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Higher-order Lipschitz mappings

Jeffery Ezearn*

*Correspondence:
ezearnje@gmail.com
Department of Mathematics,
Kwame Nkrumah University of
Science and Technology, PMB,
Kumasi, Ghana

Abstract

We study self-mappings on complete metric spaces, which we refer to as *higher-order Lipschitz mappings*. These mappings generalise Lipschitz mappings, the latter which are equivalent to first-order Lipschitz mappings studied in this paper. The main result of this paper is to extend the Banach fixed point theorem (and an often-cited generalisation) to higher-order contraction mappings. We also present results on the problem of local Lipschitzity of these higher-order Lipschitz mappings.

MSC: Primary 47H09; secondary 47H10

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1 Introduction

Let $(\mathcal{X}, d)$ be a complete metric space and let $T : \mathcal{X} \to \mathcal{X}$ be a Lipschitz mapping, that is, $d(Ty, Tx) \leq cd(y, x)$ for all $x, y \in \mathcal{X}$ where $c \geq 0$. When $0 \leq c < 1$, then T is referred to as a *contraction mapping* and when $c = 1$, then T is referred to as a *non-expansive mapping*. In this paper, we consider the following generalisation of Lipschitz mappings:

Definition 1.1 (Higher-order Lipschitz mapping) A mapping $T : \mathcal{X} \to \mathcal{X}$ on a metric space $(\mathcal{X}, d)$ is an *rth-order Lipschitz* mapping if

$$d\left(T^r y, T^r x\right) \leq \sum_{k=0}^{r-1} c_k d\left(T^k y, T^k x\right) \quad \forall x, y \in \mathcal{X}, \tag{1}$$

where r is a natural number and c_k, for all $0 \leq k \leq r-1$, are non-negative real numbers.

An example is when $\mathcal{X}$ is a finite dimensional vector space and $T : \mathcal{X} \to \mathcal{X}$ is a matrix. Indeed, by the Cayley-Hamilton theorem [1, 2], T in this case satisfies an identity

$$f(T) = T^r - a_{r-1}T^{r-1} - \cdots - a_0 = 0,$$

where $f(z) = z^r - a_{r-1}z^{r-1} - \cdots - a_0$ denotes the characteristic polynomial of T. It follows that we have the identity

$$T^r y - T^r x = a_{r-1}\left(T^{r-1}y - T^{r-1}x\right) + \cdots + a_0(y - x),$$

which upon taking norms and using the triangle inequality gives us

$$\left\| T^r y - T^r x \right\| \le c_{r-1} \left\| T^{r-1} y - T^{r-1} x \right\| + \cdots + c_0 \|y - x\|, \tag{2}$$

where $c_k := |a_k|$. Now, as in the first-order case, we classify higher-order Lipschitz mappings into three cases, thus

- T is an *rth-order contraction mapping* if the polynomial $p(z) := z^r - \sum_{k=0}^{r-1} c_k z^k$ is *stable*, that is, $|\lambda| < 1$ if $p(\lambda) = 0$.
- T is an *rth-order non-expansive mapping* if the polynomial $p(z) := z^r - \sum_{k=0}^{r-1} c_k z^k$ is *tamely unstable*, that is, there exists at least a magnitude-wise dominating root $\lambda \in \mathbb{C}$ such that $p(\lambda) = 0$ and $|\lambda| = 1$.
- T is an *rth-order expansive (Lipschitz) mapping* if the polynomial $p(z) := z^r - \sum_{k=0}^{r-1} c_k z^k$ is *wildly unstable*, that is, there exists $\lambda \in \mathbb{C}$ such that $|\lambda| > 1$ and $p(\lambda) = 0$.

Here, $\mathbb{C}$ denotes the field of complex numbers. In Section 2, we give equivalent classification based on the coefficients c_k.

In the following subsections, we review the pertinent results on the fixed point theory of Lipschitz mappings and show the relationship with the fixed point theory of the higher-order counterparts as introduced above.

1.1 Fixed point theory of contraction mappings in metric spaces

The basic result of metric fixed point theory is the Banach [3] fixed point theorem (or the contraction mapping theorem).

Theorem 1.2 (Banach fixed point) *Let $(\mathcal{X}, d)$ be a complete metric space and let $T : \mathcal{X} \to \mathcal{X}$ be a contraction mapping. Then T has a unique fixed point given by the limit of Picard iterates $x_{n+1} := Tx_n$.*

Theorem 1.2 is particularly useful in the demonstration of existence and uniqueness of solutions to certain problems in analysis and economics (see [4–6]). A survey of various extensions of Theorem 1.2 can be found in [7]; we highlight the important results related to those demonstrated in this paper. First, the higher-order contraction case when $r > 1$ and $c_k = 0$ for all $k \ge 1$ is an often-cited generalisation in many texts on Theorem 1.2; this is the case when T^r, but not T^k for all $k < r$, is a contraction mapping; that is:

Theorem 1.3 *Let $(\mathcal{X}, d)$ be a complete metric space and $T : \mathcal{X} \to \mathcal{X}$ a mapping such that T^r is a contraction for some $r > 1$. Then T has a unique fixed point given by the limit of Picard iterates $x_{n+1} := Tx_n$.*

In Section 3, we demonstrate that the conclusions of Theorems 1.2 and 1.3 extend to all higher-order contraction mappings. Both (first-order) contraction mappings and the rth-order contraction mappings defined in Theorem 1.3 are special cases of the now-proven generalised Banach contraction conjecture (see Jachymski [8], Merryfield-Stein [9] and Arvanitakis [10]).

Theorem 1.4 (Generalised Banach contraction theorem) *Let $(\mathcal{X}, d)$ be a complete metric space and let $T : \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$\min_{\substack{1\le i\le r\\ y\ne x}} \frac{d(T^i y, T^i x)}{d(y,x)} \le c$$

for some natural number r and real number $c \in (0,1)$. Then T has a unique fixed point.

In the present paper, we demonstrate the closely related result that an rth-order contraction mapping satisfies the minimising inequality

$$\min_{\substack{1\le i\le r\\ T^{i-1}y\ne T^{i-1}x}} \frac{d(T^i y, T^i x)}{d(T^{i-1}y, T^{i-1}x)} \le c \tag{3}$$

for some real number $c \in (0,1)$. Indeed inequality (3) also holds true for higher-order non-expansive and higher-order expansive mappings, respectively, for some $c = 1$ and $c > 1$.

Now an early continuous mapping generalisation of the Banach fixed point theorem is the following result due to Caccioppoli [11]:

Theorem 1.5 *Let $(\mathcal{X}, d)$ be a complete metric space and let $T : \mathcal{X} \to \mathcal{X}$ be a mapping such that*

$$d\bigl(T^n y, T^n x\bigr) \le q_n d(y,x), \tag{4}$$

where $\{q_n\}_{n\ge1}$ is a summable non-negative sequence independent of $\mathcal{X}$. Then T has a unique fixed point given by the limit of the Picard iterates $x_{n+1} := Tx_n$.

Whereas higher-order contraction mappings do not generally satisfy the hypothesis of Theorem 1.5 (note that the mappings satisfying Theorem 1.5 are necessarily uniformly continuous), we demonstrate in Section 4 that when the additional requirement of continuity is imposed on a higher-order contraction mapping, the inequality (4) holds *locally* in the sense that for every given $x_0 \in \mathcal{X}$, there exists an (open) subset $\mathcal{S} \subset \mathcal{X}$ (depending on, but not necessarily a neighbourhood of, x_0) such that for all $x \in \mathcal{S}$

$$d\bigl(T^n x, T^n x_0\bigr) \le Mc^n d(x, x_0)$$

for some constants $M \ge 1$ and $c \in (0,1)$. Indeed the same is true for higher-order non-expansive and higher-order expansive mappings, respectively, but with the constant $c = 1$ and $c > 1$.

In general, higher-order Lipschitz mappings are not reducible to lower-order Lipschitz mappings within the same metric space $(\mathcal{X}, d)$. Whereas Lipschitz mappings are (uniformly) continuous mappings, this need not be the case for general higher-order Lipschitz mappings; they may not even be continuous: for instance, the function $T : \mathbb{R} \to \mathbb{R}$ given by

$$Tx := \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x \ge 0, \end{cases}$$

with the metric induced by the usual absolute value on $\mathbb{R}$, is discontinuous at $x = 0$ but we observe that $T^2x = 0$ and so

$$\left|T^2y - T^2x\right| = 0 \leq c|y - x|$$

for any $c \geq 0$, thus making T a second-order Lipschitz (indeed, second-order contraction) mapping. The immediate cases for which a higher-order Lipschitz mapping may be of lower-order is when it is actually of lower-order on $\mathcal{X}$ itself (for instance, matrices, which are actually (first-order) Lipschitz mappings but satisfy (2) and also when it is of lower-order on $T(\mathcal{X})$; an example in the latter case is the mapping $T : \mathbb{R} \to \mathbb{R}$ given by

$$Tx := \begin{cases} 1 - x & \text{if } x < 0, \\ x & \text{if } x \geq 0, \end{cases}$$

with the metric induced by the usual absolute value on $\mathbb{R}$. Obviously T is discontinuous at $x = 0$ and noting that $T^2 = T$, then $|T^2y - T^2x| = |Ty - Tx|$; in other words, T is second-order non-expansive mapping but at the same time it is (first-order) non-expansive on the metric subspace $(T(\mathbb{R}), |\cdot|)$. We shall refer to T when it is actually of lower-order on $\mathcal{X}$ or $T(\mathcal{X})$ as a *trivial* mapping.

1.2 Fixed point theory of non-contraction mappings in Banach spaces

Now, for non-contraction mappings, complete metric spaces are in general not sufficient to guarantee the existence or uniqueness fixed points; in this regard, usually, compactness and/or convexity of subsets of normed linear spaces is required. Some noteworthy results are as follows.

Theorem 1.6 (Edelstein [12]) *Let T be a contractive mapping on a compact metric space, that is, $d(Ty, Tx) \leq d(y, x)$ with equality only if $x = y$. Then T has a unique fixed point given by the limit of Picard iteration $x_{n+1} := Tx_n$.*

Theorem 1.7 (Kirk [13]) *Let T be a non-expansive self-mapping on a weakly compact convex subset $\mathcal{C}$ of a Banach space with normal structure - that is, for any bounded non-empty convex subset $\mathcal{K} \subset \mathcal{C}$ there exists a point $x_0 \in \mathcal{K}$ such that $\sup_{x\in\mathcal{K}} \|x_0 - x\| < \operatorname{diam}(\mathcal{K}) := \sup_{x,y\in\mathcal{K}} \|x - y\|$. Then T has a fixed point.*

Theorem 1.8 (Schauder[a] [14]) *Let T be a Lipschitz self-mapping on a compact convex subset of a Banach space. Then T has a fixed point.*

In the present paper, we do not investigate the fixed point theory of higher-order Lipschitz mappings under the hypotheses in Theorems 1.6, 1.7 and 1.8 above. These are deferred to a sequel to this paper.

2 Preliminaries

First of all, we recall the following definitions:

A *nowhere dense* subset of a topological space is a set whose closure has an empty interior in the topological space; that is, it contains no open neighbourhood of its elements in the topological space.

A real matrix is *non-negative* if all its entries are non-negative real numbers; it is *positive* if all the entries are positive real numbers.

A non-negative matrix $\mathbf{A}$ is *irreducible* if for every pair of indices i, j, there exists a natural number n such that $(\mathbf{A}^n)_{ij} > 0$.

A real non-negative matrix $\mathbf{A}$ is *primitive* if there exists an integer $n \geq 1$ such that $\mathbf{A}^n$ is positive; thus, a primitive matrix is irreducible.

A polynomial $f(z)$ is *non-degenerate* if whenever $\alpha \neq \beta$ but $f(\alpha) = f(\beta) = 0$, then $\alpha \neq \zeta\beta$, where ζ is a root of unity.

An rth-order linear recurrence sequence S_n, satisfying the recursive equation $S_{n+r} = \sum_{k=0}^{r-1} c_k S_{n+k}$, is *non-degenerate* if the associated characteristic polynomial $p(z) = z^r - \sum_{k=0}^{r-1} c_k z^k$ is non-degenerate.

The following useful results are necessary for the proof of our main results. As used below and elsewhere, we employ the Kronecker delta symbol δ_{jk}, which equals 1 when $j = k$ and equals 0 if otherwise.

Theorem 2.1 *Let S_n be an rth-order linear recurrence sequence satisfying the recursion $S_{n+r} = \sum_{k=0}^{r-1} c_k S_{n+k}$. Let $p(z) = z^r - \sum_{k=0}^{r-1} c_k z^k$ be the associated characteristic polynomial having distinct* (*complex*) *roots $\lambda_1, \lambda_2, \ldots$ with respective multiplicities $\mu_1, \mu_2, \ldots$ - thus $\sum_i \mu_i = r$. Then S_n has an explicit form*

$$S_n = \sum_i p_i(n)\lambda_i^n,$$

where $p_i(n)$ is a polynomial of degree $\mu_i - 1$. Also, S_n has an implicit form

$$S_n = \sum_{j=0}^{r-1} I_j(n) S_j,$$

where $I_j(n)$ - an impulse-response sequence of S_n - is also an rth-order linear recurrence sequence satisfying the same recursion as S_n with initial values $I_j(k) = \delta_{jk}$ for all $0 \leq j, k \leq r-1$.

Proof See for instance [15], Sections 1.1.4 through 1.1.6. □

The following corollary, which follows straightforwardly from the above theorem, is what is most useful for our purposes here.

Corollary 2.2 *Using the notation of Theorem* 2.1, *define $\lambda := \max|\lambda_i|$ and $\mu := \max \mu_i$. Then $|S_n| \leq K\lambda^n n^{\mu-1}$ for some absolute constant $K > 0$ independent of n, λ, μ; moreover, if $\lambda < 1$, then $\lim_{n\to\infty} S_n = 0$.*

Theorem 2.3 (Skolem-Mahler-Lech [16–19]) *The set of zeroes of a linear recurrence sequence S_n over a field of characteristic zero comprises a finite set together with a finite number of arithmetic progressions. If S_n is non-degenerate, then the set of zeroes is finite.*

Remark The set referred to is the set of indices n for which $S_n = 0$ and in either case it may be empty.

Theorem 2.4 (Baire category [20]) *Let $\mathcal{X}$ be a complete metric space. Then $\mathcal{X}$ is not the countable union of nowhere dense closed sets.*

Theorem 2.5 (Perron-Frobenius [21, 22]) *Let* $\mathbf{A}$ *be an irreducible non-negative* $r \times r$ *matrix with spectral radius* $\rho(\mathbf{A})$. *Then the following statements hold*:

1. $\rho(\mathbf{A})$ *is an eigenvalue of* $\mathbf{A}$ *and it is uniquely dominating if* $\mathbf{A}$ *is primitive*;
2. $\min_i \sum_j \mathbf{A}_{ij} \leq \rho(\mathbf{A}) \leq \max_i \sum_j \mathbf{A}_{ij}$;
3. *Collatz-Wielandt formula*: *Let* $\mathbf{N} := \{\mathbf{v} = \{v_j \geq 0\}_{j=1}^r : \exists i, v_i \neq 0\}$. *Then* $\rho(\mathbf{A}) = \max_{\mathbf{v}\in\mathbf{N}} \min_{1\leq i\leq r, v_i\neq 0} \frac{1}{v_i}(\mathbf{Av})_i$.

Remark The Perron-Frobenius theorem is more general than this but this suffices for our purposes here. By a *uniquely dominating* eigenvalue, we imply one which is a unique maximum in absolute value.

Theorem 2.6 (Keilson-Styan inequality [23]) *Let* $\mathbf{A}$ *be a non-negative* $r \times r$ *matrix with spectral radius* $\rho(\mathbf{A})$. *Then* $\det(tI - \mathbf{A}) \leq t^r - \rho(\mathbf{A})^r$ *for all* $t \geq \rho(\mathbf{A})$.

Theorem 2.7 (Rouché) *Let* $g(z) = z^r$ *and* $h(z) = \sum_{k=0}^{r-1} a_k z^k$ *be complex-valued polynomials such that* $g(R) > \sum_{k=0}^{r-1} |a_k| R^k$ *for a real number* $R > 0$, *then the polynomial* $f(z) := g(z) - h(z)$ *has all its roots lying strictly inside the circle* $|z| = R$.

Proof This follows immediately from a more general theorem of Rouché, a proof of which can be found in Titchmarsh [24]. □

Theorem 2.8 (Bolzano's intermediate value [25]) *If a continuous real function defined on an interval is sometimes positive and sometimes negative, then it must be* 0 *at some point in the interval*.

Theorem 2.9 (Descartes' rule of signs) *Let* $f(z) := \sum_{k=0}^{r} a_k z^k$ *be an rth degree polynomial over the real numbers* a_k. *Then the number of positive real roots of* f *is bounded above by the number of sign changes of the coefficients* a_k *as one proceeds from* $k = 0$ *to* $k = r$ (*ignoring zero coefficients*).

Proof See for instance [26]. □

Proposition 2.10 *Let* $p(z) = z^r - \sum_{k=0}^{r-1} c_k z^k$, *where* $c_k \geq 0$, *be a polynomial*.

(i) *If* p *is stable, then* $0 < p(1) \leq 1$ *and there exists* $\lambda \in [0,1)$, *which is unique and positive if* $c_0 \neq 0$, *such that* $p(\lambda) = 0$.

(ii) *If* p *is tamely unstable, then* $p(1) = 0$ *and* 1 *is the only positive root of* p.

(iii) *If* p *is wildly unstable, then* $p(1) < 0$ *and there exists a unique positive* $\lambda > 1$ *such that* $p(\lambda) = 0$.

Proof First of all, that $p(1) \leq 1$ follows since $c_k \geq 0$.

For (i): Suppose to the contrary that $p(1) \leq 0$. Now we note that for real numbers t, we have $\lim_{t\to\infty} p(t) = \infty$ and as such there exists $t_1 \geq 1$ such that $p(t_1) \geq 0$. Given that p is a continuous function (on the whole of the real line), then by Bolzano's intermediate value theorem (Theorem 2.8), there exists $t_0 \in [1, t_1]$ such that $p(t_0) = 0$, contradicting the fact that by assumption we should rather have $t_0 < 1$. Finally since $p(0) = -c_0 \leq 0$ and $p(1) > 0$ then (by Bolzano's intermediate value theorem again) there exists $\lambda \in [0,1)$ such that $p(\lambda) = 0$. The uniqueness of λ when $c_0 \neq 0$, in which case λ cannot be 0, follows from Descartes' rule of signs (Theorem 2.9).

For (ii): If $1 - p(1) = \sum_{k=0}^{r-1} c_k < 1$, then from Rouché's theorem (Theorem 2.7) all roots of $p(z)$ would be strictly less than 1 in absolute value; hence $p(1) \leq 0$. Now suppose to the contrary that $p(1) < 0$; hence there exists $t_1 > 1$ such that $p(t_1) \geq 0$ and so by Bolzano's intermediate value theorem there exists $t_0 \in (1, t_1]$ such that $p(t_0) = 0$, contradicting the fact that by assumption we should rather have $t_0 \leq 1$; thus $p(1) = 0$. That 1 is the only positive root follows from Descartes' rule of signs.

For (iii): Supposing to the contrary that $p(1) \in [0,1]$, then we would have $1 - p(1) \in [0,1]$. But that would imply from Rouché's theorem that all roots of $p(z)$ are at most 1 in absolute value, which is a contradiction. Finally, since $p(1) < 0$ and $\lim_{t\to\infty} p(t) = \infty$, by Bolzano's intermediate value theorem, there exists $\lambda > 1$ such that $p(\lambda) = 0$, the uniqueness of which follows from Descartes' rule of signs. □

Corollary 2.11 *Let T be a second-order Lipschitz mapping on $(\mathcal{X}, d)$. Then the inequality* (1) *takes the form*

$$d\big(T^2y, T^2x\big) \leq \big(\lambda - \lambda'\big) d(Ty, Tx) + \lambda\lambda' d(y,x),$$

where $0 \leq \lambda' \leq \lambda$.

Proof By Proposition 2.10, one root of the polynomial $p(z) = z^2 - c_1 z - c_0$ is real and non-negative, λ say; hence given that $c_0 \geq 0$ then the other root must be real and non-positive, $-\lambda'$ say, where $\lambda' \geq 0$. We can therefore factor $p(z)$ as $p(z) = (z - \lambda)(z + \lambda') = z^2 - (\lambda - \lambda')z - \lambda\lambda'$ and given that $c_1 = \lambda - \lambda' \geq 0$, the conclusion follows. □

Lemma 2.12 *Let $p(z) := z^r - \sum_{k=0}^{r-1} c_k z^k$, where $c_k \geq 0$, be a polynomial with unique positive root λ as given by Proposition* 2.10. *Then*

1. *λ dominates all other roots of $p(z)$; furthermore, λ is a uniquely dominating root if $p(z)$ is non-degenerate.*
2. $\lambda \in [1 - p(1), (1 - p(1))^{1/r}]$, $\lambda = 1$ *and* $\lambda \in (1, 1 - p(1)]$ *if p is stable, tamely unstable and wildly unstable, respectively.*

Proof We observe that the (companion) non-negative matrix

$$\mathbf{C} := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{r-1} \end{pmatrix} \tag{5}$$

has the characteristic polynomial $p(z) := z^r - \sum_{k=0}^{r-1} c_k z^k$. Now let $I_j(n)$ be the rth-order impulse-response sequence with characteristic polynomial $p(z)$ and initial values $I_j(k) = \delta_{jk}$ for all $0 \leq j, k \leq r - 1$. If we define the column vector $\mathbf{v}(n) := [I_j(n), I_j(n+1), \dots, I_j(n+r-1)]^{\mathbf{T}}$ then $\mathbf{v}(n+1) = \mathbf{C}\mathbf{v}(n)$ and since $\mathbf{v}(0) = [\delta_{j,0}, \delta_{j,1}, \dots, \delta_{j,r-1}]^{\mathbf{T}}$, then it follows by induction that

$$I_j(n+k) = \big(\mathbf{v}(n)\big)_{k+1} = \big(\mathbf{C}\mathbf{v}(n-1)\big)_{k+1} = \cdots = \big(\mathbf{C}^n\mathbf{v}(0)\big)_{k+1} = \big(\mathbf{C}^n\big)_{k+1,j+1}$$

for all $n \geq 0$ and $0 \leq j, k \leq r - 1$. Given that $\mathbf{C}^n \geq 0$ for all $n \geq 0$, then $\mathbf{C}$ would be irreducible if for any given index $j \in [0, r - 1]$ there exists a sufficiently large natural number n such that $I_j(n) > 0$; but if this were not the case, then $I_j(n) = 0$ for all sufficiently large n and so by the Skolem-Mahler-Lech theorem (Theorem 2.3) there would exist a natural m such that $I_j(k + lm) = 0$ for all $0 \leq k \leq m - 1$ and all $l \geq 0$, contradicting the fact that $I_j(j) = 1 \neq 0$. Similarly, when $I_j(n)$ is non-degenerate, that is, when $p(z)$ is a non-degenerate polynomial, then by the Skolem-Mahler-Lech theorem, there would exist a sufficiently large natural number n_j such that $I_j(n) > 0$ for all $n \geq n_j$; consequently, the matrix $\mathbf{C}^{\max_j\{n_j\}}$ would be positive and so it follows that $\mathbf{C}$ is primitive when $p(z)$ is non-degenerate. This gives the first part of the lemma.

Now from Proposition 2.10, λ is the unique positive root of $p(z)$; consequently it follows from the first part of the Perron-Frobenius theorem (Theorem 2.5) that $\rho(\mathbf{C}) = \lambda$ and the conclusion of the first part of the lemma follows immediately. Finally we observe that $\min_i \sum_j \mathbf{C}_{ij} = \min\{1, \sum_{k=0}^{r-1} c_k\} = \min\{1, 1 - p(1)\}$ and similarly we have $\max_i \sum_j \mathbf{C}_{ij} = \max\{1, \sum_{k=0}^{r-1} c_k\} = \max\{1, 1 - p(1)\}$; but by Proposition 2.10, we have $p(1) > 0$, $p(1) = 0$ and $p(1) < 0$ if T is rth-order contraction, non-expansive and expansive, respectively; hence given that $\rho(\mathbf{C}) = \lambda$, then from the second part of the Perron-Frobenius theorem plus the fact that $\lambda^r \leq 1 - p(1)$ when $\lambda \leq 1$ (the Keilson-Styan inequality, Theorem 2.6 with $t = 1$), the second part of the lemma follows. □

Proposition 2.13 *For all* $\lambda \in [0, 1)$, $\mu \geq 1$ *and integers* $m \geq 0$,

$$\sum_{k=0}^{\infty} \lambda^{m+k}(m+k)^{\mu-1} < L\lambda^m(m+1)^{\mu-1}$$

for some absolute constant L dependent only on λ, μ.

Proof First note that the polylogarithm function $Li_{1-\mu}(\lambda) := \sum_{k=1}^{\infty} \lambda^k k^{\mu-1}$ converges (via Cauchy's root test, say) for all $\lambda \in [0, 1)$. Now if $m \in \{0, 1\}$ then

$$\sum_{k=0}^{\infty} \lambda^{m+k}(m+k)^{\mu-1} = \sum_{k=1}^{\infty} \lambda^k k^{\mu-1} = Li_{1-\mu}(\lambda),$$

from which the proposition follows in that case by setting, for instance, $L := (1 + Li_{1-\mu}(\lambda)) \max\{1, 2^{1-\mu}/\lambda\}$. Now when $m \geq 2$ and using the fact that $mk \geq m + k$ when also $k \geq 2$, then we have

$$\begin{aligned}
\sum_{k=0}^{\infty} \lambda^{m+k}(m+k)^{\mu-1} &= \lambda^m\left(m^{\mu-1} + \lambda(m+1)^{\mu-1}\right) + \sum_{k=2}^{\infty} \lambda^{m+k}(m+k)^{\mu-1} \\
&\leq \lambda^m\left(m^{\mu-1} + \lambda(m+1)^{\mu-1}\right) + \sum_{k=2}^{\infty} \lambda^{m+k}(mk)^{\mu-1} \\
&= \lambda^m\left(m^{\mu-1} + \lambda(m+1)^{\mu-1}\right) + \lambda^m m^{\mu-1} \sum_{k=2}^{\infty} \lambda^k k^{\mu-1} \\
&= \lambda^m\left(m^{\mu-1} + \lambda(m+1)^{\mu-1}\right) + \lambda^m m^{\mu-1}\left(-\lambda + Li_{1-\mu}(\lambda)\right) \\
&< \lambda^m(m+1)^{\mu-1}\left(1 + \lambda - \lambda + Li_{1-\mu}(\lambda)\right) \\
&= \lambda^m(m+1)^{\mu-1}\left(1 + Li_{1-\mu}(\lambda)\right).
\end{aligned}$$

Setting $L := (1 + Li_{1-\mu}(\lambda)) \max\{1, 2^{1-\mu}/\lambda\}$ as before, the proposition follows. □

Proposition 2.14 *Let $T : \mathcal{X} \to \mathcal{Y}$ be a mapping between metric spaces $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$. Then T is continuous at $x^* \in \mathcal{X}$ if and only if for every sequence $\{x_n\}_{n\geq 0}$ converging to x^* we find that the sequence $\{Tx_n\}_{n\geq 0}$ is Cauchy.*

Proof Indeed if T is continuous at x^*, then the conclusion of the proposition holds true, hence it suffices for us to show that $\lim_{n\to\infty} Tx_n = Tx^*$ for every sequence $\{x_n\}_{n\geq 0}$ converging to x^*. Suppose to the contrary that for some sequence $\{x_n\}_{n\geq 0}$ converging to x^* we have $\lim_{n\to\infty} Tx_n \neq Tx^*$ and we define a sequence $\{u_n\}_{n\geq 0}$ by $u_{2n} = x^*$ and $u_{2n+1} = x_n$. Clearly u_n converges to x^* but since we have

$$\lim_{n\to\infty} d_{\mathcal{Y}}(Tu_{2n+1}, Tu_{2n}) = \lim_{n\to\infty} d_{\mathcal{Y}}\bigl(Tx_n, Tx^*\bigr) \neq 0$$

we find that $\{Tu_n\}_{n\geq 0}$ is not Cauchy, which is a contradiction. Hence $\lim_{n\to\infty} Tx_n = Tx^*$ for any sequence $\{x_n\}_{n\geq 0}$ converging to x^* and so T is continuous at x^*. □

3 Main result

We now prove our main results. We begin with a direct proof of the fixed point theorem for higher-order contraction mappings; thereafter, we provide a re-metrisation argument that relates higher-order Lipschitz mappings to (first-order) Lipschitz mappings. This re-metrisation of the original metric space does not necessarily result in a complete metric space even if the former is complete; consequently, we shall need a completion of the re-metrised space and an extension of the higher-order Lipschitz mapping into the complete re-metrised space.

3.1 Higher-order contraction mappings

The main result of this subsection is as follows, which extends the conclusion of the Banach fixed point theorem (Theorem 1.2) to higher-order contraction mappings:

Theorem 3.1 (Higher-order contraction mapping theorem) *Let $(\mathcal{X}, d)$ be a complete metric space and let $T : \mathcal{X} \to \mathcal{X}$ be an rth-order contraction mapping. Then T has a unique fixed point and* $\lim_{n\to\infty} T^n x$ *converges to this fixed point for arbitrary $x \in \mathcal{X}$.*

We accomplish the proof below, but we require the following auxiliary lemma.

Lemma 3.2 *The sequence $\{T^n x\}_{n\geq 0}$ is Cauchy for all $x \in \mathcal{X}$; moreover,* $\lim_{n\to\infty} T^n y = \lim_{n\to\infty} T^n x$ *for all $x, y \in \mathcal{X}$.*

Proof For all $x := x_0, y := y_0 \in \mathcal{X}$ let $x_n := T^n x$ and $y_n := T^n y$. Then from the inequality (1), we have

$$d(y_{m+r}, x_{m+r}) \leq \sum_{k=0}^{r-1} c_k d(y_{m+k}, x_{m+k}) \tag{6}$$

for all $m \geq 0$. Now we prove by induction that

$$d(y_n, x_n) \leq \sum_{j=0}^{r-1} I_j(n) d(y_j, x_j), \tag{7}$$

where $I_j(n)$ satisfies $I_j(n+r) = \sum_{k=0}^{r-1} c_k I_j(n+k)$ and $I_j(k) = \delta_{jk}$ for all $0 \leq k \leq r-1$ (thus, $I_j(n)$ is an impulse-response sequence). Indeed inequality (7) already holds with equality when $0 \leq n := k \leq r-1$, which serves as our base cases for the induction; thus for some non-negative integer m, suppose it holds for $n = m, m+1, \ldots, m+r-1$. Then from inequality (6) we have

$$\begin{aligned}
d(y_{m+r}, x_{m+r}) &\leq \sum_{k=0}^{r-1} c_k d(y_{m+k}, x_{m+k}) \\
&\leq \sum_{k=0}^{r-1} c_k \sum_{j=0}^{r-1} I_j(m+k) d(y_j, x_j) \\
&= \sum_{j=0}^{r-1} d(y_j, x_j) \sum_{k=0}^{r-1} c_k I_j(m+k) \\
&= \sum_{j=0}^{r-1} d(y_j, x_j) I_j(m+r).
\end{aligned}$$

Thus inequality (7) holds for $n = m + r$ as well and so it holds for all $n \geq 0$. From Corollary 2.2, $I_j(n) \to 0$ as $n \to \infty$ and so from inequality (7) $d(y_n, x_n) \to 0$ as $n \to \infty$; hence from the continuity of d, we have $d(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = 0$ and thus $\lim_{n\to\infty} T^n y = \lim_{n\to\infty} T^n x$, assuming the limit exists, which we show next.

Henceforward we make the substitution $y = Tx$, thus $x_{n+1} = Tx_n$. We show that $\{x_n\}_{n\geq 0}$ is Cauchy but this is trivial if $x_1 = x_0$, that is, if x_0 is a fixed point; hence we assume that $x_1 \neq x_0$. Now let $n > m$ and letting λ and μ denote, respectively, the maximum in absolute value and multiplicity of the roots of the polynomial $p(z) = z^r - \sum_{k=0}^{r-1} c_k z^k$, then via the triangle inequality of d, inequality (7), Corollary 2.2 and Proposition 2.13, we have

$$\begin{aligned}
d(x_n, x_m) &\leq \sum_{k=0}^{n-m-1} d(x_{m+k+1}, x_{m+k}) \\
&\leq \sum_{k=0}^{n-m-1} \sum_{j=0}^{r-1} I_j(m+k) d(x_{j+1}, x_j) \\
&= \sum_{j=0}^{r-1} d(x_{j+1}, x_j) \sum_{k=0}^{n-m-1} I_j(m+k) \\
&\leq K \sum_{j=0}^{r-1} d(x_{j+1}, x_j) \sum_{k=0}^{n-m-1} \lambda^{m+k}(m+k)^{\mu-1} \\
&< K \sum_{j=0}^{r-1} d(x_{j+1}, x_j) \sum_{k=0}^{\infty} \lambda^{m+k}(m+k)^{\mu-1} \\
&< KL\lambda^m (m+1)^{\mu-1} \sum_{j=0}^{r-1} d(x_{j+1}, x_j).
\end{aligned}$$

Note that $\sum_{j=0}^{r-1} d(x_{j+1}, x_j) \neq 0$ since by assumption $x_1 \neq x_0$. Now given that $0 \leq \lambda < 1$, it follows that $\lambda^m (m+1)^{\mu-1} \to 0$ as $m \to \infty$, hence for any arbitrary $\varepsilon > 0$, we can find $N(\varepsilon)$

large enough such that

$$\lambda^m (m+1)^{\mu-1} < \frac{\varepsilon}{KL \sum_{j=0}^{r-1} d(x_{j+1}, x_j)} \quad \forall m \geq N(\varepsilon),$$

which implies that $d(x_n, x_m) < \varepsilon$ for all $n > m \geq N(\varepsilon)$ and thus the sequence $\{x_n\}_{n\geq 0}$ is indeed Cauchy. □

Proof of Theorem 3.1 Indeed by Lemma 3.2 the sequence $\{x_n := T^n x\}_{n\geq 0}$ is Cauchy for any arbitrary $x \in \mathcal{X}$ and is therefore convergent, say $\lim_{n\to\infty} T^n x = \lim_{n\to\infty} Tx_n = x^* \in \mathcal{X}$. Consequently $\lim_{n\to\infty} T^n x^* = x^*$ and so the set $S(x^*) := \{T^n x^*\}_{n\geq 0}$ is a closed subset of $\mathcal{X}$; indeed $(S(x^*), d)$ is a complete metric subspace of $\mathcal{X}$ such that $T(S(x^*)) \subseteq S(x^*)$. But for every sequence $\{s_n\}_{n\geq 0} \subseteq S(x^*)$ convergent to x^*, it follows from Lemma 3.2 that the sequence $\{Ts_n\}_{n\geq 0}$ is Cauchy; hence given that $x^* \in S(x^*)$, then by Proposition 2.14 it follows that T is continuous at x^* in $S(x^*)$ and consequently $Tx^* = x^*$. Now to see that x^* is a unique fixed point, observe that if $Ty^* = y^*$ for some $y^* \neq x^*$ then

$$\begin{aligned} 0 &< d(y^*, x^*) = d(T^r y^*, T^r x^*) \\ &\leq \sum_{k=0}^{r-1} c_k d(T^k y^*, T^k x^*) \\ &= \sum_{k=0}^{r-1} c_k d(y^*, x^*) \\ &= (1 - p(1)) d(y^*, x^*). \end{aligned}$$

But by Proposition 2.10, we have $0 < p(1) \leq 1$, which leads to the contradiction that $0 < d(y^*, x^*) \leq (1 - p(1)) d(y^*, x^*) < d(y^*, x^*)$. Equivalently, in a more straightforward fashion, if $y^* \neq x^*$ is also a fixed point, then we get the contradiction (via Lemma 3.2) that $y^* = \lim_{n\to\infty} T^n y^* = \lim_{n\to\infty} T^n x^* = x^*$. This completes the proof of Theorem 3.1. □

3.2 The general case

Now we consider the general case of higher-order Lipschitz mappings. First of all, it is worthy of note the theorem of Bessaga [27] states that whenever $\mathcal{X}$ is an arbitrary set with a self-map T satisfying the property that each iterate T^n has a unique fixed point, then for each $c \in (0,1)$, there exists a metric d_c on $\mathcal{X}$ such that $(\mathcal{X}, d_c)$ is a complete metric space and T is a contraction mapping on $(\mathcal{X}, d_c)$. Thus in light of Theorem 3.1 demonstrated in the previous subsection, we are motivated to consider a re-metrisation of the space $(\mathcal{X}, d)$ over which a higher-order Lipschitz mapping is defined.

Now let T be an rth-order Lipschitz mapping on a complete metric space $(\mathcal{X}, d)$ as given in (1) and let λ be the unique positive root of the polynomial $p(z) = z^r - \sum_{k=0}^{r-1} c_k z^k$ as guaranteed by Proposition 2.10, in particular we assume that $p(0) \neq 0$. Define a new metric on the space $\mathcal{X}$ as follows:

$$D(y, x) = \sum_{k=0}^{r-1} b_k d(T^k y, T^k x), \quad \text{where } b_k = \sum_{j=0}^{k} c_j \lambda^{j-k-1}. \tag{8}$$

That D is a metric on $\mathcal{X}$ is straightforward. Indeed, D is non-negative since d and b_k are non-negative and it is sub-additive since d is sub-additive; furthermore, $D(y,x) = 0$ if and only if $y = x$ since $b_k \neq 0$ (because, by assumption, $c_0 \neq 0$) and finally $D(y,x) = D(x,y)$.

Now we have the following lemma.

Lemma 3.3 *Let* $(\mathcal{X},d)$ *be a (not necessarily complete) metric space and let* $T : \mathcal{X} \to \mathcal{X}$ *be an rth-order Lipschitz mapping. Let D be the new metric defined in* (8). *Then*

$$D(Ty,Tx) \le \lambda D(y,x).$$

Moreover, a sequence $\{x_n\}_{n\ge1} \subset (\mathcal{X},D)$ *is Cauchy in* $(\mathcal{X},D)$ *if and only if the sequence* $\{T^k x_n\}_{n\ge1} \subset (\mathcal{X},d)$ *is Cauchy in* $(\mathcal{X},d)$ *for all* $0 \le k \le r-1$.

First, we prove the following recurrence relation for the constants b_k in (8).

Proposition 3.4 *Let* b_k *be as defined in* (8). *Then*

$$b_0 = \lambda^{-1}c_0, \qquad b_{r-1} = 1,$$
$$b_k = \lambda^{-1}(b_{k-1} + c_k), \quad 1 \le k \le r-1.$$

Proof Obviously, $b_0 = \sum_{j=0}^{0} c_j\lambda^{j-k-1} = c_0\lambda^{-1}$. Then also

$$b_{r-1} = \sum_{j=0}^{r-1} c_j\lambda^{j-r} = \lambda^{-r}\sum_{j=0}^{r-1} c_j\lambda^{j} = \lambda^{-r}\bigl(\lambda^r - p(\lambda)\bigr) = 1.$$

Finally,

$$b_k = \sum_{j=0}^{k} c_j\lambda^{j-k-1} = \lambda^{-1}\sum_{j=0}^{k-1} c_j\lambda^{j-k} + c_k\lambda^{-1} = \lambda^{-1}(b_{k-1} + c_k),$$

which completes the proof. □

Proof of Lemma 3.3 Via Proposition 3.4, we have

$$\begin{aligned}
D(Ty,Tx) &= \sum_{k=0}^{r-1} b_k d\bigl(T^{k+1}y, T^{k+1}x\bigr) \\
&= \sum_{k=1}^{r} b_{k-1} d\bigl(T^k y, T^k x\bigr) \\
&= b_{r-1} d\bigl(T^r y, T^r x\bigr) + \sum_{k=1}^{r-1} b_{k-1} d\bigl(T^k y, T^k x\bigr) \\
&\le \sum_{k=0}^{r-1} c_k d\bigl(T^k y, T^k x\bigr) + \sum_{k=1}^{r-1} b_{k-1} d\bigl(T^k y, T^k x\bigr) \\
&= c_0 d(y,x) + \sum_{k=1}^{r-1} (b_{k-1} + c_k) d\bigl(T^k y, T^k x\bigr)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda b_0 d(y,x) + \sum_{k=1}^{r-1} \lambda b_k d\bigl(T^k y, T^k x\bigr) \\
&= \lambda \sum_{k=0}^{r-1} b_k d\bigl(T^k y, T^k x\bigr) \\
&= \lambda D(y,x).
\end{aligned}
$$

Finally, since $b_k \neq 0$ for all $0 \leq k \leq r-1$, then we note from (8) that if $\lim_{n\geq m\to\infty} D(x_n,x_m) = 0$, then likewise $\lim_{n\geq m\to\infty} d(T^k x_n, T^k x_m) = 0$ for all $0 \leq k \leq r-1$; similarly, if $\lim_{n\geq m\to\infty} d(T^k x_n, T^k x_m) = 0$ for all $0 \leq k \leq r-1$, then likewise $\lim_{n\geq m\to\infty} D(x_n,x_m) = 0$, which completes the proof. □

We note in the first place that Lemma 3.3 *does not imply* that T is uniformly continuous (or even continuous) in $(\mathcal{X},d)$ as was noted in the introduction; rather, T is Lipschitz continuous (and therefore uniformly continuous) in $(\mathcal{X},D)$. Secondly, when $\lambda < 1$, then the Banach fixed point theorem (Theorem 1.2) *cannot be applied* to assert that T has a fixed point in $(\mathcal{X},D)$ unless T is continuous in $(\mathcal{X},d)$; however, the following theorem remedies the case when T is discontinuous on $(\mathcal{X},d)$. To proceed, let $(\overline{\mathcal{X}},\overline{D})$ be the canonical completion of the metric space $(\mathcal{X},D)$; that is,

$$
\overline{D}\bigl([y_n],[x_n]\bigr) := \lim_{n\to\infty} D(y_n,x_n),
$$

where $\{y_n\}_{n\geq 1}$, $\{x_n\}_{n\geq 1}$ are Cauchy sequences in $(\mathcal{X},D)$ and $[x_n]$ denotes the equivalence class of $\{x_n\}_{n\geq 1}$ in $(\mathcal{X},D)$, where $\{y_n\}_{n\geq 1}$ is equivalent to $\{x_n\}_{n\geq 1}$ if $\lim_{n\to\infty} D(y_n,x_n) = 0$.

Theorem 3.5 *Define the mapping,*

$$
\overline{T}: \overline{\mathcal{X}} \to \overline{\mathcal{X}}, \qquad [x_n] \mapsto [Tx_n].
$$

Then we have

$$
\overline{D}\bigl(\overline{T}[y_n],\overline{T}[x_n]\bigr) \leq \lambda \overline{D}\bigl([y_n],[x_n]\bigr).
$$

In particular, if $(\mathcal{X},d)$ is complete, then T has a fixed point in $(\mathcal{X},d)$ if and only if $\overline{T}$ has a fixed point in $(\overline{\mathcal{X}},\overline{D})$.

Proof Since $\{x_n\}_{n\geq 1}$ is Cauchy in $(\mathcal{X},D)$ then, by Lemma 3.3,

$$
D(Tx_n,Tx_m) \leq \lambda D(x_n,x_m)
$$

and so $\{Tx_n\}_{n\geq 1}$ is Cauchy in $(\mathcal{X},D)$; thus $\overline{T}$ is well defined. Now given Cauchy sequences $\{y_n\}_{n\geq 1}$, $\{x_n\}_{n\geq 1}$ in $(\mathcal{X},D)$, then we have

$$
\begin{aligned}
\overline{D}\bigl(\overline{T}[y_n],\overline{T}[x_n]\bigr) &= \overline{D}\bigl([Ty_n],[Tx_n]\bigr) \\
&= \lim_{n\to\infty} D(Ty_n,Tx_n) \\
&\leq \lambda \lim_{n\to\infty} D(y_n,x_n) \\
&= \lambda \overline{D}\bigl([y_n],[x_n]\bigr).
\end{aligned}
$$

Finally, if $x = Tx$ in $(\mathcal{X}, d)$, then let $[x]$ be the equivalence class of the constant sequence $\{x, x, x, \ldots\} \in (\overline{\mathcal{X}}, \overline{D})$. Then

$$\overline{T}[x] = [Tx] = [x].$$

On the other hand, if $[x_n] = \overline{T}[x_n] = [Tx_n]$ in $(\overline{\mathcal{X}}, \overline{D})$, then by Lemma 3.3, T is continuous at $x := \lim_{n\to\infty} x_n$ in $(\mathcal{X}, d)$, hence

$$Tx = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_n = x,$$

which completes the proof. □

4 Local Lipschitzity

If $r > 1$, a natural question that arises is whether there are pairs $x_0, y_0 \in \mathcal{X}$ and a constant $c > 0$ such that $d(T^n y_0, T^n x_0) \leq c^n d(y_0, x_0)$ for all $n \geq 0$; such a pair is therefore *Lipschitzian* under T, in the sense that this would be the case if T were a Lipschitz mapping with Lipschitz constant c. In this section, motivated by the existence of the unique positive real root λ of the polynomial $p(z)$ (in Definition 1.1) when $p(0) \neq 0$, we show that *near-local Lipschitzity* of open subsets exists with respect to an arbitrary point: that is, given $x_0 \in \mathcal{X}$ there exists an open subset $\mathcal{S} \subset \mathcal{X}$ and positive real number $m_0 \geq 1$ such that $d(T^n x, T^n x_0) \leq m_0 \lambda^n d(x, x_0)$ for all $x \in \mathcal{S}$. We also prove the closely related result that there exists an open subset $\mathcal{S}$ of $\mathcal{X}^2$ and real number $m_0 \geq 1$ such that $d(T^n y, T^n x) \leq m_0 \lambda^n d(y, x)$ for all $(x, y) \in \mathcal{S}$. Though plausible, we are not able to determine whether or when local Lipschitzity can occur (non-trivially) in either case, that is, whether or when m_0 can take the value 1.

Unless otherwise mentioned, we assume $p(0) \neq 0$ throughout the remaining part of this subsection.

Proposition 4.1 *Let $(\mathcal{X}, d)$ be a complete metric space and let T be an rth-order Lipschitz mapping on $\mathcal{X}$. For every pair $x \neq y \in \mathcal{X}$ define*

$$M := M(y, x) = \max_{0 \leq k \leq r-1} \lambda^{-k} \frac{d(T^k y, T^k x)}{d(y, x)}.$$

Then

$$M = \max_{n \geq 0} \lambda^{-n} \frac{d(T^n y, T^n x)}{d(y, x)}.$$

Proof We prove by induction; by definition the conclusion holds up to $n = r - 1$, hence assuming the conclusion holds up to $n + r - 1$, we have

$$\begin{aligned} d\big(T^{n+r} y, T^{n+r} x\big) &\leq \sum_{k=0}^{r-1} c_k d\big(T^{n+k} y, T^{n+k} x\big) \\ &\leq M \sum_{k=0}^{r-1} c_k \lambda^{n+k} d(Ty, Tx) \end{aligned}$$

$$= M\lambda^n d(Ty, Tx) \sum_{k=0}^{r-1} c_k \lambda^k$$
$$= M\lambda^n d(Ty, Tx)\big(\lambda^r - p(\lambda)\big)$$
$$= M\lambda^{n+r} d(Ty, Tx),$$

and so the conclusion holds for $n + r$ as well and thus it holds for all n. □

Remark Obviously, $M(y,x) \geq 1$ for every $x, y \in \mathcal{X}$ so if $M(y,x)$ can be 1 for some pair x, y then T is essentially *locally Lipschitz* on the pair x, y. We are not able to establish whether or when local Lipschitzity can occur, but the next results give near misses.

Theorem 4.2 *Let T be an rth-order Lipschitz mapping on a complete metric space $(\mathcal{X}, d)$. Then for all $x, y \in \mathcal{X}$*

$$\min_{\substack{1 \leq i \leq r \\ d(T^{i-1}y, T^{i-1}x) \neq 0}} \frac{d(T^i y, T^i x)}{d(T^{i-1}y, T^{i-1}x)} \leq \lambda.$$

In particular, $\min_{x \neq y} \frac{d(Ty,Tx)}{d(y,x)} \leq \lambda$.

Proof Define the column vector $\mathbf{v} := [d(y,x), \dots, d(T^{r-1}y, T^{r-1}x)]^{\mathbf{T}}$ and let $\mathbf{C}$ be the companion non-negative $r \times r$ matrix defined in (5). Then

$$\mathbf{Cv} = \begin{pmatrix} d(Ty, Tx) \\ d(T^2y, T^2x) \\ \vdots \\ \sum_{k=0}^{r-1} c_k d(T^r y, T^r x) \end{pmatrix}.$$

Now by definition of T in Definition 1.1 we have $(\mathbf{Cv})_r \geq d(T^r y, T^r x)$, consequently we have the inequality

$$\min_{\substack{1 \leq i \leq r \\ v_i \neq 0}} \frac{(\mathbf{Cv})_i}{\mathbf{v}_i} \geq \min_{\substack{1 \leq i \leq r \\ d(T^{i-1}y, T^{i-1}x) \neq 0}} \frac{d(T^i y, T^i x)}{d(T^{i-1}y, T^{i-1}x)}. \tag{9}$$

But by the Collatz-Wielandt formula of the Perron-Frobenius theorem (Theorem 2.5), the left-hand side of (9) attains the maximum value of $\rho(\mathbf{C})$, which equals λ from the proof of Lemma 2.12. The conclusion of the theorem then follows immediately. □

Remark We note that if the inequality in Theorem 4.2 is uniform for all $1 \leq i \leq r$ for some $x, y \in \mathcal{X}$, then the maximum bound $M(x,y)$ as defined in Proposition 4.1 would be exactly equal to 1.

Theorem 4.3 *Let T be a continuous higher-order Lipschitz mapping on a complete metric space $(\mathcal{X}, d)$. Then for every $x_0 \in \mathcal{X}$ there exist an open set $\mathcal{S} \subset \mathcal{X}$ and a natural number m_0 such that*

$$d\big(T^n x, T^n x_0\big) \leq m_0 \lambda^n d(x, x_0)$$

for all $x \in \mathcal{S}$ and $n \geq 0$.

Proof By Proposition 4.1, there exists a bound $M(x_0,x)$ for each $x,x_0 \in \mathcal{X}$ such that $d(T^n x_0, T^n x) \le M(x_0,x)\lambda^n d(x_0,x)$ for all $n \ge 0$. For all natural numbers m define the sets

$$\mathcal{X}_m := \bigl\{x \in \mathcal{X} : M(x_0,x) \le m\bigr\}.$$

Obviously if $\{y_k\}_{k\ge1} \subset \mathcal{X}$ is a sequence converging to y such that $M(x_0,y_k) \le m$, then via the continuity of the metric d and of T we have

$$\begin{aligned} d\bigl(T^n x_0, T^n y\bigr) &= \lim_{k\to\infty} d\bigl(T^n x_0, T^n y_k\bigr) \\ &\le \lim_{k\to\infty} M(x_0,y_k)\lambda^n d(x_0,y_k) \\ &\le m \lim_{k\to\infty} \lambda^n d(x_0,y_k) \\ &= m\lambda^n d(x_0,y) \end{aligned}$$

and as such the sets $\mathcal{X}_m$ are all closed. But every $x \in \mathcal{X}$ is contained in some $\mathcal{X}_m$ and so we have $\mathcal{X} = \bigcup_{m\ge1} \mathcal{X}_m$. Thus by the Baire category theorem (Theorem 2.4), there exist $m_0 \ge 1$ and an open set $S \in \mathcal{X}_{m_0}$ such that $M(x_0,x) \le m_0$ for all $x \in \mathcal{S}$. □

Theorem 4.4 *Let T be a continuous higher-order Lipschitz mapping on a complete metric space $(\mathcal{X},d)$. Then there exist an open set $\mathcal{S} \subset \mathcal{X}^2$ and a natural number m_0 such that*

$$d\bigl(T^n y, T^n x\bigr) \le m_0\lambda^n d(y,x)$$

for all $(x,y) \in \mathcal{S}$ and $n \ge 0$.

Proof The proof is similar to that given for Theorem 4.3. Here we define the sets

$$\mathcal{X}_m := \bigl\{(x,y) \in \mathcal{X}^2 : M(y,x) \le m\bigr\} \quad \forall m \in \{1,2,3,\ldots\},$$

where $M(y,x)$ is the maximum bound defined in Proposition 4.1. With respect to a chosen metric on $\mathcal{X}^2$ which agrees with the product topology on $\mathcal{X}^2$, then $\mathcal{X}^2$ is also a complete metric space. Furthermore, if $\{(x_k,y_k)\}_{k\ge1} \subset \mathcal{X}^2$ converges to (x,y) say, such that $M(y_k,x_k) \le m$ then via the continuity of the metric d and of T we have

$$\begin{aligned} d\bigl(T^n y, T^n x\bigr) &= \lim_{k\to\infty} d\bigl(T^n y_k, T^n x_k\bigr) \\ &\le \lim_{k\to\infty} M(y_k,x_k)\lambda^n d(y_k,x_k) \\ &\le m \lim_{k\to\infty} \lambda^n d(y_k,x_k) \\ &= m\lambda^n d(y,x) \end{aligned}$$

and so the sets $\mathcal{X}_m$ are closed. Since every pair $(x,y) \in \mathcal{X}^2$ is contained in some $\mathcal{X}_m$, it follows that

$$\mathcal{X}^2 = \bigcup_{m\ge1} \mathcal{X}_m.$$

Thus by the Baire category theorem, there exist $m_0 \geq 1$ and an open set $S \in \mathcal{X}_{m_0}$ such that $M(y,x) \leq m_0$ for all $(x,y) \in \mathcal{S}$. □

Now we have the following problem.

Local Lipschitzity problem Suppose T is a non-trivial rth-order Lipschitz mappings (that is, T is not of lower order on either $T(\mathcal{X})$ or $\mathcal{X}$). *Can the constants M and m_0 appearing in Proposition* 4.1 *and Theorems* 4.3 *and* 4.4 *be exactly equal to 1*?

The determination of whether or when this can be answered in the affirmative would demonstrate that there are subsets of elements in a complete metric space from which the Picard iterations are sharply convergent to the fixed point of a higher-order contraction mapping on the metric space in question.

5 Conclusion

In light of the fact that a higher-order Lipschitz mapping can be considered as a linear system of mapping in the form of the matrix inequality

$$\begin{pmatrix} d(Ty,Tx) \\ d(T^2y,T^2x) \\ \vdots \\ d(T^{r-1}y,T^{r-1}x) \\ d(T^ry,T^rx) \end{pmatrix} \leq \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ c_0 & c_1 & c_2 & \dots & c_{r-1} \end{pmatrix} \begin{pmatrix} d(y,x) \\ d(Ty,Tx) \\ \vdots \\ d(T^{r-2}y,T^{r-2}x) \\ d(T^{r-1}y,T^{r-1}x) \end{pmatrix}$$

it is rather natural to consider mappings when the above matrix inequality can be satisfied but with non-negative $r \times r$ matrix rather than the companion matrix. We recall that the tools used in the local-Lipschitzity analysis of higher-order Lipschitz mappings (the Perron-Frobenius result in particular) are at our disposal to use when we consider a non-negative matrix instead of just a companion matrix. We therefore propose the following broader kind of mappings.

Definition 5.1 Let $(\mathcal{X},d)$ be a (complete) a metric space. Then a countable collection of self-mappings $\{T_k\}_{k\geq 1}$ on $\mathcal{X}$ is said to form an *rth-order Lipschitz system of mappings* if there exists an $r \times r$ non-negative matrix $[c_{i,j}]_{i,j=1}^r$ such that the following matrix inequality is satisfied:

$$\begin{pmatrix} d(T_{k+1}y,T_{k+1}x) \\ d(T_{k+2}y,T_{k+2}x) \\ \vdots \\ d(T_{k+r}y,T_{k+r}x) \end{pmatrix} \leq \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1r} \\ c_{21} & c_{22} & \dots & c_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ c_{r,1} & c_{r,2} & \dots & c_{rr} \end{pmatrix} \begin{pmatrix} d(T_ky,T_kx) \\ d(T_{k+1}y,T_{k+1}x) \\ \vdots \\ d(T_{k+r-1}y,T_{k+r-1}x) \end{pmatrix}.$$

When $T_k := T^k$, then (essentially) we achieve an rth-order Lipschitz mapping. One can then similarly inquire as regards the fixed point theory of such Lipschitz system of mappings.

Competing interests
The author declares that there are no competing interests.

Endnote

[a] Schauder's theorem is more general than this and relates to continuous self-mappings on compact convex subsets of Banach spaces.

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