convex. When the subdifferential of f is single-valued, it coincides with the gradient $\partial f = \nabla f$, see [27].

We remark that if E is a reflexive Banach space, then we have

(i) f is essentially smooth if and only if f^* is essentially strictly convex, see [28];
(ii) $(\partial f)^{-1} = \partial f^*$, see [29];
(iii) f is Legendre if and only if f^* is Legendre, see [28];
(iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\operatorname{ran} \nabla f = \operatorname{dom} \nabla f^* = \operatorname{int}\operatorname{dom} f^*$ and $\operatorname{ran} \nabla f^* = \operatorname{dom} \nabla f = \operatorname{int}\operatorname{dom} f$, see [28].

The following result is useful in the next section.

Lemma 2.3 (see Bauschke *et al.* [28]) *Suppose $x \in E$ and $y \in \operatorname{int}\operatorname{dom} f$. If f is essentially strictly convex, then $D_f(x,y) = 0 \Leftrightarrow x = y$.*

When E is a smooth and strictly convex Banach space, one important and interesting example of a Legendre function is $f(x) = \frac{1}{p}\|x\|^p$ $(1 < p < \infty)$. In this case the gradient ∇f of f coincides with the generalized duality mapping of E, *i.e.*, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces.

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of f at $x \in \operatorname{dom} f$ is the function $\nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty]$ defined by

$$\nu_f(x,t) := \inf\{D_f(y,x) : y \in \operatorname{dom} f, \|y - x\| = t\}. \tag{2.17}$$

The function f is called totally convex at x if $\nu_f(x,t) > 0$, whenever $t > 0$. The function f is called totally convex if it is totally convex at any point $x \in \operatorname{int}\operatorname{dom} f$. The function f is said to be totally convex on bounded sets if $\nu_f(B,t) > 0$ for any nonempty bounded subset

B of E and $t > 0$, where the modulus of total convexity of the function f on the set B is the function $\nu_f : \operatorname{int}\operatorname{dom} f \times [0,+\infty) \to [0,+\infty]$ defined by

$$\nu_f(B,t) := \inf\{\nu_f(x,t) : x \in B \cap \operatorname{dom} f\}. \tag{2.18}$$

Recall that the function f is said to be sequentially consistent [26] if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n\to\infty} D_f(y_n,x_n) = 0 \quad\Rightarrow\quad \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.19}$$

The following lemmas will be useful in the proof of the next section.

Lemma 2.4 (see Butnariu and Iusem [30]) *The function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 2.5 (see Reich and Sabach [18]) *Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n,x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.*

Recall the following definitions.

Definition 2.6 Let C be a subset of E and let $T : C \to C$ be a mapping.

(1) T is said to be Bregman relatively nonexpansive if $\widehat{F}(T) = F(T) \neq \emptyset$ and

$$D_f(p,Tx) \le D_f(p,x), \quad \forall x \in C, p \in F(T). \tag{2.20}$$

(2) T is said to be Bregman weak relatively nonexpansive if $\widetilde{F}(T) = F(T) \neq \emptyset$ and

$$D_f(p,Tx) \le D_f(p,x), \quad \forall x \in C, p \in F(T). \tag{2.21}$$

(3) T is said to be Bregman quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$D_f(p,Tx) \le D_f(p,x), \quad \forall x \in C, p \in F(T). \tag{2.22}$$

(4) T is said to be Bregman quasi-strictly pseudo-contractive [23] if there exists a constant $k \in [0,1)$ and $F(T) \neq \emptyset$ such that

$$D_f(p,Tx) \le D_f(p,x) + kD_f(x,Tx), \quad \forall x \in C, p \in F(T). \tag{2.23}$$

(5) A mapping $T : C \to C$ is said to be closed if for any sequence $\{x_n\} \subset C$ with $x_n \to x \in C$ and $Tx_n \to y \in C$ as $n \to \infty$, then $Tx = y$.

Remark 2.7 From the definitions, the following facts are obtained easily.

(1) Bregman relatively nonexpansive mappings, Bregman weak relatively nonexpansive mappings, Bregman quasi-nonexpansive mappings, and Bregman quasi-strict pseudo-contractions are more general than relatively nonexpansive mappings, relatively weak nonexpansive mappings, hemi-relatively nonexpansive mappings, and quasi-ϕ-strictly pseudo-contractions, respectively.

(2) The class of Bregman quasi-strictly pseudo-contractions is more general than the class of Bregman relatively nonexpansive mappings, the class of Bregman weak relatively nonexpansive mappings, and the class of Bregman quasi-nonexpansive mappings.

Now, we give some examples of Bregman quasi-strict pseudo-contractions.

Example 2.8 (see Reich and Sabach [31]) Let E be a real reflexive Banach space, $A : E \to 2^{E^*}$ be a maximal monotone mapping and $f : E \to (-\infty, +\infty]$ be a uniformly Fréchet differentiable and bounded on bounded subsets of E such that $A^{-1}0 \neq \emptyset$, then the resolvent

$$\mathrm{Res}_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x) \tag{2.24}$$

is closed and Bregman relatively nonexpansive from E onto $D(A)$, so is a closed Bregman quasi-strict pseudo-contraction.

Example 2.9 Let E be a smooth Banach space, and define $f(x) = \|x\|^2$ all $x \in E$. Let $x_0 \neq 0$ be any element of E, $T : E \to E$ be defined as follows:

$$T(x) = \begin{cases} (\frac{1}{2} + \frac{1}{2^{n+1}})x_0, & \text{if } x = (\frac{1}{2} + \frac{1}{2^n})x_0; \\ -x, & \text{if } x \neq (\frac{1}{2} + \frac{1}{2^n})x_0 \end{cases} \tag{2.25}$$

for all $n \geq 1$. Then T is a Bregman quasi-strict pseudo-contraction.

Proof Since $\nabla f(y) = 2Jy$, the Bregman distance with respect to f

$$\begin{aligned} D_f(x, y) &= f(x) - f(y) - \langle \nabla f(y), x - y \rangle \\ &= \|x\|^2 - \|y\|^2 - 2\langle Jy, x - y\rangle \\ &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E. \end{aligned} \tag{2.26}$$

Therefore, we have from (2.3) that

$$D_f(x, Tx) = \|x\|^2 - 2\langle x, JTx\rangle + \|Tx\|^2 \geq \left(\|x\| - \|Tx\|\right)^2 \geq 0, \quad \forall x \in E. \tag{2.27}$$

From the definition of T, it is obvious that $F(T) = \{0\}$. In addition, it is easy to see that $\|Tx\| \leq \|x\|$, which implies that

$$\|Tx\|^2 - \|x\|^2 \leq 2\langle 0, JTx - Jx\rangle = 2\langle p, JTx - Jx\rangle, \quad \forall p \in F(T), x \in E. \tag{2.28}$$

It follows from the above inequality that

$$\|p\|^2 - 2\langle p, JTx\rangle + \|Tx\|^2 \leq \|p\|^2 - 2\langle p, Jx\rangle + \|x\|^2, \quad \forall p \in F(T), x \in E. \tag{2.29}$$

By using (2.2), (2.3), and (2.5), we have

$$D_f(p, Tx) \leq D_f(p, x) + kD_f(x, Tx), \quad \forall p \in F(T), x \in E, \tag{2.30}$$

which implies that T is a Bregman quasi-strict pseudo-contraction. □

Example 2.10 (see Ugwunnadi *et al.* [23]) Let $E = \mathbb{R}$ and define $T, f : [-1, 0] \to \mathbb{R}$ by $f(x) = x$ and $Tx = 2x$ for all $x \in [-1, 0]$. Then T is a Bregman quasi-strict pseudo-contraction but not a quasi-ϕ-strict pseudo-contraction.

3 Main results

In this section, we state and prove our main theorem.

Theorem 3.1 *Let E be a real reflexive Banach space, C be a nonempty, closed, and convex subset of E. Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E, and $T : C \to C$ be a closed and Bregman quasi-strict pseudo-contraction such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P^f_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : D_f(x_n, Tx_n) \le \frac{1}{1-\kappa}\langle \nabla f(x_n) - \nabla f(Tx_n), x_n - z\rangle\}, \\ x_{n+1} = P^f_{C_{n+1}}(x_0), \quad n \ge 0, \end{cases} \tag{3.1}$$

where $\kappa \in [0, 1)$. Then the sequence $\{x_n\}$ converges strongly to $\widehat{p} = P^f_{F(T)}(x_0)$, where $P^f_{F(T)}$ is the Bregman projection of E onto $F(T)$.

Proof The proof is split into seven steps.

Step 1: Show that $F(T)$ is closed and convex.

First, we prove that $F(T)$ is closed. Let $\{p_n\}$ be a sequence in $F(T)$ with $p_n \to p$ as $n \to \infty$. One has $Tp_n = p_n \to p$ as $n \to \infty$. By the closedness of T, one has $Tp = p$. This implies that $F(T)$ is closed.

Next we prove that $F(T)$ is convex. For any $p_1, p_2 \in F(T)$, $t \in (0, 1)$, putting $p = tp_1 + (1 - t)p_2$, we prove that $p \in F(T)$. From the definition of Bregman distance $D_f(x, y)$, one has

$$\begin{aligned} D_f(p, Tp) &= f(p) - f(Tp) - \langle \nabla f(Tp), p - Tp\rangle \\ &= f(p) - f(Tp) - t\langle \nabla f(Tp), p_1 - Tp\rangle - (1 - t)\langle \nabla f(Tp), p_2 - Tp\rangle \\ &= f(p) + t\bigl[f(p_1) - f(Tp) - \langle \nabla f(Tp), p_1 - Tp\rangle\bigr] \\ &\quad + (1 - t)\bigl[f(p_2) - f(Tp) - \langle \nabla f(Tp), p_2 - Tp\rangle\bigr] - tf(p_1) - (1 - t)f(p_2) \\ &= f(p) + tD_f(p_1, Tp) + (1 - t)D_f(p_2, Tp) - tf(p_1) - (1 - t)f(p_2). \end{aligned} \tag{3.2}$$

Since T is Bregman quasi-strictly pseudocontractive, one has

$$\begin{aligned} &tD_f(p_1, Tp) + (1 - t)D_f(p_2, Tp) \\ &\quad \le t\bigl\{D_f(p_1, p) + \kappa D_f(p, Tp)\bigr\} + (1 - t)\bigl\{D_f(p_2, p) + \kappa D_f(p, Tp)\bigr\} \\ &\quad = t\bigl\{f(p_1) - f(p) - \langle \nabla f(p), p_1 - p\rangle + \kappa D_f(p, Tp)\bigr\} \\ &\qquad + (1 - t)\bigl\{f(p_2) - f(p) - \langle \nabla f(p), p_2 - p\rangle + \kappa D_f(p, Tp)\bigr\} \\ &\quad = tf(p_1) + (1 - t)f(p_2) - f(p) + \kappa D_f(p, Tp). \end{aligned} \tag{3.3}$$

Substituting (3.3) into (3.2), one obtains

$$\begin{aligned} D_f(p,Tp) &= f(p) + tD_f(p_1,Tp) + (1-t)D_f(p_2,Tp) - tf(p_1) - (1-t)f(p_2) \\ &\le f(p) + tf(p_1) + (1-t)f(p_2) - f(p) + \kappa D_f(p,Tp) - tf(p_1) - (1-t)f(p_2) \\ &= \kappa D_f(p,Tp), \end{aligned} \tag{3.4}$$

which implies that $D_f(p,Tp) \le 0$, and from Lemma 2.3, it follows that $Tp = p$. Therefore, $F(T)$ is also convex. Hence $F(T)$ is closed and convex, and $P^f_{F(T)}(x_0)$ is well defined for every $x_0 \in C$.

Step 2: Show that C_n is closed and convex for all $n \ge 1$.

In fact, for $n = 1$, $C_1 = C$ is closed and convex. By induction, assume that C_k is closed and convex for some $k \in \mathbb{N}$. For $z \in C_{k+1}$, one obtains

$$D_f(x_k,Tx_k) \le \frac{1}{1-\kappa}\langle \nabla f(x_k) - \nabla f(Tx_k), x_k - z\rangle. \tag{3.5}$$

It is easy to see that C_{k+1} is also closed and convex. Then, for all $n \ge 1$, C_n is closed and convex. Furthermore, $P^f_{C_n}(x_0)$ is well defined for every $x_0 \in C$ and $n \ge 1$.

Step 3: Show that $F(T) \subset C_n$ for all $n \ge 1$.

It is obvious that $F(T) \subset C = C_1$. Suppose that $F(T) \subset C_k$ for some $k \in \mathbb{N}$, for all $p \in F(T)$, one has $p \in C_k$. Since T is Bregman quasi-strictly pseudocontractive, one has

$$D_f(p,Tx_k) \le D_f(p,x_k) + \kappa D_f(x_k,Tx_k). \tag{3.6}$$

On the other hand, in view of the three point identity of the Bregman distance, one has

$$D_f(p,Tx_k) = D_f(p,x_k) + D_f(x_k,Tx_k) + \langle \nabla f(x_k) - \nabla f(Tx_k), p - x_k\rangle. \tag{3.7}$$

Substituting (3.7) into (3.6), one has

$$D_f(p,x_k) + D_f(x_k,Tx_k) + \langle \nabla f(x_k) - \nabla f(Tx_k), p - x_k\rangle \le D_f(p,x_k) + \kappa D_f(x_k,Tx_k), \tag{3.8}$$

that is,

$$D_f(x_k,Tx_k) \le \frac{1}{1-\kappa}\langle \nabla f(x_k) - \nabla f(Tx_k), x_k - p\rangle, \tag{3.9}$$

which implies that $p \in C_{k+1}$. From this it follows that $F(T) \subset C_n$ for all $n \ge 1$.

Step 4: Show that $\lim_{n\to\infty} D_f(x_n,x_0)$ *exists.*

In fact, since $x_n = P^f_{C_n}(x_0)$, from Lemma 2.2(b), one has

$$\langle \nabla f(x_0) - \nabla f(x_n), y - x_n\rangle \le 0, \quad \forall y \in C_n, \tag{3.10}$$

and since $F(T) \subset C_n$ for all $n \ge 1$, we arrive at

$$\langle \nabla f(x_0) - \nabla f(x_n), p - x_n\rangle \le 0, \quad \forall p \in F(T). \tag{3.11}$$

From Lemma 2.2(c), one has

$$D_f(x_n, x_0) = D_f\big(P^f_{C_n}(x_0), x_0\big) \le D_f(p, x_0) - D_f\big(p, P^f_{C_n}(x_0)\big) \le D_f(p, x_0) \tag{3.12}$$

for each $p \in F(T)$ and for each $n \ge 1$. Therefore, $\{D_f(x_n, x_0)\}$ is bounded. In view of Lemma 2.5, one has $\{x_n\}$ is also bounded.

On the other hand, noticing that $x_n = P^f_{C_n}(x_0)$ and $x_{n+1} = P^f_{C_{n+1}}(x_0) \in C_{n+1} \subset C_n$, one has $D_f(x_n, x_0) \le D_f(x_{n+1}, x_0)$ for all $n \ge 1$. This implies that $\{D_f(x_n, x_0)\}$ is a nondecreasing sequence. Therefore, $\lim_{n\to\infty} D_f(x_n, x_0)$ exists.

Step 5: Show that $x_n \to \widehat{p}$ as $n \to \infty$.

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup \widehat{p} \in C = C_1$. Since C_n is closed and convex and $C_{n+1} \subset C_n$, this implies that C_n is weakly closed and $\widehat{p} \in C_n$ for all $b \ge 1$. In view of $x_{n_i} = P^f_{C_{n_i}}(x_0)$, one has

$$D_f(x_{n_i}, x_0) \le D_f(\widehat{p}, x_0), \quad \forall n_i \ge 1. \tag{3.13}$$

Since f is a lower semi-continuous function on the convex set C, it is weakly lower semi-continuous on C. Hence we have

$$\begin{aligned}\liminf_{i\to\infty} D_f(x_{n_i}, x_0) &= \liminf_{i\to\infty}\big\{f(x_{n_i}) - f(x_0) - \big\langle \nabla f(x_0), x_{n_i} - x_0\big\rangle\big\} \\ &\ge f(\widehat{p}) - f(x_0) - \big\langle \nabla f(x_0), \widehat{p} - x_0\big\rangle \\ &= D_f(\widehat{p}, x_0).\end{aligned} \tag{3.14}$$

Therefore, one has

$$D_f(\widehat{p}, x_0) \le \liminf_{i\to\infty} D_f(x_{n_i}, x_0) \le \limsup_{i\to\infty} D_f(x_{n_i}, x_0) \le D_f(\widehat{p}, x_0), \tag{3.15}$$

which implies that

$$\lim_{i\to\infty} D_f(x_{n_i}, x_0) = D_f(\widehat{p}, x_0), \tag{3.16}$$

and so $f(x_{n_i}) \to f(\widehat{p})$ as $n \to \infty$. Since f is uniformly continuous, one has

$$\lim_{i\to\infty} x_{n_i} = \widehat{p}. \tag{3.17}$$

On the other hand, noticing that $\{D_f(x_n, x_0)\}$ is convergent, this together with (3.16) implies that

$$\lim_{n\to\infty} \big\{D_f(x_n, x_0)\big\} = D_f(\widehat{p}, x_0). \tag{3.18}$$

Therefore $f(x_n) \to f(\widehat{p})$ as $n \to \infty$. Since f is uniformly continuous, one has

$$\lim_{n\to\infty} x_n = \widehat{p}. \tag{3.19}$$

Step 6: Show that $\widehat{p} = T\widehat{p}$.

Since $x_{n+1} = P^f_{C_{n+1}}(x_0) \in C_{n+1}$, from (3.1) one has

$$D_f(x_n, Tx_n) \leq \frac{1}{1-k}\bigl\langle \nabla f(x_n) - \nabla f(Tx_n), x_n - x_{n+1}\bigr\rangle, \tag{3.20}$$

which together with $\lim_{n\to\infty} x_n = \widehat{p}$ implies that

$$\lim_{n\to\infty} D_f(x_n, Tx_n) = 0. \tag{3.21}$$

Noticing that f is totally convex on bounded subsets of E, from Lemma 2.4 f is sequentially consistent. It follows that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.22}$$

Since $x_n \to \widehat{p}$ as $n \to \infty$, by the closedness of T, we have $T\widehat{p} = \widehat{p}$.

Step 7: Show that $\widehat{p} = P^f_{F(T)}(x_0)$.

Taking $n \to \infty$ in (3.11), one has

$$\bigl\langle \nabla f(x_0) - \nabla f(\widehat{p}), p - \widehat{p}\bigr\rangle \leq 0, \quad \forall p \in F(T). \tag{3.23}$$

In view of Lemma 2.2(a), one has $\widehat{p} = P^f_{F(T)}(x_0)$. This completes the proof of Theorem 3.1. □

Since the class of Bregman quasi-nonexpansive mappings is Bregman quasi-strict pseudo-contractive, the following corollary is obtained by using Theorem 3.1.

Corollary 3.2 *Let E be a real reflexive Banach space, C be a nonempty, closed, and convex subset of E. Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E, and $T : C \to C$ be a closed and Bregman quasi-nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P^f_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : D_f(x_n, Tx_n) \leq \langle \nabla f(x_n) - \nabla f(Tx_n), x_n - z\rangle\}, \\ x_{n+1} = P^f_{C_{n+1}}(x_0), \quad n \geq 0. \end{cases} \tag{3.24}$$

Then the sequence $\{x_n\}$ converges strongly to $\widehat{p} = P^f_{F(T)}(x_0)$, where $P^f_{F(T)}$ is the Bregman projection of E onto $F(T)$.

Setting $f(x) = \|x\|^2$ for all $x \in E$, then $\nabla f(x) = 2Jx$ for all $x \in E$. Hence $D_f(x, y)$ reduces to the Lyapunov function $\phi(x, y) = \|2\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$, the Bregman projection $P^f_C(x)$ reduces to the generalized projection Π_C from E onto C and the Bregman quasi-nonexpansive mapping reduces to the hemi-relatively nonexpansive mapping. So, by utilizing Corollary 3.2, the following corollary is obtained.

Corollary 3.3 *Let E be a real reflexive, smooth, and strictly convex Banach space, C be a nonempty, closed, and convex subset of E. Suppose that $T : C \to C$ is a closed hemi-relatively nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : \phi(x_n, Tx_n) \leq 2\langle J(x_n) - J(Tx_n), x_n - z\rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad n \geq 0. \end{cases} \tag{3.25}$$

Then the sequence $\{x_n\}$ converges strongly to $\widehat{p} = \Pi_{F(T)}(x_0)$, where $\Pi_{F(T)}$ is the generalized projection of E onto $F(T)$.

Similar to Corollary 3.3, the following corollary can be obtained from Theorem 3.1.

Corollary 3.4 *Let E be a real reflexive, smooth, and strictly convex Banach space, C be a nonempty, closed, and convex subset of E. Suppose that $T : C \to C$ is a closed and quasi-ϕ-strict pseudo-contraction such that $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : \phi(x_n, Tx_n) \leq \frac{2}{1-\kappa}\langle J(x_n) - J(Tx_n), x_n - z\rangle\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \quad n \geq 0. \end{cases} \tag{3.26}$$

Then the sequence $\{x_n\}$ converges strongly to $\widehat{p} = \Pi_{F(T)}(x_0)$, where $\Pi_{F(T)}$ is the generalized projection of E onto $F(T)$.

4 Applications

4.1 Application to equilibrium problems

Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E. Let $g : C \times C \to \mathbb{R}$ be a bifunction that satisfies the following conditions:

(A1) $g(x,x) = 0$ for all $x \in C$;

(A2) g is monotone, *i.e.*, $g(x,y) + g(y,x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$,

$$\limsup_{t \downarrow 0} g\bigl(tz + (1-t)x, y\bigr) \leq g(x,y); \tag{4.1}$$

(A4) for each $x \in C$, $g(x, \cdot)$ is convex and lower semicontinuous.

The so-called equilibrium problem corresponding to g is to find $\bar{x} \in C$ such that

$$g(\bar{x}, y) \geq 0, \quad \forall y \in C. \tag{4.2}$$

The set of its solutions is denoted by $EP(g)$. The resolvent of a bifunction $g : C \times C \to \mathbb{R}$ is the operator $\mathrm{Res}_g^f : E \to 2^C$ defined by

$$\mathrm{Res}_g^f(x) = \{z \in C : g(z,y) + \langle \nabla f(z) - \nabla f(x), y - z\rangle \geq 0, \forall y \in C\}. \tag{4.3}$$

The resolvent operator Res_g^f has the following properties:

(1) Res_g^f is single-valued;

(2) The set of fixed points of Res_g^f is the solution set of the corresponding equilibrium problem, *i.e.*, $F(\mathrm{Res}_g^f) = EP(g)$;

(3) Res_g^f is a closed Bregman quasi-nonexpansive mapping, so is a closed Bregman quasi-strict pseudo-contraction.

Theorem 4.1 *Let E be a real reflexive Banach space, C be a nonempty, closed, and convex subset of E. Let $g : C \times C \to \mathbb{R}$ be a bifunction that satisfies conditions* (A1)-(A4) *such that $EP(g) \neq \emptyset$. Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E, and $\mathrm{Res}_g^f : E \to 2^C$ be a resolvent operator defined as* (4.3). *Let $\{x_n\}$ be a sequence generated by the following iterative algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}^f(x_0), \\ C_{n+1} = \{z \in C_n : D_f(x_n, \mathrm{Res}_g^f(x_n)) \leq \frac{1}{1-\kappa}\langle \nabla f(x_n) - \nabla f(\mathrm{Res}_g^f(x_n)), x_n - z\rangle\}, \\ x_{n+1} = P_{C_{n+1}}^f(x_0), \quad n \geq 0, \end{cases} \tag{4.4}$$

where $\kappa \in [0,1)$. Then the sequence $\{x_n\}$ converges strongly to $\widehat{p} = P_{EP(g)}^f(x_0)$, where $P_{EP(g)}^f$ is the Bregman projection of E onto $EP(g)$.

Proof Since Res_g^f is a closed Bregman quasi-strict pseudo-contraction, by applying Theorem 3.1, the sequence $\{x_n\}$ converges strongly to $\widehat{p} = P_{EP(g)}^f(x_0)$. □

4.2 Application to variational inequality problems

Let C be a nonempty subset of a real reflexive Banach space E with dual E^*. Let $A : C \subseteq E \to E^*$ be a nonlinear mapping. The variational inequality problem for a nonlinear mapping A and its domain C is to find $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x}\rangle \geq 0, \quad \forall y \in C. \tag{4.5}$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

Recall the definition of Bregman inverse strongly monotone operators which was introduced by Butnariu and Kassay [32]. We assume that the Legendre function f satisfies the following range condition:

$$\mathrm{ran}(\nabla f - A) \subseteq \mathrm{ran}(\nabla f). \tag{4.6}$$

The operator $A : E \to 2^E$ is called Bregman inverse strongly monotone if

$$(\operatorname{dom} A) \cap (\operatorname{int} \operatorname{dom} f) \neq \emptyset \tag{4.7}$$

and for any $x, y \in \operatorname{int} \operatorname{dom} f$ and each $u \in Ax$, $v \in Ay$, we have

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v)\rangle \geq 0. \tag{4.8}$$

For any operator $A : E \to 2^{E^*}$, the anti-resolvent $A^f : E \to 2^E$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A). \tag{4.9}$$

Observe that $\operatorname{dom} A^f \subseteq (\operatorname{dom} A) \cap (\operatorname{int} \operatorname{dom} f)$ and $\operatorname{ran} A^f \subseteq \operatorname{int} \operatorname{dom} f$.

The following result which points out the connection between the fixed point set of $P_C^f \circ A^f$ and the solution set of the variational inequality corresponding to the Bregman inverse strongly monotone operator A was introduced by Reich and Sabach [31].

Lemma 4.2 *Let $f : E \to (-\infty, +\infty]$ be a Legendre and totally convex function which satisfies the range condition $\operatorname{ran}(\nabla f - A) \subseteq \operatorname{ran}(\nabla f)$. Let $A : E \to E^*$ be a Bregman inverse strongly monotone mapping. If C is a nonempty, closed, and convex subset of $(\operatorname{dom} A) \cap (\operatorname{int} \operatorname{dom} f)$, then* (1) *$VI(C, A) = F(P_C^f \circ A^f)$*; (2) *$P_C^f \circ A^f$ is a Bregman relatively nonexpansive mapping, so is a closed Bregman quasi-strict pseudo-contraction.*

Theorem 4.3 *Let E be a real reflexive Banach space, C be a nonempty, closed, and convex subset of E. Let $A : E \to E^*$ be a Bregman inverse strongly monotone operator such that $C \subset (\operatorname{dom} A)$ and $VI(C, A) \neq \emptyset$. Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Assume that the range condition $\operatorname{ran}(\nabla f - A) \subseteq \operatorname{ran}(\nabla f)$ is satisfied for A. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}^f(x_0), \\ C_{n+1} = \{z \in C_n : D_f(x_n, P_C^f \circ A^f(x_n)) \\ \qquad \leq \frac{1}{1-\kappa}\langle \nabla f(x_n) - \nabla f(P_C^f \circ A^f(x_n)), x_n - z\rangle\}, \\ x_{n+1} = P_{C_{n+1}}^f(x_0), \quad n \geq 0, \end{cases} \tag{4.10}$$

where $\kappa \in [0, 1)$. Then the sequence $\{x_n\}$ converges strongly to $\widehat{p} = P_{VI(C,A)}^f(x_0)$, where $P_{VI(C,A)}^f$ is the Bregman projection of E onto $VI(C, A)$.

4.3 Application to zero point problem of maximal monotone operators

Let E be a real reflexive Banach space, $A : E \to 2^{E^*}$ be a maximal monotone operator. The problem of finding an element $x \in E$ such that $0^* \in Ax$ is very important in optimization theory and related fields.

Recall that the resolvent of A, denoted by $\mathrm{Res}_A^f : E \to 2^E$, is defined as follows:

$$\mathrm{Res}_A^f(x) = (\nabla f + A)^{-1} \circ \nabla f(x). \tag{4.11}$$

From Example 2.8, we know that Res_A^f is a closed Bregman quasi-strict pseudo-contraction. So the following result is obtained easily by applying Theorem 3.1.

Theorem 4.4 *Let E be a real reflexive Banach space with the dual E^*, $A : E \to 2^{E^*}$ be a maximal monotone operator with $A^{-1}(0^*) \neq \emptyset$. Let $f : E \to \mathbb{R}$ be a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $\mathrm{Res}_A^f : E \to 2^E$ be the resolvent with respect to A. Let $\{x_n\}$ be a sequence generated by the following iterative algorithm*:

$$\begin{cases} x_0 \in C \text{ chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1}^f(x_0), \\ C_{n+1} = \{z \in C_n : D_f(x_n, \mathrm{Res}_A^f(x_n)) \leq \frac{1}{1-\kappa}\langle \nabla f(x_n) - \nabla f(\mathrm{Res}_A^f(x_n)), x_n - z\rangle\}, \\ x_{n+1} = P_{C_{n+1}}^f(x_0), \quad n \geq 0, \end{cases} \tag{4.12}$$

where $\kappa \in [0,1)$. Then the sequence $\{x_n\}$ converges strongly to $\widehat{p} = P_{A^{-1}(0^*)}^f(x_0)$, where $P_{A^{-1}(0^*)}^f$ is the Bregman projection of E onto $A^{-1}(0^*)$.

5 Numerical examples

In this section, we use a numerical example to demonstrate the convergence of Theorem 3.1.

Let $E = R$, $C = [-1,1]$, $f(x) = \frac{2}{3}x^2$, $Tx = -2x$, $k \in [\frac{1}{3},1)$. From the definition of T, it is obvious that 0 is the unique fixed point of T, that is, $F(T) = \{0\}$, one can easily prove that

$$D(0, Tx) \leq D(0,x) + kD(x, Tx), \quad \forall x \in [-1,1], k \in \left[\frac{1}{3}, 1\right) \tag{5.1}$$

(see Example 3.3 of Ugwunnadi *et al.* [23]). Hence, T is a closed and Bregman quasi-strict pseudo-contraction. In fact, one can prove that from the definition of Bregman distance, we have

$$\begin{aligned} D_f(x_n, Tx_n) &= f(x_n) - f(Tx_n) - \langle \nabla f(Tx_n), x_n - Tx_n\rangle \\ &= \frac{2}{3}x_n^2 - \frac{8}{3}x_n^2 + \left\langle \frac{8}{3}x_n, 3x_n\right\rangle \\ &= 6x_n^2. \end{aligned} \tag{5.2}$$

On the other hand, we compute that

$$\langle \nabla f(x_n) - \nabla f(Tx_n), x_n - z\rangle = \langle 4x_n, x_n - z\rangle = 4x_n^2 - 4x_n z. \tag{5.3}$$

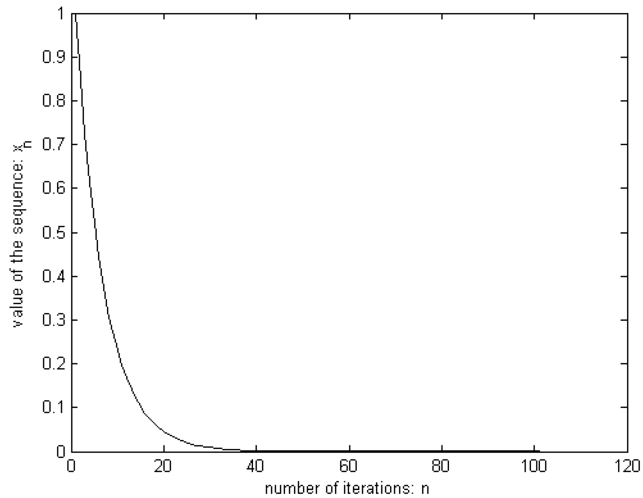

Figure 1 $x_0 = 1$, $k = 0.9$, **the convergence process of the sequence** $\{x_n\}$ **generated by (5.5).**

From (5.2) and (5.3), the C_{n+1} of algorithm (3.1) can be evolved into the following:

$$C_{n+1} = \left\{ z \in C_n : z \le \frac{3k-1}{2} x_n \right\}. \tag{5.4}$$

Therefore, algorithm (3.1) can be simplified as

$$\begin{cases} x_0 \in [-1,1] \text{ chosen arbitrarily}, \\ C_1 = C = [-1,1], \\ x_1 = P^f_{C_1}(x_0), \\ C_{n+1} = \{z \in C_n : z \le \frac{3k-1}{2} x_n\}, \\ x_{n+1} = P^f_{C_{n+1}}(x_0) = \frac{3k-1}{2} x_n, \quad n \ge 0. \end{cases} \tag{5.5}$$

So, the sequence $\{x_n\}$ converges strongly to $\widehat{p} = P^f_{F(T)}(x_0) = 0$ by using Theorem 3.1.

Take the initial point $x_0 = 1$, $k = 0.9$, the numerical experiment result using software Matlab 7.0 is given in Figure 1, which shows that the iteration process of the sequence $\{x_n\}$ converges to 0.

Competing interests

The author declares that they have no competing interests.

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