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Coincidence point theorems for generalized contractions with application to integral equations

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Abstract
In this article, we introduce a new type of contraction and prove certain coincidence point theorems which generalize some known results in this area. As an application, we derive some new fixed point theorems for F-contractions. The article also includes an example which shows the validity of our main result and an application in which we prove an existence and uniqueness of a solution for a general class of Fredholm integral equations of the second kind.

MSC: 46S40; 47H10; 54H25

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1 Introduction and preliminaries

The Banach contraction principle [1] is one of the earliest and most important results in fixed point theory. Because of its application in many disciplines such as computer science, chemistry, biology, physics, and many branches of mathematics, a lot of authors have improved, generalized, and extended this classical result in nonlinear analysis; see, *e.g.*, [2–10] and the references therein. In 2012, Azam [3] obtained the existence of a coincidence point of a mapping and a relation under a contractive condition in the context of metric space. For coincidence point results see also [11]. Consistent with Azam, we begin with some basic known definitions and results which will be used in the sequel. Throughout this article, $\mathbb{N}$, $\mathbb{R}^+$, $\mathbb{R}$ denote the set of all natural numbers, the set of all positive real numbers, and the set of all real numbers, respectively.

Let A and B be arbitrary nonempty sets. A relation R from A to B is a subset of $A \times B$ and is denoted $R : A \rightsquigarrow B$. The statement $(x, y) \in R$ is read 'x is R-related to y', and is denoted xRy. A relation $R : A \rightsquigarrow B$ is called left-total if for all $x \in A$ there exists a $y \in B$ such that xRy, that is, R is a multivalued function. A relation $R : A \rightsquigarrow B$ is called right-total if for all $y \in B$ there exists an $x \in A$ such that xRy. A relation $R : A \rightsquigarrow B$ is known as functional, if xRy, xRz implies that $y = z$, for $x \in A$ and $y, z \in B$. A mapping $T : A \to B$ is a relation from A to B which is both functional and left-total.

For $R : A \rightsquigarrow B$, $E \subset A$ we define

$$R(E) = \{y \in B : xRy \text{ for some } x \in E\},$$

$$\operatorname{dom}(R) = \{x \in A : R(\{x\}) \neq \phi\},$$
$$\operatorname{Range}(R) = \{y \in B : y \in R(\{x\}) \text{ for some } x \in \operatorname{dom}(R)\}.$$

For convenience, we denote $R(\{x\})$ by $R\{x\}$. The class of relations from A to B is denoted by $\mathcal{R}(A,B)$. Thus the collection $\mathcal{M}(A,B)$ of all mappings from A to B is a proper subcollection of $\mathcal{R}(A,B)$. An element $w \in A$ is called a coincidence point of $T : A \to B$ and $R : A \rightsquigarrow B$ if $Tw \in R\{w\}$. In the following we always suppose that X is a nonempty set and (Y,d) is a metric space. For $R : X \rightsquigarrow Y$ and $u, v \in \operatorname{dom}(R)$, we define

$$D(R\{u\}, R\{v\}) = \inf_{uRx, vRy} d(x,y).$$

Wardowski [12] introduced and studied a new contraction called an F-contraction to prove a fixed point result as a generalization of the Banach contraction principle.

Definition 1 Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying the following conditions:

(F_1) F is strictly increasing;
(F_2) for all sequence $\{\alpha_n\} \subseteq R^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F_3) there exists $0 < k < 1$ such that $\lim_{n\to 0^+} \alpha^k F(\alpha) = 0$.

Consistent with Wardowski [12], we denote by $\digamma$ the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the conditions (F_1)-(F_3).

Definition 2 [12] Let (X,d) be a metric space. A self-mapping T on X is called an F-contraction if there exists $\tau > 0$ such that for $x, y \in X$

$$d(Tx, Ty) > 0 \quad \Longrightarrow \quad \tau + F(d(Tx, Ty)) \leq F(d(x,y)),$$

where $F \in \digamma$.

Theorem 3 [12] *Let (X,d) be a complete metric space and T:$X \to X$ be a self-mapping. If there exists $\tau > 0$ such that for all $x, y \in X$: $d(Tx, Ty) > 0$ implies*

$$\tau + F(d(Tx, Ty)) \leq F(d(x,y)),$$

where $F \in \digamma$, then T has a unique fixed point.

Abbas *et al.* [13] further generalized the concept of an F-contraction and proved certain fixed and common fixed point results. Hussain and Salimi [14] introduced some new type of contractions called α-GF-contractions and established Suzuki-Wardowski type fixed point theorems for such contractions. For more details on F-contractions, we refer the reader to [11, 13–20].

In this paper, we obtain coincidence points of mappings and relations under a new type of contractive condition in a metric space. Moreover, we discuss an illustrative example to highlight the realized improvements.

2 Main results

Now we state and prove the main results of this section.

Theorem 4 *Let X be a nonempty set and (Y,d) be a metric space. Let $T : X \to Y, R : X \rightsquigarrow Y$ be such that R is left-total,* $\mathrm{Range}(T) \subseteq \mathrm{Range}(R)$ *and* $\mathrm{Range}(T)$ *or* $\mathrm{Range}(R)$ *is complete. If there exist a mapping $F : \mathbb{R}^+ \to \mathbb{R}$ and $\tau > 0$ such that*

$$d(Tx,Ty) > 0 \quad \Longrightarrow \quad \tau + F\bigl(d(Tx,Ty)\bigr) \le F\bigl(D\bigl(R\{x\},R\{y\}\bigr)\bigr) \tag{2.1}$$

for all $x,y \in X$, then there exists $w \in X$ such that $Tw \in R\{w\}$.

Proof Let $x_0 \in X$ be an arbitrary but fixed element. We define the sequences $\{x_n\} \subset X$ and $\{y_n\} \subset \mathrm{Range}(R)$. Let $y_1 = Tx_0$, $\mathrm{Range}(T) \subseteq \mathrm{Range}(R)$. We may choose $x_1 \in X$ such that x_1Ry_1, since R is left-total. Let $y_2 = Tx_1$, since $\mathrm{Range}(T) \subseteq \mathrm{Range}(R)$. If $Tx_0 = Tx_1$, then we have x_1Ry_2. This implies that x_1 is the required point that is $Tx_1 \in R\{x_1\}$. So we assume that $Tx_0 \neq Tx_1$, then from (2.1) we get

$$\tau + F\bigl(d(y_1,y_2)\bigr) = \tau + F\bigl(d(Tx_0,Tx_1)\bigr) \le F\bigl(D\bigl(R\{x_0\},R\{x_1\}\bigr)\bigr). \tag{2.2}$$

We may choose $x_2 \in X$ such that x_2Ry_2, since R is left-total. Let $y_3 = Tx_2$, since $\mathrm{Range}(T) \subseteq \mathrm{Range}(R)$. If $Tx_1 = Tx_2$, then we have x_2Ry_3. This implies that $Tx_2 \in R\{x_2\}$ and x_2 is the coincidence point. So $Tx_1 \neq Tx_2$, then from (2.1), we get

$$\tau + F\bigl(d(y_2,y_3)\bigr) = \tau + F\bigl(d(Tx_1,Tx_2)\bigr) \le F\bigl(D\bigl(R\{x_1\},R\{x_2\}\bigr)\bigr). \tag{2.3}$$

By induction, we can construct sequences $\{x_n\} \subset X$ and $\{y_n\} \subset \mathrm{Range}(R)$ such that

$$y_n = Tx_{n-1} \quad \text{and} \quad x_nRy_n \tag{2.4}$$

for all $n \in \mathbb{N}$. If there exists $n_0 \in \mathbb{N}$ for which $Tx_{n_0-1} = Tx_{n_0}$. Then $x_{n_0}Ry_{n_0+1}$. Thus $Tx_{n_0} \in R\{x_{n_0}\}$ and the proof is finished. So we suppose now that $Tx_{n-1} \neq Tx_n$ for every $n \in \mathbb{N}$. Then from (2.2), (2.3), and (2.4), we deduce that

$$\tau + F\bigl(d(y_n,y_{n+1})\bigr) = \tau + F\bigl(d(Tx_{n-1},Tx_n)\bigr) \le F\bigl(D\bigl(R\{x_{n-1}\},R\{x_n\}\bigr)\bigr) \tag{2.5}$$

for all $n \in \mathbb{N}$. Since x_nRy_n and $x_{n+1}Ry_{n+1}$, by the definition of D, we get $D(R\{x_{n-1}\},R\{x_n\}) \le d(y_{n-1},y_n)$. Thus from (2.5), we have

$$\tau + F\bigl(d(y_n,y_{n+1})\bigr) \le F\bigl(d(y_{n-1},y_n)\bigr), \tag{2.6}$$

which further implies that

$$\begin{aligned} F\bigl(d(y_n,y_{n+1})\bigr) &\le F\bigl(d(y_{n-1},y_n)\bigr) - \tau \le F\bigl(d(y_{n-2},y_{n-1})\bigr) - 2\tau \le \cdots \\ &\le F\bigl(d(y_0,y_1)\bigr) - n\tau. \end{aligned} \tag{2.7}$$

From (2.7), we obtain

$$\lim_{n\to\infty} F\big(d(y_n,y_{n+1})\big) = -\infty. \tag{2.8}$$

Then from (F_2), we get

$$\lim_{n\to\infty} d(y_n,y_{n+1}) = 0. \tag{2.9}$$

Now from (F_3), there exists $0 < k < 1$ such that

$$\lim_{n\to\infty} \big[d(y_n,y_{n+1})\big]^k F\big(d(y_n,y_{n+1})\big) = 0. \tag{2.10}$$

By (2.7), we have

$$\begin{aligned} &d(y_n,y_{n+1})^k F\big(d(y_n,y_{n+1})\big) - d(y_n,y_{n+1})^k F\big(d(y_0,y_1)\big) \\ &\quad\le d(y_n,y_{n+1})^k \big[F\big(d(y_0,y_1) - n\tau\big) - F\big(d(y_0,y_1)\big)\big] \\ &\quad= -n\tau\big[d(y_n,y_{n+1})\big]^k \le 0. \end{aligned} \tag{2.11}$$

By taking the limit as $n \to \infty$ in (2.11) and applying (2.9) and (2.10), we have

$$\lim_{n\to\infty} n\big[d(y_n,y_{n+1})\big]^k = 0. \tag{2.12}$$

It follows from (2.12) that there exists $n_1 \in \mathbb{N}$ such that

$$n\big[d(y_n,y_{n+1})\big]^k \le 1 \tag{2.13}$$

for all $n > n_1$. This implies

$$d(y_n,y_{n+1}) \le \frac{1}{n^{1/k}} \tag{2.14}$$

for all $n > n_1$. Now we prove that $\{y_n\}$ is a Cauchy sequence. For $m > n > n_1$ we have

$$d(y_n,y_m) \le \sum_{i=n}^{m-1} d(y_i,y_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}. \tag{2.15}$$

Since $0 < k < 1$, $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges. Therefore, $d(y_n,y_m) \to 0$ as $m,n \to \infty$. Thus we proved that $\{y_n\}$ is a Cauchy sequence in Range(R). Completeness of Range(R) ensures that there exists $z \in$ Range(R) such that $y_n \to z$ as $n \to \infty$. Now since R is left-total, wRz for some $w \in X$. Now

$$\begin{aligned} F\big(d(y_n,Tw)\big) &= F\big(d(Tx_{n-1},Tw)\big) \le F\big(D\big(R\{x_{n-1}\},R\{w\}\big)\big) - \tau \\ &< F\big(d(y_{n-1},z)\big) - \tau. \end{aligned}$$

Since $\lim_{n\to\infty} d(y_{n-1},z) = 0$, by ($F_2$), we have $\lim_{n\to\infty} F(d(y_{n-1},z)) = -\infty$. This implies that $\lim_{n\to\infty} F(d(y_n,Tw)) = -\infty$, which further implies that $\lim_{n\to\infty} d(y_n,Tw) = 0$. Hence

$d(z, Tw) = 0$. It follows that $z = Tw$. Hence $Tw \in R\{w\}$. In the case when Range(T) is complete. Since Range(T) $\subseteq$ Range(R), there exists an element $z^* \in$ Range(R) such that $y_n \to z^*$. The remaining part of the proof is the same as in previous case. □

Example 5 Let $X = Y = \mathbb{R}$, $d(x,y) = |x - y|$. Define $T : \mathbb{R} \to \mathbb{R}$, $R : \mathbb{R} \rightsquigarrow \mathbb{R}$ as follows:

$$Tx = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q}', \end{cases}$$

$$R = \bigl(\mathbb{Q}\times[0,4]\bigr) \cup \bigl(\mathbb{Q}'\times[7,9]\bigr).$$

Then Range(T) $= \{0,1\} \subset$ Range(R) $= [0,4] \cup [7,9]$. Let $F(t) = \ln(t)$ and $\tau = 1$.

For $x \in \mathbb{Q}$, $y \in \mathbb{Q}'$ or $y \in \mathbb{Q}$, $x \in \mathbb{Q}'$, $d(Tx, Ty) > 0$ implies that

$$\tau + F\bigl(d(Tx, Ty)\bigr) \le F\bigl(D\bigl(R\{x\}, R\{y\}\bigr)\bigr).$$

Thus all conditions of the above theorem are satisfied and 1 is the coincidence point of T and R.

From Theorem 4 we deduce the following result immediately.

Theorem 6 *Let X be a nonempty set and (Y, d) be a metric space. Let $T, R : X \to Y$ be two mappings such that* Range(T) $\subseteq$ Range(R) *and* Range(T) *or* Range(R) *is complete. If there exist a mapping $F : \mathbb{R}^+ \to \mathbb{R}$ and $\tau > 0$ such that*

$$\tau + F\bigl(d(Tx, Ty)\bigr) \le F\bigl(d(Rx, Ry)\bigr)$$

for all $x, y \in X$, then T and R have a coincidence point in X. Moreover, if either T or R is injective, then R and T have a unique coincidence point in X.

Proof By Theorem 4, we see that there exists $w \in X$ such that $Tw = Rw$, where

$$Rw = \lim_{n\to\infty} Rx_n = \lim_{n\to\infty} Tx_{n-1}, \quad x_0 \in X.$$

For uniqueness, assume that $w_1, w_2 \in X$, $w_1 \neq w_2$, $Tw_1 = Rw_1$, and $Tw_2 = Rw_2$. Then $\tau + F(d(Tw_1, Tw_2)) \le F(d(Rw_1, Rw_2))$ for some $\tau > 0$. If R or T is injective, then

$$d(Rw_1, Rw_2) > 0$$

and

$$\tau + F\bigl(d(Rw_1, Rw_2)\bigr) = \tau + F\bigl(d(Tw_1, Tw_2)\bigr) \le F\bigl(d(Rw_1, Rw_2)\bigr),$$

a contradiction to the fact that $\tau > 0$. □

Remark 7 If in the above theorem we choose $X = Y$, $R = I$ (the identity mapping on X), we obtain Theorem 3, which is Theorem 3.1 of Wardowski [12].

Corollary 8 *Let $T : X \to Y$, $R : X \rightsquigarrow Y$ be such that R is left-total,* $\mathrm{Range}(T) \subseteq \mathrm{Range}(R)$ *and* $\mathrm{Range}(T)$ *or* $\mathrm{Range}(R)$ *is complete. If there exists $\lambda \in [0,1)$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \lambda D\bigl(R\{x\}, R\{y\}\bigr),$$

then there exists $w \in X$ such that $Tw \in R\{w\}$.

Proof Consider the mapping $F(t) = \ln(t)$, for $t > 0$. Then obviously F satisfies (F_1)-(F_3). From Theorem 4, we obtain the desired conclusion. □

Corollary 9 *Let X be nonempty set and (Y, d) be a metric space. $T, R : X \to Y$ be two mappings such that* $\mathrm{Range}(T) \subseteq \mathrm{Range}(R)$ *and* $\mathrm{Range}(T)$ *or* $\mathrm{Range}(R)$ *is complete. If there exists a $\lambda \in [0,1)$ such that for all $x, y \in X$*

$$d(Tx, Ty) \leq \lambda d(Rx, Ry),$$

then R and T have a coincidence point in X. Moreover, if either T or R is injective, then R and T have a unique coincidence point in X.

Proof Consider the mapping $F(t) = \ln(t)$, for $t > 0$. Then obviously F satisfies (F_1)-(F_3). From Theorem 6, we obtain the desired conclusion. □

Remark 10 If in the above corollary we choose $X = Y$ and $R = I$ (the identity mapping on X), we obtain the Banach contraction theorem.

In this way, we recall the concept of F-contractions for multivalued mappings and proved Suzuki-type fixed point theorem for such contractions. Nadler [10] invented the concept of a Hausdorff metric H induced by metric d on X as follows:

$$H(A, B) = \max\Bigl\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\Bigr\}$$

for every $A, B \in CB(X)$. He extended the Banach contraction principle to multivalued mappings. Since then many authors have studied fixed points for multivalued mappings. Very recently, Sgroi and Vetro extended the concept of the F-contraction for multivalued mappings (see also [21]).

Theorem 11 *Let (X, d) be a metric space and let $T : X \to CB(X)$. Assume that there exist a function $F \in \mathcal{F}$ that is continuous from the right and $\tau \in \mathbb{R}^+$ such that*

$$\frac{1}{2} d(x, Tx) \leq d(x, y) \quad \Longrightarrow \quad 2\tau + F\bigl(H(Tx, Ty)\bigr) \leq F\bigl(d(x, y)\bigr) \tag{2.16}$$

for all $x, y \in X$. Then T has a fixed point.

Proof Let $x_0 \in X$ be an arbitrary point of X and choose $x_1 \in Tx_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T and the proof is completed. Assume that $x_1 \notin Tx_1$, then $Tx_0 \neq Tx_1$. Now

$$\frac{1}{2} d(x_0, Tx_0) \leq \frac{1}{2} d(x_0, x_1) < d(x_0, x_1).$$

From the assumption, we have

$$2\tau + F\big(H(Tx_0, Tx_1)\big) \leq F\big(d(x_0, x_1)\big).$$

Since F is continuous from the right, there exists a real number $h > 1$ such that

$$F\big(hH(Tx_0, Tx_1)\big) \leq F\big(H(Tx_0, Tx_1)\big) + \tau.$$

Now, from

$$d(x_1, Tx_1) \leq H(Tx_0, Tx_1) < hH(Tx_0, Tx_1),$$

we deduce that there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leq hH(Tx_0, Tx_1).$$

Consequently, we get

$$F\big(d(x_1, x_2)\big) \leq F\big(hH(Tx_0, Tx_1)\big) < F\big(H(Tx_0, Tx_1)\big) + \tau,$$

which implies that

$$\begin{aligned} 2\tau + F\big(d(x_1, x_2)\big) &\leq 2\tau + F\big(H(Tx_0, Tx_1)\big) + \tau \\ &\leq F\big(d(x_0, x_1)\big) + \tau. \end{aligned}$$

Thus

$$\tau + F\big(d(x_1, x_2)\big) \leq F\big(d(x_0, x_1)\big).$$

Continuing in this manner, we can define a sequence $\{x_n\} \subset X$ such that $x_n \notin Tx_n$, $x_{n+1} \in Tx_n$ and

$$\tau + F\big(d(x_n, x_{n+1})\big) \leq F\big(d(x_{n-1}, x_n)\big)$$

for all $n \in \mathbb{N} \cup \{0\}$. Therefore

$$\begin{aligned} F\big(d(x_n, x_{n+1})\big) &\leq F\big(d(x_{n-1}, x_n)\big) - \tau \leq F\big(d(x_{n-2}, x_{n-1})\big) - 2\tau \leq \cdots \\ &\leq F\big(d(x_0, x_1)\big) - n\tau \end{aligned} \tag{2.17}$$

for all $n \in \mathbb{N}$. Since $F \in \mathcal{F}$, by taking the limit as $n \to \infty$ in (2.17) we have

$$\lim_{n\to\infty} F\big(d(x_n, x_{n+1})\big) = -\infty \quad \Longleftrightarrow \quad \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.18}$$

Now from (F_3), there exists $0 < k < 1$ such that

$$\lim_{n\to\infty} \big[d(x_n, x_{n+1})\big]^k F\big(d(x_n, x_{n+1})\big) = 0. \tag{2.19}$$

By (2.17), we have

$$\begin{aligned} & d(x_n, x_{n+1})^k F\big(d(x_n, x_{n+1})\big) - d(x_n, x_{n+1})^k F\big(d(x_0, x_1)\big) \\ & \quad \leq d(x_n, x_{n+1})^k \big[F\big(d(x_0, x_1) - n\tau\big) - F\big(d(x_0, x_1)\big)\big] \\ & \quad = -n\tau \big[d(x_n, x_{n+1})\big]^k \leq 0. \end{aligned} \tag{2.20}$$

By taking the limit as $n \to \infty$ in (2.20) and applying (2.18) and (2.19), we have

$$\lim_{n\to\infty} n\big[d(x_n, x_{n+1})\big]^k = 0. \tag{2.21}$$

It follows from (2.21) that there exists $n_1 \in \mathbb{N}$ such that

$$n\big[d(x_n, x_{n+1})\big]^k \leq 1 \tag{2.22}$$

for all $n > n_1$. This implies

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/k}} \tag{2.23}$$

for all $n > n_1$. Now we prove that $\{x_n\}$ is a Cauchy sequence. For $m > n > n_1$ we have

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}. \tag{2.24}$$

Since $0 < k < 1$, $\sum_{i=1}^{\infty} \frac{1}{i^{1/k}}$ converges. Therefore, $d(x_n, x_m) \to 0$ as $m, n \to \infty$. Thus $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, there exists $z \in X$ such that such that $x_n \to z$ as $n \to +\infty$. Now, we prove that z is a fixed point of T. If there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that $x_{n_k} \in Tz$ for all $k \in \mathbb{N}$. Since Tz is closed and $x_n \to z$ as $n \to +\infty$, we get $z \in Tz$ and the proof is completed. So we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} \notin Tz$ for all $n \in \mathbb{N}$ with $n \geq n_0$. This implies that $Tx_{n-1} \neq Tz$ for all $n \geq n_0$. We first show that

$$d(z, Tx) \leq d(z, x)$$

for all $x \in X \backslash \{z\}$. Since $x_n \to z$, there exists $n_0 \in \mathbb{N}$ such that

$$d(z, x_n) \leq \frac{1}{3} d(z, x)$$

for all $n \in \mathbb{N}$ with $n \geq n_0$. Then we have

$$\begin{aligned} \frac{1}{2} d(x_n, Tx_n) < d(x_n, Tx_n) & \leq d(x_n, x_{n+1}) \\ & \leq d(x_n, z) + d(z, x_{n+1}) \\ & \leq \frac{2}{3} d(x, z) = d(x, z) - \frac{1}{3} d(x, z) \\ & \leq d(x, z) - d(z, x_n) \leq d(x, x_n). \end{aligned}$$

Thus, by assumption, we get

$$2\tau + F\bigl(H(Tx_n, Tx)\bigr) \le F\bigl(d(x_n, x)\bigr). \tag{2.25}$$

Since F is continuous from the right, there exists a real number $h > 1$ such that

$$F\bigl(hH(Tx_n, Tx)\bigr) < F\bigl(H(Tx_n, Tx)\bigr) + \tau.$$

Now, from

$$d(x_{n+1}, Tx) \le H(Tx_n, Tx) < hH(Tx_n, Tx),$$

we obtain

$$F\bigl(d(x_{n+1}, Tx)\bigr) \le F\bigl(hH(Tx_n, Tx)\bigr) < F\bigl(H(Tx_n, Tx)\bigr) + \tau.$$

Thus we have

$$\begin{aligned} 2\tau + F\bigl(d(x_{n+1}, Tx)\bigr) &\le 2\tau + F\bigl(H(Tx_n, Tx)\bigr) + \tau \\ &\le F\bigl(d(x_n, x)\bigr) + \tau. \end{aligned}$$

Since F is strictly increasing, we have

$$d(x_{n+1}, Tx) < d(x_n, x).$$

Letting n tend to $+\infty$, we obtain

$$d(z, Tx) \le d(z, x)$$

for all $x \in X \setminus \{z\}$. We next prove that

$$2\tau + F\bigl(H(Tz, Tx)\bigr) \le F\bigl(d(z, x)\bigr)$$

for all $x \in X$. Since $F \in \mathcal{F}$, we take $x \neq z$. Then for every $n \in \mathbb{N}$, there exists $y_n \in Tx$ such that

$$d(z, y_n) \le d(z, Tx) + \frac{1}{n} d(z, x).$$

So we have

$$\begin{aligned} d(x, Tx) &\le d(x, y_n) \\ &\le d(x, z) + d(z, y_n) \\ &\le d(x, z) + d(z, Tx) + \frac{1}{n} d(z, x) \\ &\le d(x, z) + d(x, z) + \frac{1}{n} d(z, x) \\ &= \left(2 + \frac{1}{n}\right) d(x, z) \end{aligned}$$

for all $n \in \mathbb{N}$ and hence $\frac{1}{2}d(x, Tx) \leq d(x, z)$. Thus by assumption, we get

$$2\tau + F\big(H(Tz, Tx)\big) \leq F\big(d(z, x)\big).$$

Thus

$$\begin{aligned} 2\tau + F\big(d(x_{n+1}, Tz)\big) &\leq 2\tau + F\big(H(Tx_n, Tz)\big) \\ &\leq F\big(d(x_n, z)\big). \end{aligned}$$

Since F is strictly increasing, we have $d(x_{n+1}, Tz) < d(x_n, z)$. Letting $n \to \infty$, we get $d(z, Tz) \leq 0$. Since Tz is closed, we obtain $z \in Tz$. Thus z is fixed point of T. □

3 Applications

Fixed point theorems for contractive operators in metric spaces are widely investigated and have found various applications in differential and integral equations (see [9, 15, 22, 23] and references therein). In this section we discuss the existence and uniqueness of solution of a general class of the following Volterra type integral equations under various assumptions on the functions involved. Let $C[0, \Theta]$ denote the space of all continuous functions on $[0, \Theta]$, where $\Theta > 0$ and for an arbitrary $\|x\|_\tau = \sup_{t \in [0,\Theta]}\{|x(t)|e^{-\tau t}\}$, where $\tau > 0$ is taken arbitrary. Note that $\|\cdot\|_\tau$ is a norm equivalent to the supremum norm, and $(C([0, \Theta], \mathbb{R}), \|\cdot\|_\tau)$ endowed with the metric d_τ defined by

$$d_\tau(x, y) = \sup_{t \in [0,\Theta]} \big\{\big|x(t) - y(t)\big|e^{-\tau t}\big\}$$

for all $x, y \in C([0, \Theta], \mathbb{R})$ is a Banach space; see also [19].

Consider the integral equation

$$(fy)(t) = \int_0^t K\big(t, s, hx(s)\big)\, ds + g(t), \tag{3.1}$$

where $x : [0, \Theta] \to \mathbb{R}$ is unknown, $g : [0, \Theta] \to \mathbb{R}$, and $h, f : \mathbb{R} \to \mathbb{R}$ are given functions. The kernel K of the integral equation is defined on $[0, \Theta] \times [0, \Theta]$.

Theorem 12 *Assume that the following conditions are satisfied*:

(i) $K : [0, \Theta] \times [0, \Theta] \times \mathbb{R} \to \mathbb{R}$, $g : [0, \Theta] \to \mathbb{R}$ *and* $f : \mathbb{R} \to \mathbb{R}$ *are continuous*,

(ii) $\int_0^t K(t, s, \cdot) : \mathbb{R} \to \mathbb{R}$ *is increasing*, *for all* $t, s \in [0, \Theta]$,

(iii) *there exists* $\tau \in (0, +\infty)$ *such that*

$$\big|K\big(t, s, hx(s)\big) - K\big(t, s, hy(s)\big)\big| \leq \tau\big|hx(s) - hy(s)\big|$$

for all $t, s \in [0, \Theta]$ *and* $hx, hy \in \mathbb{R}$,

(iv) *if* f *is injective*, *then for* $\tau > 0$ *there exists* $e^{-\tau} \in \mathbb{R}^+$ *such that for all* $x, y \in \mathbb{R}$;

$$|hx - hy| \leq e^{-\tau}|fx - fy|$$

and $\{fx : x \in C([0,\Theta],\mathbb{R})\}$ *is complete. Then there exists* $w \in C([0,\Theta],\mathbb{R})$ *such that for* $x_0 \in C([0,\Theta],\mathbb{R})$ *and* $x_n(t) = fx_{n-1}(t)$

$$fw(t) = \lim_{n\to\infty} fx_n(t) = \lim_{n\to\infty}\left[g(t) + \int_0^t K\big(t,s,hx_{n-1}(s)\big)\,ds\right]$$

and w *is the unique solution of* (3.1).

Proof Let $X = Y = C([0,\Theta],\mathbb{R})$ and

$$d_\tau(x,y) = \sup_{t\in[0,\Theta]}\left\{\left|x(t) - y(t)\right|e^{-\tau t}\right\}$$

for all $x, y \in X$. Let $T, R : X \to X$ be defined as follows:

$$(Tx)(t) = g(t) + \int_0^t K\big(t,s,hx(s)\big)\,ds \quad \text{and} \quad Rx = fx.$$

Then by assumptions $RX = \{Rx : x \in X\}$ is complete. Let $x^* \in TX$, then $x^* = Tx$ for $x \in X$ and $x^*(t) = Tx(t)$. By the assumptions there exists $y \in X$ such that $Tx(t) = fy(t)$, hence $RX \subseteq TX$. Since

$$\begin{aligned}
\left|(Tx)(t) - (Ty)(t)\right| &= \left|\int_0^t \left[K\big(t,s,hx(s)\big)\right]ds - \int_0^t \left[K\big(t,s,hy(s)\big)\right]ds\right| \\
&\le \int_0^t \left|K\big(t,s,hx(s)\big) - K\big(t,s,hy(s)\big)\right|ds \\
&\le \int_0^t \tau\left|hx(s) - hy(s)\right|ds \\
&\le \int_0^t \tau e^{-\tau}\left|fx(s) - fy(s)\right|ds \\
&= \int_0^t \tau e^{-\tau}\left|(Rx)(s) - (Ry)(s)\right|e^{-\tau s}e^{\tau s}\,ds \\
&\le \tau e^{-\tau}\|Rx - Ry\|_\tau \int_0^t e^{\tau s}\,ds \\
&\le \tau e^{-\tau}\|Rx - Ry\|_\tau \frac{e^{\tau t}}{\tau} \\
&= e^{-\tau}\|Rx - Ry\|_\tau e^{\tau t}.
\end{aligned}$$

This implies that

$$\left|(Tx)(t) - (Ty)(t)\right|e^{\tau t} \le e^{-\tau}\|Rx - Ry\|_\tau,$$

or equivalently

$$d_\tau(Tx, Ty) \le e^{-\tau} d_\tau(Rx, Ry).$$

Taking logarithms, we have

$$\ln\big(d_\tau(Tx, Ty)\big) \le \ln\big(e^{-\tau} d_\tau(Rx, Ry)\big).$$

After routine calculations, one can easily get

$$\tau + \ln\bigl(d_\tau(Tx, Ty)\bigr) \le \ln\bigl(d_\tau(Rx, Ry)\bigr).$$

Now, we observe that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(t) = \ln(t)$ for each $t \in C([0,\Theta],\mathbb{R})$ and $\tau > 0$ is in $\mathcal{F}$. Thus all conditions of Theorem 6 are satisfied. Hence, there exists a unique $w \in X$ such that

$$fw(t) = \lim_{n\to\infty} Rx_n(t) = \lim_{n\to\infty} Tx_{n-1}(t) = T(w)(t), \quad x_0 \in X,$$

for all t, which is the unique solution of (3.1). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final paper.

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