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Some fixed point theorems in Menger PbM-spaces with an application

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Abstract
In this paper, we establish the structure of Menger PbM-spaces as a generalization of Menger PM-spaces. We present some fixed point theorems for a new class of contractive mappings in the framework of Menger PbM-spaces. We also provide examples to illustrate the results presented herein. Then we utilize our main result to obtain the existence and uniqueness of a solution for a Volterra type integral equation.

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1 Introduction and preliminaries

The concept of a Menger probabilistic metric space (briefly, Menger PM-space) was initiated by Menger [1]. The idea of Menger was to use a distribution function instead of a nonnegative number for the value of a metric. The notion of a probabilistic metric space corresponds to the situation when we do not know exactly the distance between two points. Thus, one thinks of the distance between two points x and y as being probabilistic with $F_{x,y}(t)$ representing the probability that the distance between x and y is less than t.

In 1972, Sehgal and Bharucha-Reid [2] obtained a generalization of the Banach contraction principle on a complete Menger PM-space, which is a milestone in developing fixed point theory in a Menger PM-space. After that, Schweizer and Sklar [3] studied the properties of Menger PM-spaces and gave some basic results on these spaces.

In recent times, the study on existence of fixed points for mappings satisfying generalized contractive type conditions in Menger PM-spaces has attracted much attention (see [4–7]). This study was initiated by Ćirić in [8]; more details in [9]. Also a nice overview of this research can be found in the book of Hadžić and Pap [10].

On the other hand, the notion of a b-metric space was studied by Czerwik [11, 12] and many fixed point results were obtained for single and multivalued mappings by Czerwik and many other authors (see [13–16] and references cited therein).

In this paper, motivated by [5, 11], we establish the structure of Menger PbM-spaces and obtain fixed point results for classes of mappings that extend the notion of generalized β-type contractive mappings introduced by Gopal *et al.* [5] in Menger PbM-spaces. We also give some examples to show that our fixed point theorems for the new type of contractive mappings are independent. Then we use our main results to obtain the existence and uniqueness of a solution for a Volterra type integral equation.

In the following, we provide some notations, definitions and auxiliary facts will be used later in this paper. Throughout this paper, $\mathbb{R}^+$ denotes the set of nonnegative real numbers.

Definition 1.1 [12, 17] Let X be a nonempty set, and let the functional $d : X \times X \to [0, \infty)$ satisfy:

(b1) $d(x, y) = 0$ if and only if $x = y$,

(b2) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(b3) there exists a real number $s \geq 1$ such that $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

Then d is called a b-metric on X and a pair (X, d) is called a b-metric space with coefficient s.

Definition 1.2 [18] Let (X, d) be a b-metric space. Then a sequence $\{x_n\}$ in X is called:

(i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$; in this case, we write $\lim_{n\to\infty} x_n = x$;

(ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$. The b-metric space (X, d) is complete if every Cauchy sequence in X converges in X.

Remark 1.3 [18] Notice that in a b-metric space (X, d) the following assertions hold:

(i) a convergent sequence has a unique limit;

(ii) each convergent sequence is Cauchy;

(iii) $(X, \underset{\to}{d})$ is an L-space (see [19, 20]);

(iv) in general, a b-metric is not continuous;

(v) in general, a b-metric does not induce a topology on X.

Example 1.4 [16] Let $X = [0, \infty)$ and define $d : X \times X \to [0, \infty)$ as

$$d(x, y) = |x - y|^2 \quad \text{for all } x, y \in X.$$

Then (X, d) is a complete b-metric space with coefficient $s = 2 > 1$, but it is not a usual metric space.

Definition 1.5 [21] Let (X, d) and (X', d') be two b-metric spaces with coefficient s and s', respectively. A mapping $T : X \to X'$ is called continuous if for each sequence $\{x_n\}$ in X, which converges to $x \in X$ with respect to d, then $\{Tx_n\}$ converges to Tx with respect to d'.

We recall the following definitions in the class of Menger PM-spaces.

Definition 1.6 [5] A binary operation $T : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous t-norm if the following conditions hold:

(i) T is commutative and associative,

(ii) T is continuous,

(iii) $T(a, 1) = a$ for all $a \in [0, 1]$,

(iv) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0, 1]$.

The following are three basic continuous t-norms:

(1) The minimum t-norm, say T_M, defined by $T_M(a,b) = \min\{a,b\}$.

(2) The product t-norm, say T_P, defined by $T_P(a,b) = ab$.

(3) The Lukasiewicz t-norm, say T_L, defined by $T_L(a,b) = \max\{a+b-1,0\}$.

These t-norms are related in the following way: $T_L \leq T_P \leq T_M$.

Definition 1.7 [6] A function $F : (-\infty,+\infty) \to [0,1]$ is called a distribution function if it is non-decreasing and left-continuous with $\lim_{t\to-\infty} F(t) = 0$. If in addition $F(0) = 0$, then F is called a distance distribution function.

Definition 1.8 [6] A distance distribution function F satisfying $\lim_{t\to+\infty} F(t) = 1$ is called a Menger distance distribution function. The set of all Menger distance distribution functions is denoted by $\mathcal{D}^+$. A special Menger distance distribution function is given by

$$\mathcal{H}(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0. \end{cases}$$

Definition 1.9 [5] A Menger probabilistic metric space (briefly, Menger PM-space) is a triple (X,F,T) where X is a nonempty set, T is a continuous t-norm, and F is a mapping from $X \times X$ into $\mathcal{D}^+$ such that, if $F_{x,y}$ denotes the value of F at the pair (x,y), the following conditions hold:

(PM1) $F_{x,y}(t) = \mathcal{H}(t)$ if and only if $x = y$,

(PM2) $F_{x,y}(t) = F_{y,x}(t)$,

(PM3) $F_{x,y}(t+s) \geq T(F_{x,z}(t), F_{z,y}(s))$ for all $x,y,z \in X$ and $s,t \geq 0$.

Definition 1.10 [5] Let (X,F,T) be a Menger PM-space. Then:

(i) A sequence $\{x_n\}$ in X is said to be convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer N such that $F_{x_n,x}(\varepsilon) > 1-\lambda$, whenever $n \geq N$.

(ii) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer N such that $F_{x_n,x_m}(\varepsilon) > 1-\lambda$, whenever $n,m \geq N$.

(iii) A Menger PM-space is said to be complete if every Cauchy sequence in X is convergent to a point in X.

According to [3], the (ε,λ)-topology in a Menger PM-space (X,F,T) is introduced by the family of neighborhoods N_x of a point $x \in X$ given by $N_x = \{N_x(\varepsilon,\lambda) : \varepsilon > 0, \lambda \in (0,1)\}$, where $N_x(\varepsilon,\lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1-\lambda\}$.

The (ε,λ)-topology is a Hausdorff topology. In this topology a function f is continuous in $x_0 \in X$ if and only if $f(x_n) \to f(x_0)$, for every sequence $x_n \to x_0$.

Example 1.11 [22] Let (X,d) be a metric space. Define a mapping $F : X \times X \to \mathcal{D}^+$ by

$$F(x,y)(t) = F_{x,y}(t) = \mathcal{H}\big(t - d(x,y)\big), \quad \forall x,y \in X, t \in \mathbb{R}.$$

Then (X,F,T_M) is a Menger PM-space induced by (X,d). If (X,d) is complete, then (X,F,T_M) is complete.

Definition 1.12 [5] A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Φ-function if it satisfies the following conditions:

(i) $\phi(t) = 0$ if and only if $t = 0$,
(ii) $\phi(t)$ is strictly monotone increasing and $\phi(t) \to \infty$ as $t \to \infty$,
(iii) ϕ is left-continuous in $(0, \infty)$,
(iv) ϕ is continuous at 0.

In the sequel, the class of all Φ-functions will be denoted by Φ.

We conclude this section recalling the following fixed point theorem of Gopal *et al.*, see [5]. Before this, we quote some definitions.

Definition 1.13 [5] Let (X, F, T) be a Menger PM-space and $f : X \to X$ be a given mapping. We say that f is a generalized β-type contractive mapping if there exists a function $\beta : X \times X \times (0, \infty) \to (0, \infty)$ such that

$$\begin{aligned}\beta(x,y,t)F_{fx,fy}\bigl(\phi(t)\bigr) \geq \min\biggl\{&F_{x,y}\biggl(\phi\biggl(\frac{t}{c}\biggr)\biggr), F_{x,fx}\biggl(\phi\biggl(\frac{t}{c}\biggr)\biggr), F_{y,fy}\biggl(\phi\biggl(\frac{t}{c}\biggr)\biggr),\\ &F_{x,fy}\biggl(2\phi\biggl(\frac{t}{c}\biggr)\biggr), F_{y,fx}\biggl(2\phi\biggl(\frac{t}{c}\biggr)\biggr)\biggr\},\end{aligned}$$

for all $x, y \in X$ and for all $t > 0$, where $\phi \in \Phi$ and $c \in (0, 1)$.

Definition 1.14 [5] Let (X, F, T) be a Menger PM-space, $f : X \to X$ be a given mapping and $\beta : X \times X \times (0, \infty) \to (0, \infty)$ be a function. We say that f is β-admissible if

$$x, y \in X, \text{for all } t > 0, \quad \beta(x, y, t) \leq 1 \quad \Rightarrow \quad \beta(fx, fy, t) \leq 1.$$

Theorem 1.15 [5] *Let (X, F, T) be a complete Menger PM-space with continuous t-norm T which satisfies $T(a, a) \geq a$ with $a \in [0, 1]$. Let $f : X \to X$ be a generalized β-type contractive mapping satisfying the following conditions*:

(i) *f is β-admissible,*
(ii) *there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$ for all $t > 0$,*
(iii) *if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$.*

Then f has a fixed point.

We denote by $\mathrm{Fix}(f)$ the set of fixed points of f. Consider the following condition:

(J) For all $u, v \in \mathrm{Fix}(f)$ and for all $t > 0$ there exists $z \in X$ such that $\beta(z, fz, t) \leq 1$ with $\beta(u, z, t) \leq 1$ and $\beta(v, z, t) \leq 1$.

Theorem 1.16 [5] *Adding condition* (J) *to the hypotheses of Theorem* 1.15, *we find that f has a unique fixed point.*

2 Main results

In this section, we introduce the notion of a Menger PbM-space and describe some of its properties.

Definition 2.1 A Menger probabilistic b-metric space (briefly, Menger PbM-space) with coefficient α is a triple (X,F,T) where X is a nonempty set, T is a continuous t-norm, F is a mapping from $X \times X$ into $\mathcal{D}^+$ (for $x,y \in X$, we denote $F(x,y)$ by $F_{x,y}$), and α is a real number in $(0,1]$ such that the following conditions hold:

(PbM1) $F_{x,y}(t) = \mathcal{H}(t)$ for all $t \in \mathbb{R}$, if and only if $x = y$,

(PbM2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x,y \in X$ and $t \in \mathbb{R}$,

(PbM3) $F_{x,y}(t+s) \geq T(F_{x,z}(\alpha t), F_{z,y}(\alpha s))$ for all $x,y,z \in X$, and $t,s \geq 0$.

It should be noted that the class of Menger PbM-spaces is larger than the class of Menger PM-spaces, since a Menger PbM-space is a Menger PM-space when $\alpha = 1$.

Definition 2.2 Let (X,F,T) be a Menger PbM-space. Then a sequence $\{x_n\}$ in X is called:

(i) convergent to x in X (often denoted by $x_n \to x$) if for any given $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer $N = N(\varepsilon,\lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ whenever $n \geq N$, which is equivalent to $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ for all $t > 0$;

(ii) Cauchy if for any given $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists a positive integer $N = N(\varepsilon,\lambda)$ such that $F_{x_n,x_m}(\varepsilon) > 1 - \lambda$ whenever $n,m \geq N$.

The Menger PbM-space (X,F,T) is said to be complete if every Cauchy sequence in X is convergent in X.

Remark 2.3 In a Menger PbM-space (X,F,T) the following assertions hold:

(i) a convergent sequence has a unique limit,

(ii) in general, a Menger PbM-space is not a topological space.

In the following we present examples which show that introducing a Menger PbM-space instead of Menger PM-space is meaningful.

Example 2.4 Let $X = \mathbb{R}^+$. Define $F : X \times X \to \mathcal{D}^+$ by

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|^2}, & \text{if } t > 0, \\ 0, & \text{if } t \leq 0, \end{cases}$$

for all $x,y \in X$. It is easy to show that (X,F,T_M) is a complete Menger PbM-space with $\alpha = \frac{1}{2}$. However, it is not a Menger PM-space. We show that (PM3) does not hold. To prove this, let $x = 3$, $y = 1$, $z = 2$, $t_1 = 1$, and $t_2 = 2$. Then $F_{x,y}(t_1 + t_2) = \frac{3}{7}$, $F_{x,z}(t_1) = \frac{1}{2}$, and $F_{z,y}(t_2) = \frac{2}{3}$, hence $F_{x,y}(t_1 + t_2) = \frac{3}{7} \ngeq \frac{1}{2} = T_M(F_{x,z}(t_1), F_{z,y}(t_2))$.

Example 2.5 Let (X,d) be a b-metric space with coefficient $s \geq 1$. Define a mapping $F : X \times X \to \mathcal{D}^+$ as in Example 1.11. Then (X,F,T_M) is a Menger PbM-space with $\alpha = \frac{1}{s}$. We know that $t_1 + t_2 - d(x,y) \geq t_1 - sd(x,z) + t_2 - sd(z,y)$, for each $x,y,z \in X$, $t_1,t_2 \in \mathbb{R}$, and hence by the properties of $\mathcal{H}$, we get

$$\begin{aligned} \mathcal{H}\bigl(t_1 + t_2 - d(x,y)\bigr) &\geq \mathcal{H}\bigl(t_1 - sd(x,z) + t_2 - sd(z,y)\bigr) \\ &\geq \min\bigl\{\mathcal{H}\bigl(t_1 - sd(x,z)\bigr), \mathcal{H}\bigl(t_2 - sd(z,y)\bigr)\bigr\} \\ &= \min\left\{\mathcal{H}\left(\frac{t_1}{s} - d(x,z)\right), \mathcal{H}\left(\frac{t_2}{s} - d(z,y)\right)\right\}. \end{aligned}$$

It gives (PbM3). Furthermore, a straightforward computation shows that if (X,d) is complete, then (X,F,T_M) is complete.

Now assume that (X,d) is as in Example 1.4. Then by the above comment, (X,F,T_M) is a complete Menger PbM-space with $\alpha = \frac{1}{2}$. We claim that (X,F,T_M) is not a Menger PM-space. Indeed, (PM3) does not hold. To see this, let $x = 1$, $y = 0$, $z = \frac{1}{3}$, $t_1 = \frac{1}{2}$, and $t_2 = \frac{1}{3}$. Then $F_{x,y}(t_1 + t_2) = 0$ and $F_{x,z}(t_1) = F_{z,y}(t_2) = 1$, hence $F_{x,y}(t_1 + t_2) = 0 \ngeq 1 = T_M(F_{x,z}(t_1), F_{z,y}(t_2))$.

Note that the above examples are Menger PbM-spaces (but are not Menger PM-spaces) if T_M substitutes with T_P or T_L.

The following result is used in our next considerations. It is a generalization of [4], Lemma 2.5 in Menger PbM-spaces.

Lemma 2.6 *Let (X,F,T) be a Menger PbM-space with coefficient α. Then the function F is a lower semi-continuous function of points, i.e., for every fixed $t > 0$ and every two convergent sequences $\{x_n\}$, $\{y_n\}$ in X such that $x_n \to x$ and $y_n \to y$ it follows that* $\lim_{n\to\infty} \inf F_{x_n,y_n}(t) = F_{x,y}(t)$.

Proof Let $t > 0$ and $\varepsilon > 0$ be given. Since $F_{x,y}$ is left-continuous at t so there exists δ_1 such that $0 < \delta_1 < t$ and $F_{x,y}(t) - F_{x,y}(t - \delta_1) < \varepsilon$. Suppose h is an arbitrary fixed real number with $0 < 2h < \delta_1$, then $F_{x,y}(t) - F_{x,y}(t - 2h) < \varepsilon$. Using again left-continuity of $F_{x,y}$ at $t - \delta_1$, there exists $\delta_2 > 0$ such that $F_{x,y}(t - \delta_1) - F_{x,y}(t - \delta_2) < \varepsilon$. By repeating this argument we can find $k \in \mathbb{N}$, $\delta_i, \delta_{i+1} > 0$ $(i = 1, \ldots, k)$ in which $F_{x,y}(t - \delta_i) - F_{x,y}(t - \delta_{i+1}) < \varepsilon$ and $\alpha^2 t - 2\alpha^2 h \in (t - \delta_k, t - \delta_{k+1})$. We deduce that

$$\begin{aligned} F_{x,y}(t) - F_{x,y}\big(\alpha^2 t - 2\alpha^2 h\big) &= \big(F_{x,y}(t) - F_{x,y}(t - \delta_1)\big) + \big(F_{x,y}(t - \delta_1) - F_{x,y}(t - \delta_2)\big) + \cdots \\ &\quad + \big(F_{x,y}(t - \delta_k) - F_{x,y}\big(\alpha^2 t - 2\alpha^2 h\big)\big) \\ &< (k+1)\varepsilon. \end{aligned} \tag{1}$$

Set $F_{x,y}(\alpha^2 t - 2\alpha^2 h) = a$. Taking into account continuity of T and $T(a,1) = a$, there is a real number l in $(0,1)$, fulfills

$$T(a,l) > a - \frac{\varepsilon}{3} \quad \text{and} \quad T\left(a - \frac{\varepsilon}{3}, l\right) > a - \frac{2\varepsilon}{3}. \tag{2}$$

On the other hand, since $x_n \to x$ and $y_n \to y$, there exists an integer $M_{h,l}$ such that

$$F_{x_n,x}\big(\alpha^2 h\big) > l \quad \text{and} \quad F_{y_n,y}(\alpha h) > l, \tag{3}$$

whenever $n > M_{h,l}$. Now, by (PbM3)

$$F_{x_n,y_n}(t) \geq T\big(F_{x_n,y}(\alpha t - \alpha h), F_{y,y_n}(\alpha h)\big) \tag{4}$$

and

$$F_{x_n,y}(\alpha t - \alpha h) \geq T\big(F_{x_n,x}\big(\alpha^2 h\big), F_{x,y}\big(\alpha^2 t - 2\alpha^2 h\big)\big). \tag{5}$$

From (2), (3), and (5), we obtain

$$F_{x_n,y}(\alpha t - \alpha h) \geq T(a,l) > a - \frac{\varepsilon}{3}. \tag{6}$$

Thus, on combining (1), (2), (3), (4), and (6), we get

$$F_{x_n,y_n}(t) \geq T\left(a - \frac{\varepsilon}{3}, l\right) > a - \frac{2\varepsilon}{3} > F_{x,y}(t) - \frac{(3k+5)\varepsilon}{3}.$$

This completes the proof. □

3 Generalized β-γ-type contractive mappings

In this section, we generalize the results obtained by Gopal *et al.* [5] for the wider class of generalized β-γ-type contractive mappings in Menger PbM-spaces.

Definition 3.1 Let (X,F,T) be a Menger PbM-space with coefficient α and $f: X \to X$ be a given mapping. We say that f is a generalized β-γ-type contractive mapping of degree k ($k \in \mathbb{N}$), if there exist two functions $\beta: X \times X \times (0,\infty) \to (0,\infty)$ and $\gamma: X \times X \times (0,\infty) \to (0,\infty)$ such that

$$\begin{aligned}\beta\left(x,y,\alpha^k t\right) F_{fx,fy}\left(\alpha^k \phi(t)\right) \geq \gamma\left(fx,fy,\alpha^{k-1}\frac{t}{c}\right) \min\Bigg\{ & F_{x,y}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right), \\ & F_{x,fx}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right), F_{y,fy}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right), \\ & F_{x,fy}\left(2\alpha^{k-2}\phi\left(\frac{t}{c}\right)\right), F_{y,fx}\left(2\alpha^{k-2}\phi\left(\frac{t}{c}\right)\right)\Bigg\},\end{aligned} \tag{7}$$

for all $x,y \in X$ and for all $t > 0$, where $\phi \in \Phi$ and $c \in (0,1)$. Further, the mapping f is called a generalized β-γ-type contractive mapping if it is a generalized β-γ-type contractive mapping of degree k for each $k \in \mathbb{N}$.

Definition 3.2 Let (X,F,T) be a Menger PbM-space, $f: X \to X$ be a mapping, and $\beta: X \times X \times (0,\infty) \to (0,\infty)$ and $\gamma: X \times X \times (0,\infty) \to (0,\infty)$ be two functions. We say that f is (β,γ)-admissible if $x,y \in X$, for all $t > 0$, $\beta(x,y,t) \leq 1 \Rightarrow \beta(fx,fy,t) \leq 1$ and $\gamma(x,y,t) \geq 1 \Rightarrow \gamma(fx,fy,t) \geq 1$.

Imitating the proof of [4], Lemma 2.9, we can easily obtain the following lemma.

Lemma 3.3 *Let (X,F,T) be a Menger PbM-space with coefficient α. Let ϕ be a Φ-function. Then the following statement holds*:

If for $x,y \in X$, $c \in (0,1)$, and $k \in \mathbb{N}$ we have $F_{x,y}(\alpha^k \phi(t)) \geq F_{x,y}(\alpha^{k-1}\phi(\frac{t}{c}))$ for all $t > 0$, then $x = y$.

Our first main result is the following.

Theorem 3.4 *Let (X,F,T) be a complete Menger PbM-space with coefficient α, which satisfies $T(a,a) \geq a$ with $a \in [0,1]$. Let $f: X \to X$ be a generalized β-γ-type contractive mapping satisfying the following conditions*:

(i) *f is (β,γ)-admissible*,

(ii) *there exists* $x_0 \in X$ *such that* $\beta(x_0, fx_0, t) \le 1$ *and* $\gamma(x_0, fx_0, t) \ge 1$ *for all* $t > 0$,

(iii) *if* $\{x_n\}$ *is a sequence in* X *such that* $\beta(x_{n-1}, x_n, t) \le 1$ *and* $\gamma(x_n, x_{n+1}, t) \ge 1$ *for all* $n \in \mathbb{N}$, *and for all* $t > 0$ *and* $x_n \to x$ *as* $n \to \infty$, *then* $\beta(x_{n-1}, x, t) \le 1$ *and* $\gamma(x_n, fx, t) \ge 1$ *for all* $n \in \mathbb{N}$ *and for all* $t > 0$.

Then f *has a fixed point.*

Proof Since $T(a,a) \ge a$ for all $a \in [0,1]$, $T \ge T_M$. Let $x_0 \in X$ be such that (ii) holds and define a sequence $\{x_n\}$ in X so that $x_{n+1} = fx_n$, for all $n = 0, 1, \ldots$. We suppose $x_{n+1} \neq x_n$ for all $n = 0, 1, \ldots$, otherwise f has trivially a fixed point. From (i), (ii), and by induction, we get $\beta(x_{n-1}, x_n, t) \le 1$ and $\gamma(x_n, x_{n+1}, t) \ge 1$ for all $n \in \mathbb{N}$ and all $t > 0$. Taking into account the continuity of ϕ at zero, we can find $r > 0$ such that $t > \phi(r)$ and therefore we have

$$\begin{aligned}
F_{x_n,x_{n+1}}(t) &\ge \beta\big(x_{n-1}, x_n, \alpha^k r\big) F_{fx_{n-1},fx_n}\big(\alpha^k \phi(r)\big) \\
&\ge \gamma\left(x_n, x_{n+1}, \alpha^{k-1}\frac{r}{c}\right) \min\left\{F_{x_{n-1},fx_{n-1}}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{x_{n-1},x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right),\right. \\
&\qquad \left. F_{x_n,fx_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{x_{n-1},fx_n}\left(2\alpha^{k-2}\phi\left(\frac{r}{c}\right)\right), F_{x_n,fx_{n-1}}\left(2\alpha^{k-2}\phi\left(\frac{r}{c}\right)\right)\right\} \\
&\ge \min\left\{F_{x_{n-1},x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{x_n,x_{n+1}}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right)\right\}.
\end{aligned}$$

We will show that

$$F_{x_n,x_{n+1}}\big(\alpha^k \phi(r)\big) \ge F_{x_{n-1},x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right). \tag{8}$$

If we assume that $F_{x_n,x_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c}))$ is the minimum, then from Lemma 3.3, we get $x_n = x_{n+1}$, which is a contradiction with the assumption $x_n \neq x_{n+1}$ and so $F_{x_{n-1},x_n}(\alpha^{k-1}\phi(\frac{r}{c}))$ is the minimum *i.e.*, inequality (8) holds. Now from (8), one obtains that

$$F_{x_n,x_{n+1}}\big(\alpha^k t\big) \ge F_{x_n,x_{n+1}}\big(\alpha^k \phi(r)\big) \ge F_{x_{n-1},x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right) \ge \cdots \ge F_{x_0,x_1}\left(\alpha^{k-n}\phi\left(\frac{r}{c^n}\right)\right),$$

that is,

$$F_{x_n,x_{n+1}}\big(\alpha^k t\big) \ge F_{x_0,x_1}\left(\alpha^{k-n}\phi\left(\frac{r}{c^n}\right)\right),$$

for arbitrary $n \in \mathbb{N}$. Next, let $m, n \in \mathbb{N}$ with $m > n$, then by (PbM3) and strictly increasing of ϕ we have

$$\begin{aligned}
&F_{x_n,x_m}\big((m-n)t\big) \\
&\quad \ge \min\big\{F_{x_n,x_{n+1}}(\alpha t), \ldots, F_{x_{m-1},x_{m-2}}\big(\alpha^{m-n-1}t\big), F_{x_{m-1},x_m}\big(\alpha^{m-n-1}t\big)\big\} \\
&\quad \ge \min\left\{F_{x_0,x_1}\left(\alpha^{1-n}\phi\left(\frac{r}{c^n}\right)\right), \ldots, F_{x_0,x_1}\left(\alpha^{1-n}\phi\left(\frac{r}{c^{m-2}}\right)\right), F_{x_0,x_1}\left(\alpha^{-n}\phi\left(\frac{r}{c^{m-1}}\right)\right)\right\} \\
&\quad \ge \min\left\{F_{x_0,x_1}\left(\alpha^{1-n}\phi\left(\frac{r}{c^n}\right)\right), \ldots, F_{x_0,x_1}\left(\alpha^{1-n}\phi\left(\frac{r}{c^{m-2}}\right)\right), F_{x_0,x_1}\left(\alpha^{1-n}\phi\left(\frac{r}{c^{m-1}}\right)\right)\right\} \\
&\quad = F_{x_0,x_1}\left(\alpha^{1-n}\phi\left(\frac{r}{c^n}\right)\right).
\end{aligned}$$

Since $\alpha^{1-n}\phi(\frac{r}{c^n}) \to \infty$ as $n \to \infty$, for fixed $\varepsilon \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $F_{x_0,x_1}(\alpha^{1-n}\phi(\frac{r}{c^n})) > 1-\varepsilon$, whenever $n \geq n_0$. This implies that, for every $m > n \geq n_0$,

$$F_{x_n,x_m}\big((m-n)t\big) > 1-\varepsilon.$$

Since $t > 0$ and $\varepsilon \in (0,1)$ are arbitrary, we deduce that $\{x_n\}$ is a Cauchy sequence in the complete Menger PbM-space (X,F,T). Then there exists $u \in X$ such that $x_n \to u$ as $n \to \infty$. We are going to show that u is a fixed point of f. Using (PbM3), we have

$$\begin{aligned} F_{fu,u}(t) &\geq T\big(F_{fu,x_n}\big(\alpha\phi(r)\big), F_{x_n,u}\big(\alpha t - \alpha\phi(r)\big)\big) \\ &\geq \min\big\{F_{fu,x_n}\big(\alpha\phi(r)\big), F_{x_n,u}\big(\alpha t - \alpha\phi(r)\big)\big\}. \end{aligned}$$

Note that, if $x_n = fu$ for infinitely many values of n, then $u = fu$, and hence the proof is finished. Therefore, we assume that $x_n \neq fu$ for all $n \in \mathbb{N}$. Now, since $x_n \to u$, then, for any arbitrary $\varepsilon \in (0,1)$ and n large enough, we get $F_{x_n,u}(\alpha t - \alpha\phi(r)) > 1-\varepsilon$. Hence, $F_{fu,u}(t) \geq \min\{F_{fu,x_n}(\alpha\phi(r)), 1-\varepsilon\}$. Since $\varepsilon > 0$ is arbitrary, we have $F_{fu,u}(t) \geq F_{fu,x_n}(\alpha\phi(r))$. Next, using (iii) we get

$$\begin{aligned} F_{u,fu}(t) &\geq F_{x_n,fu}\big(\alpha\phi(r)\big) \\ &= F_{fx_{n-1},fu}\big(\alpha\phi(r)\big) \\ &\geq \beta(x_{n-1},u,\alpha r)F_{fx_{n-1},fu}\big(\alpha\phi(r)\big) \\ &\geq \gamma\left(fx_{n-1},fu,\frac{r}{c}\right)\min\left\{F_{x_{n-1},u}\left(\phi\left(\frac{r}{c}\right)\right), F_{x_{n-1},x_n}\left(\phi\left(\frac{r}{c}\right)\right),\right. \\ &\qquad \left. F_{u,fu}\left(\phi\left(\frac{r}{c}\right)\right), F_{x_{n-1},fu}\left(\frac{2}{\alpha}\phi\left(\frac{r}{c}\right)\right), F_{u,x_n}\left(\frac{2}{\alpha}\phi\left(\frac{r}{c}\right)\right)\right\} \\ &\geq \min\left\{F_{x_{n-1},u}\left(\phi\left(\frac{r}{c}\right)\right), F_{u,fu}\left(\phi\left(\frac{r}{c}\right)\right), F_{x_{n-1},x_n}\left(\phi\left(\frac{r}{c}\right)\right)\right\}. \end{aligned}$$

It follows that

$$\begin{aligned} F_{u,fu}(t) &\geq \lim_{n\to\infty}\inf F_{x_n,fu}\big(\alpha\phi(r)\big) \\ &\geq \lim_{n\to\infty}\inf\min\left\{F_{x_{n-1},u}\left(\phi\left(\frac{r}{c}\right)\right), F_{u,fu}\left(\phi\left(\frac{r}{c}\right)\right), F_{x_{n-1},x_n}\left(\phi\left(\frac{r}{c}\right)\right)\right\} \\ &\geq \min\left\{1-\varepsilon, F_{u,fu}\left(\phi\left(\frac{r}{c}\right)\right), 1-\varepsilon\right\}. \end{aligned}$$

Finally, since $\varepsilon \in (0,1)$ is arbitrary, then $F_{fu,u}(\alpha\phi(r)) \geq F_{u,fu}(\phi(\frac{r}{c}))$. From Lemma 3.3, we conclude that $u = fu$ and so we achieve our goal. □

In the following we present an example of a generalized β-γ-type contractive mapping, which is not a generalized β-type contractive mapping.

Example 3.5 Let $X = [\frac{1}{4},\infty)$ and F be as in Example 2.4, then (X,F,T_M) is a complete Menger PbM-space, with $\alpha = \frac{1}{2}$. Define the mapping $f : X \to X$ and functions β and γ

from $X \times X \times (0,\infty)$ into $(0,\infty)$ as follows:

$$fx = \begin{cases} 1, & \text{if } x \in [\frac{1}{4}, 1], \\ 2, & \text{otherwise}, \end{cases}$$

$$\beta(x,y,t) = \frac{1}{2},$$

$$\gamma(x,y,t) = \begin{cases} \frac{1}{2}, & \text{if } x,y \in [\frac{1}{4},1], \text{ or } x,y \notin [\frac{1}{4},1], \\ \frac{1}{2^5}, & \text{otherwise}, \end{cases}$$

for all $t > 0$. Now, we consider $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\phi(t) = t$ and let $c = \frac{1}{2}$. To prove that f is a generalized β-γ-type contractive mapping, it suffices to check the following condition:

$$\begin{aligned} &\beta\bigl(x,y,\alpha^k t\bigr) F_{fx,fy}\bigl(\alpha^k \phi(t)\bigr) \\ &\quad \geq \gamma\left(fx,fy,\alpha^{k-1}\frac{t}{c}\right) \min\left\{F_{x,y}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right), F_{x,fx}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right),\right. \\ &\qquad \left. F_{y,fy}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right), F_{x,fy}\left(2\alpha^{k-2}\phi\left(\frac{t}{c}\right)\right), F_{y,fx}\left(2\alpha^{k-2}\phi\left(\frac{t}{c}\right)\right)\right\}. \end{aligned}$$

We distinguish three cases:

Case I. If $x,y \in [\frac{1}{4},1]$ or $x,y \notin [\frac{1}{4},1]$, then the left-hand side of above inequality is equal to $\frac{1}{2}$ and $\gamma(fx,fy,\alpha^{k-1}\frac{t}{c}) = \gamma(1,1,\frac{t}{2^{k-2}}) = \gamma(2,2,\frac{t}{2^{k-2}}) = \frac{1}{2}$. Hence, the inequality obviously true.

Case II. If $x \notin [\frac{1}{4},1]$ and $y \in [\frac{1}{4},1]$, then

$$\begin{aligned} \beta\bigl(x,y,\alpha^k t\bigr) F_{fx,fy}\bigl(\alpha^k \phi(t)\bigr) &= \frac{t}{2t + 2^{k+1}} \geq \gamma\left(fx,fy,\alpha^{k-1}\frac{t}{c}\right) F_{fx,y}\left(2\alpha^{k-2}\phi\left(\frac{t}{c}\right)\right) \\ &= \frac{t}{2^5 t + 2^{k+1}|y-2|^2} \end{aligned}$$

and hence the inequality is again true.

Case III. If $x \in [\frac{1}{4},1]$ and $y \notin [\frac{1}{4},1]$, then

$$\begin{aligned} \beta\bigl(x,y,\alpha^k t\bigr) F_{fx,fy}\bigl(\alpha^k \phi(t)\bigr) &= \frac{t}{2t + 2^{k+1}} \geq \gamma\left(fx,fy,\alpha^{k-1}\frac{t}{c}\right) F_{x,fy}\left(2\alpha^{k-2}\phi\left(\frac{t}{c}\right)\right) \\ &= \frac{t}{2^5 t + 2^{k+1}|x-2|^2} \end{aligned}$$

and hence the inequality is again true.

Also, if we take $\gamma(x,y,t) = 1$ for all $x,y \in X$ and all $t > 0$, then f is not a generalized β-type contractive mapping. Indeed, for $x = 1$, $y = 2$, and $t = 2^k$ we have

$$\frac{1}{4} \geq \min\left\{\frac{2}{2+c}, 1, 1, \frac{8}{8+c}, \frac{8}{8+c}\right\} = \frac{2}{2+c}.$$

This gives $c \geq 6$, a contradiction.

Example 3.6 Let X, F, f be as in Example 3.5. Define the functions $\beta : X \times X \times (0,\infty) \to (0,\infty)$ and $\gamma : X \times X \times (0,\infty) \to (0,\infty)$ as follows:

$$\beta(x,y,t) = \begin{cases} 1, & \text{if } x,y \in [\frac{1}{4},1], \\ \frac{4t+4}{4t+|x-y|^2}, & \text{otherwise,} \end{cases}$$

$$\gamma(x,y,t) = \begin{cases} 1, & \text{if } x = y = 1, \\ \frac{t+4}{t+4|x-y|^2}, & \text{otherwise,} \end{cases}$$

for all $t > 0$.

Now, we consider $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\phi(t) = t$ and let $c = \frac{1}{2}$. Then f is a generalized β-γ-type contractive mapping.

We distinguish three cases:

Case I. If $x,y \in [\frac{1}{4},1]$, then the left-hand side of above inequality is equal to 1 and $\gamma(fx,fy,\alpha^{k-1}\frac{t}{c}) = \gamma(1,1,\frac{t}{2^{k-2}}) = 1$. Hence, the inequality is obviously true.

Case II. If $x,y \notin [\frac{1}{4},1]$, then $\beta(x,y,\alpha^k t)F_{fx,fy}(\alpha^k\phi(t)) = \gamma(fx,fy,\alpha^{k-1}\frac{t}{c})F_{x,y}(\alpha^{k-1}\phi(\frac{t}{c})) = \frac{t+2^k}{t+2^{k-2}|x-y|^2}$ and hence the inequality is again true.

Case III. If $x \in [\frac{1}{4},1]$ and $y \notin [\frac{1}{4},1]$ or $x \notin [\frac{1}{4},1]$ and $y \in [\frac{1}{4},1]$, then we have the same result as case II.

On the other hand, f does not satisfy inequality (7) if we assume that $\beta(x,y,t) = \gamma(x,y,t) = 1$ for all $x,y \in X$ and all $t > 0$. Indeed, for $x = 1$ and $y = 2$ we get

$$\frac{t}{t+2^k} \geq \min\left\{\frac{t}{t+2^{k-1}c}, 1, 1, \frac{t}{t+2^{k-3}c}, \frac{t}{t+2^{k-3}c}\right\} = \frac{t}{t+2^{k-1}c},$$

which gives $c \geq 2$, a contradiction.

Under an additional hypothesis on f, from Theorem 3.4, we obtain the uniqueness of the fixed point.

(J′) For all $u,v \in \operatorname{Fix}(f)$ and for all $t > 0$ there exists $z \in X$ such that $\beta(z,fz,t) \leq 1$ with $\beta(u,z,t) \leq 1$, and $\beta(v,z,t) \leq 1$ and $\gamma(z,fz,t) \geq 1$ with $\gamma(u,z,t) \geq 1$ and $\gamma(v,z,t) \geq 1$.

Theorem 3.7 *Adding condition* (J′) *to the hypotheses of Theorem* 3.4, *we find that* f *has a unique fixed point.*

Proof Let $u,v \in X$ be such that $u = fu$ and $v = fv$. From condition (J′), there exists $z \in X$ such that $\beta(z,fz,t) \leq 1$ with $\beta(u,z,t) \leq 1$ and $\beta(v,z,t) \leq 1$, and $\gamma(z,fz,t) \geq 1$ with $\gamma(u,z,t) \geq 1$ and $\gamma(v,z,t) \geq 1$. By virtue of the fact that f is (β,γ)-admissible, we deduce that

$$\beta(fz,f^2z,t) \leq 1, \qquad \beta(u,fz,t) \leq 1, \qquad \beta(v,fz,t) \leq 1,$$

and

$$\gamma(fz,f^2z,t) \geq 1, \qquad \gamma(u,fz,t) \geq 1, \qquad \gamma(v,fz,t) \geq 1.$$

By induction, we derive

$$\beta(z_n, z_{n+1}, t) \le 1, \qquad \beta(u, z_n, t) \le 1, \qquad \beta(v, z_n, t) \le 1,$$
$$\gamma(z_{n+1}, z_{n+2}, t) \ge 1, \qquad \gamma(u, z_{n+1}, t) \ge 1, \qquad \gamma(v, z_{n+1}, t) \ge 1,$$

for all $t > 0$, where $z_n = f^n z$ ($n \in \mathbb{N}$). By continuity of ϕ, there exists $r > 0$ such that $t > \phi(r)$ and therefore by (PbM1) and (PbM3) we have

$$\begin{aligned} F_{u,z_{n+1}}(t) &\ge F_{u,z_{n+1}}\bigl(\alpha^k\phi(r)\bigr) \\ &= F_{fu,fz_n}\bigl(\alpha^k\phi(r)\bigr) \\ &\ge \beta\bigl(u, z_n, \alpha^k r\bigr)F_{fu,fz_n}\bigl(\alpha^k\phi(r)\bigr) \\ &\ge \gamma\left(fu, fz_n, \alpha^{k-1}\frac{r}{c}\right)\min\left\{F_{u,z_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{u,fu}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right),\right. \\ &\qquad \left. F_{z_n,fz_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{u,fz_n}\left(2\alpha^{k-2}\phi\left(\frac{r}{c}\right)\right), F_{z_n,fu}\left(2\alpha^{k-2}\phi\left(\frac{r}{c}\right)\right)\right\} \\ &\ge \min\left\{F_{u,z_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{z_n,z_{n+1}}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right)\right\}, \end{aligned}$$

where $k \in \mathbb{N}$. Now, we consider following cases:

Case I. If $F_{z_n,z_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c}))$ is the minimum, then by (7), (PbM1), and (PbM3), it follows that

$$\begin{aligned} &F_{u,z_{n+1}}\bigl(\alpha^k\phi(r)\bigr) \\ &\quad\ge F_{z_n,z_{n+1}}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right) \\ &\quad= F_{fz_{n-1},fz_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right) \\ &\quad\ge \beta\left(z_{n-1}, z_n, \alpha^{k-1}\frac{r}{c}\right)F_{fz_{n-1},fz_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right) \\ &\quad\ge \gamma\left(z_n, z_{n+1}, \alpha^{k-2}\frac{r}{c^2}\right)\min\left\{F_{z_{n-1},z_n}\left(\alpha^{k-2}\phi\left(\frac{r}{c^2}\right)\right), F_{z_{n-1},fz_{n-1}}\left(\alpha^{k-2}\phi\left(\frac{r}{c^2}\right)\right),\right. \\ &\qquad \left. F_{z_n,fz_n}\left(\alpha^{k-2}\phi\left(\frac{r}{c^2}\right)\right), F_{z_{n-1},fz_n}\left(2\alpha^{k-3}\phi\left(\frac{r}{c^2}\right)\right), F_{z_n,fz_{n-1}}\left(2\alpha^{k-3}\phi\left(\frac{r}{c^2}\right)\right)\right\} \\ &\quad\ge \min\left\{F_{z_{n-1},z_n}\left(\alpha^{k-2}\phi\left(\frac{r}{c^2}\right)\right), F_{z_n,z_{n+1}}\left(\alpha^{k-2}\phi\left(\frac{r}{c^2}\right)\right)\right\}. \end{aligned}$$

Now, if $F_{z_n,z_{n+1}}(\alpha^{k-2}\phi(\frac{r}{c^2}))$ is the minimum for some $n \in \mathbb{N}$, then by Lemma 3.3, we deduce that $z_n = z_{n+1}$. Since $F_{u,z_{n+1}}(\alpha^k\phi(r)) \ge F_{z_n,z_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c})) = 1$, $u = z_{n+1}$. Consequently $\beta(v, u, t) \le 1$ and $\gamma(fv, fu, t) \ge 1$ for all $t > 0$ and so by (7), (PbM1), and (PbM3) we have

$$\begin{aligned} F_{v,u}\bigl(\alpha^k\phi(t)\bigr) &\ge \beta\bigl(v, u, \alpha^k t\bigr)F_{fv,fu}\bigl(\alpha^k\phi(t)\bigr) \\ &\ge \gamma\left(fv, fu, \alpha^{k-1}\frac{t}{c}\right)\min\left\{F_{v,u}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right), F_{v,fv}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right),\right. \end{aligned}$$

$$F_{u,fu}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right), F_{v,fu}\left(2\alpha^{k-2}\phi\left(\frac{t}{c}\right)\right), F_{u,fv}\left(2\alpha^{k-2}\phi\left(\frac{t}{c}\right)\right)\right\}$$
$$\geq F_{v,u}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right).$$

Again, by Lemma 3.3, we conclude that $u = v$.

On the other hand, if $F_{z_{n-1},z_n}(\alpha^{k-2}\phi(\frac{r}{c^2}))$ is the minimum, then

$$F_{z_n,z_{n+1}}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right) \geq F_{z_{n-1},z_n}\left(\alpha^{k-2}\phi\left(\frac{r}{c^2}\right)\right) \geq \cdots \geq F_{z_0,z_1}\left(\alpha^{k-(n+1)}\phi\left(\frac{r}{c^{n+1}}\right)\right),$$

and, letting $n \to \infty$, we get $F_{z_n,z_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c})) \to 1$. Therefore $\lim_{n\to\infty}F_{u,z_{n+1}}(t) = 1$, which implies that $z_{n+1} \to u$ as $n \to \infty$. A similar method shows that $z_{n+1} \to v$, for $n \to \infty$. Since the limit is unique, $u = v$.

Case II. Suppose that $F_{u,z_n}(\alpha^{k-1}\phi(\frac{r}{c}))$ is the minimum, then we get

$$F_{u,z_{n+1}}\left(\alpha^k\phi(r)\right) \geq F_{u,z_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right) \geq F_{u,z_{n-1}}\left(\alpha^{k-2}\phi\left(\frac{r}{c^2}\right)\right) \geq \cdots$$
$$\geq F_{u,z_0}\left(\alpha^{k-(n+1)}\phi\left(\frac{r}{c^{n+1}}\right)\right).$$

Letting $n \to \infty$, we obtain $\lim_{n\to\infty} F_{u,z_{n+1}}(\alpha^k\phi(r)) = 1$, that is, $z_{n+1} \to u$ as $n \to \infty$. A similar argument shows that $z_{n+1} \to v$, for $n \to \infty$. Now, uniqueness of the limit gives us $u = v$, and the proof is complete. □

Our last existence theorem is a version of [5], Theorem 3.4, for generalized β-γ-type contractive mappings in Menger PbM-spaces.

Theorem 3.8 *Let (X, F, T) be a complete Menger PbM-space with coefficient α, and $f : X \to X$ be a mapping. Assume that there exist $\beta : X \times X \times (0,\infty) \to (0,\infty)$ and $\gamma : X \times X \times (0,\infty) \to (0,\infty)$ such that the following conditions hold*:

(i)
$$\beta\left(x,y,\alpha^k t\right)F_{fx,fy}\left(\alpha^k\phi(t)\right)$$
$$\geq \gamma\left(fx,fy,\alpha^{k-1}\frac{t}{c}\right)\min\left\{F_{x,y}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right), F_{x,fx}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right),\right.$$
$$\left.F_{y,fy}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right), F_{y,fx}\left(\alpha^{k-1}\phi\left(\frac{t}{c}\right)\right)\right\},$$

for all $x, y \in X$, for all $t > 0$ and for all $k \in \mathbb{N}$, where $c \in (0,1)$ and $\phi \in \Phi$;

(ii) *f is (β,γ)-admissible*;

(iii) *there exists $x_0 \in X$ such that $\beta(x_0, fx_0, t) \leq 1$ and $\gamma(x_0, fx_0, t) \geq 1$ for all $t > 0$*;

(iv) *for each sequence $\{x_n\}$ in X such that $\beta(x_{n-1}, x_n, t) \leq 1$ and $\gamma(x_n, x_{n+1}, t) \geq 1$, for all $n \in \mathbb{N}$ and for all $t > 0$, there exists $k_0 \in \mathbb{N}$ such that $\beta(x_{m-1}, x_{n-1}, t) \leq 1$ and $\gamma(x_m, x_n, t) \geq 1$, for all $m, n \in \mathbb{N}$ with $m > n \geq k_0$ and for all $t > 0$*;

(v) *if $\{x_n\}$ is a sequence in X such that $\beta(x_{n-1}, x_n, t) \leq 1$ and $\gamma(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$ and $x_n \to x$ as $n \to \infty$, then $\beta(x_{n-1}, x, t) \leq 1$ and $\gamma(x_n, fx, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$.*

Then f has a fixed point. If in addition, condition (J′) *holds, then f has a unique fixed point.*

Proof Let $x_0 \in X$ be such that (iii) holds. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = fx_n$ for all $n = 0, 1, \ldots$. We suppose that $x_{n+1} \neq x_n$ for all $n = 0, 1, \ldots$, otherwise f has trivially a fixed point. By (ii) and (iii), and applying induction, we get $\beta(x_{n-1}, x_n, t) \leq 1$ and $\gamma(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and for all $t > 0$. By continuity of ϕ at zero, we can find $r > 0$ such that $t > \phi(r)$, thus $\beta(x_{n-1}, x_n, \alpha^k r) \leq 1$ and $\gamma(x_n, x_{n+1}, \alpha^{k-1}\frac{r}{c}) \geq 1$, where $k \in \mathbb{N}$. It follows from conditions (i) and (PbM1) that

$$\begin{aligned}
F_{x_n,x_{n+1}}(t) &\geq F_{x_n,x_{n+1}}\big(\alpha^k \phi(r)\big) \\
&\geq \beta\big(x_{n-1}, x_n, \alpha^k r\big) F_{fx_{n-1},fx_n}\big(\alpha^k \phi(r)\big) \\
&\geq \gamma\left(x_n, x_{n+1}, \alpha^{k-1}\frac{r}{c}\right) \min\left\{F_{x_{n-1},x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{x_{n-1},x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right),\right. \\
&\qquad \left. F_{x_n,x_{n+1}}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{x_n,x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right)\right\} \\
&\geq \gamma\left(x_n, x_{n+1}, \alpha^{k-1}\frac{r}{c}\right) \min\left\{F_{x_{n-1},x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{x_n,x_{n+1}}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right)\right\} \\
&\geq \min\left\{F_{x_{n-1},x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right), F_{x_n,x_{n+1}}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right)\right\}.
\end{aligned}$$

Next, if $F_{x_n,x_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c}))$ is the minimum, then $F_{x_n,x_{n+1}}(\alpha^k\phi(r)) \geq F_{x_n,x_{n+1}}(\alpha^{k-1}\phi(\frac{r}{c}))$ and so by Lemma 3.3, $x_n = x_{n+1}$, which contradicts the assumption $x_n \neq x_{n+1}$. Now if $F_{x_{n-1},x_n}(\alpha^{k-1}\phi(\frac{r}{c}))$ is the minimum, then

$$F_{x_n,x_{n+1}}(t) \geq F_{x_n,x_{n+1}}\big(\alpha^k\phi(r)\big) \geq F_{x_{n-1},x_n}\left(\alpha^{k-1}\phi\left(\frac{r}{c}\right)\right) \geq \cdots \geq F_{x_0,x_1}\left(\alpha^{k-n}\phi\left(\frac{r}{c^n}\right)\right).$$

Letting $n \to \infty$, then

$$F_{x_n,x_{n+1}}(t) \to 1. \tag{9}$$

We claim that $\{x_n\}$ is a Cauchy sequence. Suppose the contrary. Then there exist $\varepsilon > 0$, $\lambda \in (0,1)$ for which we can find subsequences $\{x_{m(s)}\}$ and $\{x_{n(s)}\}$ of $\{x_n\}$ such that $n(s)$ is the smallest index for which

$$s < m(s) < n(s), \qquad F_{x_{m(s)},x_{n(s)}}(\varepsilon) \leq 1 - \lambda, \qquad F_{x_{m(s)},x_{n(s)-1}}(\varepsilon) > 1 - \lambda. \tag{10}$$

By the properties of ϕ there exists $\varepsilon_1 > 0$ such that

$$\phi(\varepsilon_1) < \varepsilon. \tag{11}$$

From (10) and (11), we deduce that $F_{x_{m(s)},x_{n(s)}}(\alpha\phi(\varepsilon_1)) \leq 1 - \lambda$, so $\{x_n\}$ is not Cauchy sequence with respect to $\alpha\phi(\varepsilon_1)$ and λ. Thus there exist increasing sequences of integers $m(s)$ and $n(s)$, such that $n(s)$ is the smallest index for which

$$s < m(s) < n(s), \qquad F_{x_{m(s)},x_{n(s)}}\big(\alpha\phi(\varepsilon_1)\big) \leq 1 - \lambda, \qquad F_{x_{m(s)},x_{n(s)-1}}\big(\alpha\phi(\varepsilon_1)\big) > 1 - \lambda. \tag{12}$$

Take a real number η such that $0 < \eta < \phi(\frac{\varepsilon_1}{c}) - \phi(\varepsilon_1)$. From (12) it follows that

$$F_{x_{m(s)},x_{n(s)-1}}\left(\alpha\phi\left(\frac{\varepsilon_1}{c}\right) - \alpha\eta\right) > 1 - \lambda.$$

Then, for any $0 < \lambda_1 < \lambda < 1$, by (9) it is possible to find a positive integer N_1 such that for all $s > N_1$, we have

$$F_{x_{m(s)-1},x_{m(s)}}(\alpha\eta) > 1 - \lambda_1, \qquad F_{x_{n(s)-1},x_{n(s)}}(\alpha\eta) > 1 - \lambda_1. \tag{13}$$

By (13) and also applying (PbM3), we have

$$\begin{aligned} F_{x_{m(s)-1},x_{n(s)-1}}\left(\phi\left(\frac{\varepsilon_1}{c}\right)\right) &\geq T\left(F_{x_{m(s)-1},x_{m(s)}}(\alpha\eta), F_{x_{m(s)},x_{n(s)-1}}\left(\alpha\phi\left(\frac{\varepsilon_1}{c}\right) - \alpha\eta\right)\right) \\ &> T(1-\lambda_1, 1-\lambda). \end{aligned}$$

Since λ_1 is arbitrary and T is continuous, it follows that

$$F_{x_{m(s)-1},x_{n(s)-1}}\left(\phi\left(\frac{\varepsilon_1}{c}\right)\right) > 1 - \lambda. \tag{14}$$

A direct consequence of (13) is

$$\begin{aligned} F_{x_{m(s)-1},x_{m(s)}}\left(\phi\left(\frac{\varepsilon_1}{c}\right)\right) &\geq F_{x_{m(s)-1},x_{m(s)}}\left(\alpha\phi\left(\frac{\varepsilon_1}{c}\right)\right) \\ &\geq F_{x_{m(s)-1},x_{m(s)}}(\alpha\eta) > 1 - \lambda_1 > 1 - \lambda. \end{aligned} \tag{15}$$

A similar relation holds when one substitutes $x_{m(s)-1}$ and $x_{m(s)}$ with $x_{n(s)-1}$ and $x_{n(s)}$, respectively. On the other hand, we observe that

$$\begin{aligned} F_{x_{m(s)},x_{n(s)-1}}\left(\phi\left(\frac{\varepsilon_1}{c}\right)\right) &\geq F_{x_{m(s)},x_{n(s)-1}}\left(\alpha\phi\left(\frac{\varepsilon_1}{c}\right)\right) \\ &\geq F_{x_{m(s)},x_{n(s)-1}}\left(\alpha\phi\left(\frac{\varepsilon_1}{c}\right) - \alpha\eta\right) > 1 - \lambda. \end{aligned} \tag{16}$$

Applying assumptions (i), (iv), and (12), (14), (15), (16) we get

$$\begin{aligned} 1 - \lambda &\geq F_{x_{m(s)},x_{n(s)}}\big(\alpha\phi(\varepsilon_1)\big) = F_{fx_{m(s)-1},fx_{n(s)-1}}\big(\alpha\phi(\varepsilon_1)\big) \\ &\geq \beta(x_{m(s)-1}, x_{n(s)-1}, \alpha\varepsilon_1) F_{fx_{m(s)-1},fx_{n(s)-1}}\big(\alpha\phi(\varepsilon_1)\big) \\ &\geq \gamma\left(fx_{m(s)-1}, fx_{n(s)-1}, \frac{\varepsilon_1}{c}\right) \min\left\{F_{x_{m(s)-1},x_{n(s)-1}}\left(\phi\left(\frac{\varepsilon_1}{c}\right)\right), F_{x_{m(s)-1},x_{m(s)}}\left(\phi\left(\frac{\varepsilon_1}{c}\right)\right),\right. \\ &\qquad \left. F_{x_{n(s)-1},x_{n(s)}}\left(\phi\left(\frac{\varepsilon_1}{c}\right)\right), F_{x_{n(s)-1},x_{m(s)}}\left(\phi\left(\frac{\varepsilon_1}{c}\right)\right)\right\} \\ &> \gamma\left(fx_{m(s)-1}, fx_{n(s)-1}, \frac{\varepsilon_1}{c}\right)\{1-\lambda, 1-\lambda, 1-\lambda, 1-\lambda\} \\ &\geq 1 - \lambda. \end{aligned}$$

This is a contradiction; therefore $\{x_n\}$ is a Cauchy sequence in the complete Menger PbM-space. Thus $x_n \to u$ as $n \to \infty$ for some $u \in X$.

Now, we show that u is a fixed point of f. We have

$$F_{fu,u}(t) \geq T\big(F_{fu,x_n}\big(\alpha\phi(r)\big), F_{x_n,u}\big(\alpha t - \alpha\phi(r)\big)\big). \tag{17}$$

Since ϕ is continuous, there exists $r > 0$ such that $t > \phi(r)$. Further, since $u = \lim_{n\to\infty} x_n$, then, for arbitrary $\delta \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we get

$$F_{x_n,u}\big(\alpha t - \alpha\phi(r)\big) > 1 - \delta. \tag{18}$$

Hence, from (17) and (18), we find that

$$F_{fu,u}(t) \geq T\big(F_{fu,x_n}\big(\alpha\phi(r)\big), 1 - \delta\big).$$

Since $\delta > 0$ is arbitrary and T is continuous, we can write $F_{fu,u}(t) \geq F_{fu,x_n}(\alpha\phi(r))$. Without loss of generality we may assume that $x_n \neq fu$ for all $n \in \mathbb{N}$, otherwise if for infinitely many values of n, $x_n = fu$, then $u = fu$, and hence the proof is finished. Applying (i) and (v), we derive

$$\begin{aligned}
F_{u,fu}(t) &\geq F_{x_n,fu}\big(\alpha\phi(r)\big) \\
&\geq \beta(x_{n-1}, u, \alpha r) F_{fx_{n-1},fu}\big(\alpha\phi(r)\big) \\
&\geq \gamma\left(fx_{n-1}, fu, \frac{r}{c}\right) \min\left\{F_{x_{n-1},u}\left(\phi\left(\frac{r}{c}\right)\right), F_{x_{n-1},fx_{n-1}}\left(\phi\left(\frac{r}{c}\right)\right),\right. \\
&\quad \left. F_{u,fu}\left(\phi\left(\frac{r}{c}\right)\right), F_{u,fx_{n-1}}\left(\phi\left(\frac{r}{c}\right)\right)\right\} \\
&\geq \min\left\{F_{x_{n-1},u}\left(\phi\left(\frac{r}{c}\right)\right), F_{x_{n-1},fx_{n-1}}\left(\phi\left(\frac{r}{c}\right)\right), F_{u,fu}\left(\phi\left(\frac{r}{c}\right)\right), F_{u,fx_{n-1}}\left(\phi\left(\frac{r}{c}\right)\right)\right\}.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, we get $F_{fu,u}(\alpha\phi(r)) \geq F_{u,fu}(\phi(\frac{r}{c}))$. Thus $u = fu$ by Lemma 3.3. Hence f has a fixed point. Furthermore, if (J′) holds, then by using a similar technique as in the proof of Theorem 3.7 one can see that u is a unique fixed point of f. □

4 Application to integral equation

As an application of our results, we will consider the following Volterra type integral equation:

$$x(t) = g(t) + \int_0^t \Omega\big(t, s, x(s)\big)\, ds, \tag{19}$$

for all $t \in [0, k']$, where $k' > 0$.

Let $C([0,k'],\mathbb{R})$ be the space of all continuous functions defined on $[0,k']$ endowed with the b-metric

$$d(x,y) = \max_{t\in[0,k']} \big|x(t) - y(t)\big|^2, \quad x, y \in C\big(\big[0,k'\big],\mathbb{R}\big).$$

Alternatively the space $C([0,k'],\mathbb{R})$ can be endowed with the b-metric

$$d_B(x,y) = \max_{t\in[0,k']} \left(\left|x(t)-y(t)\right|^2 e^{-2Lt}\right), \quad x,y \in C\left(\left[0,k'\right],\mathbb{R}\right), L>0.$$

One can see that d and d_B are complete b-metrics with $s = 2$. We define the mapping $F : C([0,k'],\mathbb{R}) \times C([0,k'],\mathbb{R}) \to \mathcal{D}^+$ by

$$F_{x,y}(t) = \mathcal{H}\left(t - d_B(x,y)\right), \quad t>0, x,y \in C\left(\left[0,k'\right],\mathbb{R}\right). \tag{20}$$

We know that $(C([0,k'],\mathbb{R}), F, T_M)$ is a complete Menger PbM-space with coefficient $\alpha = \frac{1}{2}$.

Now we discuss the existence of a solution for the Volterra type integral equation (19).

Theorem 4.1 *Let* $(C([0,k'],\mathbb{R}), F, T_M)$ *be the Menger PbM-space and* $\Omega \in C([0,k'] \times [0,k'] \times \mathbb{R}, \mathbb{R})$ *be an operator satisfying the following conditions*:

(i) $\|\Omega\|_\infty = \sup_{t,s\in[0,k'],x\in C([0,k'],\mathbb{R})} \mid \Omega(t,s,x(s)) \mid < \infty$,

(ii) *there exists* $L > 0$ *such that for all* $x,y \in C([0,k'],\mathbb{R})$ *and all* $t,s \in [0,k']$ *we obtain*

$$\begin{aligned} &\left|\Omega\left(t,s,fx(s)\right) - \Omega\left(t,s,fy(s)\right)\right| \\ &\quad \le \frac{L}{\sqrt{2}} \max\left\{\left|x(s)-y(s)\right|, \left|x(s)-fx(s)\right|, \left|y(s)-fy(s)\right|, \left|y(s)-fx(s)\right|\right\}, \end{aligned}$$

where $f : C([0,k'],\mathbb{R}) \to C([0,k'],\mathbb{R})$ *is defined by*

$$fx(t) = g(t) + \int_0^t \Omega\left(t,s,fx(s)\right) ds, \quad g \in C\left(\left[0,k'\right],\mathbb{R}\right).$$

Then the Volterra type integral equation (19) *has a unique solution* $x^* \in C([0,k'],\mathbb{R})$.

Proof For each $x,y \in C([0,k'],\mathbb{R})$ we consider $d_B(x,y) = \max_{t\in[0,k']}(|x(t) - y(t)|^2 e^{-2Lt})$, where L satisfies condition (ii). As we mentioned above $(C([0,k'],\mathbb{R}), F, T_M)$ is a complete Menger PbM-space with coefficient $\alpha = \frac{1}{2}$. Therefore, for all $x,y \in C([0,k'],\mathbb{R})$, we get

$$\begin{aligned} d_B(fx,fy) &= \max_{t\in[0,k']} \left(\left|fx(t)-fy(t)\right|^2 e^{-2Lt}\right) \\ &= \max_{t\in[0,k']} \left(\left|\int_0^t \Omega\left(t,s,fx(s)\right) - \Omega\left(t,s,fy(s)\right) ds\right|^2 e^{-2Lt}\right) \\ &\le \frac{L^2}{2} \max\left\{d_B(x,y), d_B(x,fx), d_B(y,fy), d_B(y,fx)\right\} \max_{t\in[0,k']} \left(\int_0^t e^{L(s-t)} ds\right)^2 \\ &= \frac{1}{2}\left(1-e^{-Lk'}\right)^2 \max\left\{d_B(x,y), d_B(x,fx), d_B(y,fy), d_B(y,fx)\right\}. \end{aligned}$$

Putting $c = (1-e^{-Lk'})^2$, by using (20), for any $r > 0$ and $k \in \mathbb{N}$ we derive

$$\begin{aligned} F_{fx,fy}\left(\frac{r}{2^k}\right) &= \mathcal{H}\left(\frac{r}{2^k} - d_B(fx,fy)\right) \\ &\ge \mathcal{H}\left(\frac{r}{2^k} - \frac{c}{2} \max\left\{d_B(x,y), d_B(x,fx), d_B(y,fy), d_B(y,fx)\right\}\right) \end{aligned}$$

$$= \mathcal{H}\left(\frac{r}{2^{k-1}c} - \max\left\{d_B(x,y), d_B(x,fx), d_B(y,fy), d_B(y,fx)\right\}\right)$$
$$= \min\left\{F_{x,y}\left(\frac{r}{2^{k-1}c}\right), F_{x,fx}\left(\frac{r}{2^{k-1}c}\right), F_{y,fy}\left(\frac{r}{2^{k-1}c}\right), F_{y,fx}\left(\frac{r}{2^{k-1}c}\right)\right\},$$

for all $x, y \in C([0,k'],\mathbb{R})$. Therefore by Theorem 3.8 with $\phi(r) = r$ for all $r > 0$ and $\beta(x,y,t) = \gamma(x,y,t) = 1$ for all $x, y \in C([0,k'],\mathbb{R})$ and $t > 0$, we deduce that the operator f has a unique fixed point $x^* \in C([0,k'],\mathbb{R})$, which is the unique solution of the integral equation (19). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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