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Reich's iterated function systems and well-posedness via fixed point theory

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Abstract

In this paper, we prove the existence of the attractors for Reich's iterated function systems by virtue of a Banach-like fixed point theorem. As a result, under the condition that the Reich contractions discussed are continuous, we give an affirmative answer to an open question posed by Singh *et al.* in 2009. In addition, we formulate a collage theorem for Reich's iterated function systems.

MSC: 47H10; 54HA25

Keywords: Reich's iterated function systems; attractor; Reich contractions; complete metric spaces; fixed point theory

1 Introduction

Fixed point theory (see, for instance, [1–12]) plays an important role in iterated function systems as well as fractals (see [11–25]). As is well known, the simplest fractals are the self-similar sets, which are deemed to be the 'fixed points' of the Hutchinson-Barnsley operators in the setting of hyperspaces of compact sets of the original spaces endowed with the Pompeiu-Hausdorff metric. In order to describe such Hutchinson-Barnsley operators, one need utilize iterated function systems (see, for example, [11–15]). The concept of iterated function systems was introduced by Hutchinson in 1981 (see [13]) and popularized by Barnsley in 1998 (see [14]) as a natural generalization of the well-known Banach contraction principle. They represent one way of defining fractals as attractors of certain discrete dynamical systems. Moreover, they can be effectively applied to quantum physics, wavelets analysis, computer graphics and other applied sciences (see, for example, [26–32]). Therefore, it is no wonder that they have been attracting widespread attention of mathematicians and others for recent years (see, for example, [33–36]).

Since 1981, an increasing number of fractals has been yielded by iterated function systems (IFSs). For years, IFSs have become powerful tools for construction and analysis of new typical fractal sets. In order to construct a fractal, one usually draws support from known fixed point results obtained in the setting of appropriate spaces (see, for instance, [11–25]). In [23], Kashyap *et al.* obtained a new fractal from the Krasnoselskii fixed point theorem, generating the classical fractal set, which was introduced by Mandelbrot in 1982. Moreover, some open problems on IFS have been posed for the sake of theory and applications. In 2009, Singh *et al.* [24] posed an open question on IFS: In the IFS introduced by Hutchinson, can one replace the Banach contractions by the Reich contractions? In 2010, Sahu and Chakraborty [25] introduced K-iterated function system and obtained some interesting results related to the above-mentioned open question. In this paper, following Hutchinson and Sahu and Chakraborty, we present some new iterated function systems by using the so-called generalized contractive mappings (namely, the Hardy-Rogers type operators; see, for instance, [10, 11]), which will cover a much larger range of mappings. We obtain the attractors for a number of new iterated function systems by virtue of a Banach-like fixed point theorem concerning such generalized contractive mappings. It should be mentioned that among these new iterated function systems is Reich's iterated function system. As a result, under the condition that the Reich contractions discussed are continuous, we give an affirmative answer to the above-mentioned open question posed by Singh *et al.* In addition, we show that the attractor, either for Hutchinson's iterated function system or for the K-iterated function system, turns out to be the same as for Reich's iterated function system which can be deduced by means of such a Banach-like fixed point theorem concerning such generalized contractive mappings. At last, we formulate a collage theorem for Reich's iterated function systems.

2 Iterated function systems and an open question

In this section we first recall some well known aspects of iterated function system used in the sequel (more complete and rigorous treatments may be found in [13–15]). Then we present an interesting open question regarding an iterated function system associated with the Reich contractions, which was posed by Singh *et al.* in 2009.

Definition 2.1 ([1, 4]) Let X denote a metric space with distance function d and T be a mapping from X into itself. Then T is called a (Banach) contraction if there is a constant $0 \le s < 1$ such that, for all $x, y \in X$,

$$d(Tx, Ty) \le sd(x, y).$$

Definition 2.2 ([5, 6, 10, 11, 24]) Let X denote a metric space with distance function d and T be a mapping from X into itself. If there exists a constant α with $0 < \alpha < \frac{1}{2}$ such that, for all $x, y \in X$,

$$d\big(T(x), T(y)\big) \le \alpha\big[d\big(x, T(x)\big) + d\big(y, T(y)\big)\big],$$

then T is called a Kannan contraction.

Definition 2.3 ([6–8, 11, 24]) Let X denote a metric space with distance function d. Suppose that the mapping $T : X \to X$ satisfies

$$d(Tx, Ty) \le Ld(x, y) + M\big[d(x, Tx) + d(y, Ty)\big] \tag{2.1}$$

for all $x, y \in X$, where L, M are constants with $L, M \ge 0$, $0 < L + 2M < 1$. Then T is called a Reich contraction.

Note that if $M = 0$, then the Reich contraction is reduced to a Banach contraction; if $L = 0$, then the Reich contraction is a Kannan contraction.

The following is the classical Banach contraction principle.

Theorem 2.1 ([1, 4]) *Let $T : X \to X$ be a contraction on a complete metric space (X, d). Then T possesses exactly one fixed point $x^* \in X$. Moreover, for any point $x \in X$, the sequence $\{T^{\circ n}(x) : n = 0, 1, 2, \ldots\}$ converges to $x^* \in X$. That is $\lim_{n\to\infty} T^{\circ n}(x) = x^*$, for each $x \in X$, where $T^{\circ n}$ denotes the n-fold composition of T.*

In the famous paper [13], Hutchinson proved that, given a set of contractions in a complete metric space X, there exists a unique nonempty compact set $A \subset X$, named the attractor or fractal for the iterated function systems.

IFS generally employs contractive maps over a complete metric space (X, d), where Banach's celebrated result mentioned above guarantees the existence and uniqueness of the fixed point known as 'attractor' or 'fractal'. This can be done since a Hutchinson-Barnsley operator is also a contraction over $H(X)$, where $H(X)$ denotes the space whose points are the compact subsets of X.

We now give some basic definitions and theorems concerning an iterated function system, which are used in the proofs below.

Let (X, d) be a complete metric space and $H(X)$ denote the space whose points are the nonempty compact subsets of X. Let $x, y \in X$ and let $A, B \in H(X)$. Then the Pompeiu-Hausdorff distance from the set A to the set B is defined as

$$h(A, B) = d(A, B) \vee d(B, A),$$

where the notation $u \vee v$ means the maximum of the pair of real numbers u and v, and $d(A, B)$ is defined as

$$d(A, B) = \max\bigl\{d(x, B) : x \in A\bigr\},$$

where $d(x, B) = \min\{d(x, y) : y \in B\}$.

Note that the function h is the metric defined on the space $H(X)$.

In IFS, the contractive maps act on the members of a Hausdorff space, *i.e.*, the compact subsets of X. Thus, an iterated function system is defined as follows:

Hutchinson's iterated function system consists of a complete metric space (X, d) together with a finite set of continuous contractions $T_n : X \to X$ with respect to contractivity factor s_n, $n = 1, 2, 3, \ldots, N$.

Similarly, a K-iterated function system consists of a complete metric space (X, d) together with a finite set of continuous Kannan contractions $T_n : X \to X$ with K-contractivity factor α_n, $n = 1, 2, 3, \ldots, N$.

We now present the existence of the attractor for Hutchinson's IFS, *i.e.*, the following theorem, which was given by Hutchinson or Barnsley (see [13, 14]).

Theorem 2.2 ([13, 14]) *Let $\{X : T_n, n = 1, 2, 3, \ldots, N\}$ be an iterated function system with contractivity factor s. Then the transformation $W : H(X) \to H(X)$ defined by $W(B) = \bigcup_{n=1}^{N} T_n(B)$ for all $B \in H(X)$ is a contraction on the complete metric space $(H(X), h(d))$ with contractivity factor s. That is,*

$$h\bigl(W(B), W(C)\bigr) \le sh(B, C).$$

Its unique fixed point, which is also called an attractor, $A \in H(X)$ obeys

$$A = W(A) = \bigcup_{n=1}^{N} T_n(A)$$

and is given by $A = \lim_{n\to\infty} W^{\circ n}(B)$ for any $B \in H(X)$, where $W^{\circ n}$ denotes the n-fold composition of W.

In 2009, Singh *et al.* [24] posed an open question regarding IFS associated with the Reich contractions as follows.

Question 2.1 ([24]) Can one replace the Banach contractions in Theorem 2.2 (above) by the Reich contractions?

This precisely means whether the following is valid.

Theorem A ([24]) *Let (X,d) be a complete metric space and $f_i : X \to X$ be Reich contractions, i.e.,*

$$d\big(f_i(x),f_i(y)\big) \leq L_i d(x,y) + M_i\big[d\big(x,f_i(x)\big) + d\big(y,f_i(y)\big)\big] \tag{R}$$

for all $x,y \in X$, where L_i, M_i are constants with $L_i, M_i \geq 0$, $L_i + 2M_i < 1$, $i = 1,2,\ldots,n$. Then there exists a unique nonempty compact subset A of X that satisfies $A = \bigcup_{i=1}^{n} f_i(A)$.

We remark that, in the condition (R) above, if $M_i = 0$, then it reduces to the well-known Banach contraction condition and Theorem A reduces to above Theorem 2.2 (*i.e.*, the existence theorem for attractor of Hutchinson's IFS). Further, if $L_i = 0$ and $0 \leq M_i < 1/2$, then the condition (R) is the well-known Kannan contraction. Singh *et al.* in [24] wrote that Theorem A is correct for $L_i = 0$ and $0 \leq M_i < 1/3$. They indicated that a proof for the general case, viz., $L_i = 0$ and $0 \leq M_i < 1/2$, is awaited. In 2010, Sahu and Chakraborty [25] gave a proof for such a general case as follows.

Theorem 2.3 ([25]) *Let $\{X : T_n, n = 1,2,3,\ldots,N\}$ be a K-iterated function system with K-contractivity factor α. Then the transformation $W : H(X) \to H(X)$ defined by $W(B) = \bigcup_{n=1}^{N} T_n(B)$ for all $B \in H(X)$ is a continuous Kannan contraction on the complete metric space $(H(X), h(d))$ with contractivity factor s. That is,*

$$h\big(W(B), W(C)\big) \leq \alpha\big[h\big(B, T(B)\big) + h\big(C, T(C)\big)\big].$$

Its unique fixed point, which is also called an attractor, $A \in H(X)$ obeys

$$A = W(A) = \bigcup_{n=1}^{N} T_n(A)$$

and is given by $A = \lim_{n\to\infty} W^{\circ n}(B)$ for any $B \in H(X)$, where $W^{\circ n}$ denotes the n-fold composition of W.

In the subsequent section, we will prove that Theorem A above holds under the condition that the Reich contractions discussed are continuous, giving an affirmative answer to Question 2.1 with such an assumption.

3 New iterated function systems and an affirmative answer to the open question

In this section, we attempt to explore the possibility of improvement in IFS by replacing the Banach contraction condition or the Kannan contraction condition by a more general condition which is called, for convenience, the generalized contraction condition. One can easily see that such a condition generalizes not only the Banach contraction condition, but also the Kannan contraction condition. As a consequence, we can replace the Banach contractions in Theorem 2.2 (above) by the Reich contractions, thus give an affirmative answer to the above-mentioned open question (*i.e.*, Question 2.1).

Now, we extend the Banach contraction and the Kannan contraction to the so-called generalized contractive mapping as follows.

Definition 3.1 ([10, 11]) Let (X, d) be a metric space. The mapping $T : X \to X$ is called a generalized contractive mapping if it satisfies the following generalized contraction condition:

$$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(Tx, x) + a_3 d(Ty, y) + a_4 d(x, Ty) + a_5 d(y, Tx) \tag{3.1}$$

for all $x, y \in X$, where $a_i \geq 0$ $(i = 1, 2, 3, 4, 5)$ satisfy

$$a_1 + a_2 + a_3 + a_4 + a_5 < 1.$$

By virtue of symmetry in (3.1), we may only consider the case that $a_2 = a_3$ and $a_4 = a_5$. Thus, in the sequel, without loss of generality, we will assume that $T : X \to X$ is a generalized contractive mapping satisfying

$$d(Tx, Ty) \leq a d(x, y) + c\big(d(Tx, x) + d(Ty, y)\big) + e\big(d(x, Ty) + d(y, Tx)\big) \tag{3.2}$$

for all $x, y \in X$, where $a, c, e \geq 0$ satisfy

$$a + 2c + 2e < 1.$$

Remark 3.1 It is easily seen that any of the Banach contractions, the Kannan contractions and the Reich contractions is a special case of the generalized contractive mapping.

Now let us first present the fixed point theorem for the generalized contractive mappings.

Remark 3.2 It should be noticed that the so-called generalized contraction mentioned in Definition 3.1 above is also called Hardy-Rogers type operator. Petrusel recently pointed out in his interesting paper that any Hardy-Rogers type operator is a Ćirić type operator, but the reverse implications do not hold (see [11] for details).

Proposition 3.1 ([10, 11]) *Let (X,d) be a complete metric space. Suppose that the mapping $T : X \to X$ is a generalized contractive mapping satisfying* (3.2). *Then T has a unique fixed point p in X. Moreover, for any $x_0 \in X$, the Picard iterate $\{x_n\}$, where $x_n = Tx_{n-1}$ $(n = 1,2,\ldots)$, converges to the fixed point p and satisfies $d(x_n,p) \leq \frac{b^n}{1-b}d(Tx_0,x_0)$, where $b = \frac{a+c+e}{1-c-e}$.*

For the sake of completeness, we now give a sketch of the proof by three steps.

Step 1. We first show the existence of the fixed point. Suppose that x_0 is an arbitrary point in X. Set

$$x_{n+1} = Tx_n, \quad n = 0,1,2,\ldots.$$

We can easily show that $\{x_n\}$ is a Cauchy sequence. In fact, by (3.2), we can get

$$d(x_{n+1},x_n) \leq bd(x_n,x_{n-1}) \leq b^2 d(x_{n-1},x_{n-2}) \leq \cdots \leq b^n d(Tx_0,x_0),$$

where $b = \frac{a+c+e}{1-c-e}$.

Hence, for any $n \geq 0$, $m \geq 1$, we have

$$\begin{aligned} d(x_{n+m},x_n) &\leq d(x_{n+m},x_{n+m-1}) + d(x_{n+m-1},x_{n+m-2}) + \cdots + d(x_{n+1},x_n) \\ &\leq \bigl(b^{n+m-1} + b^{n+m-2} + \cdots + b^n\bigr)d(Tx_0,x_0) \\ &\leq \frac{b^n}{1-b}d(Tx_0,x_0). \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in (X,d). So there exists $p \in X$ such that $x_n \to p$ as $n \to \infty$. From (3.2), we have

$$\begin{aligned} d(x_n,Tp) &= d(Tx_{n-1},Tp) \\ &\leq ad(x_{n-1},p) + cd\bigl(d(x_n,x_{n-1}) + d(Tp,p)\bigr) + e\bigl(d(x_{n-1},Tp) + d(x_n,p)\bigr). \end{aligned}$$

Taking $n \to \infty$, we get

$$d(Tp,p) \leq (c+e)d(p,Tp).$$

Then, we get $(1-c-e)d(q,Tp) \leq 0$, which implies that $d(p,Tp) = 0$ since $1-c-e > 0$. So we have $p = Tp$.

Step 2. We then show the uniqueness of the fixed point. Assume that there exists another point $q \in X$ such that $Tq = q$, then by (3.2) we see

$$\begin{aligned} d(p,q) &= d(Tp,Tq) \\ &\leq ad(p,q) + c\bigl(d(Tp,p) + d(Tq,q)\bigr) + e\bigl(d(p,Tq) + d(Tp,q)\bigr) \\ &= ad(p,q) + 2ed(p,q) \end{aligned}$$

which gives $(1-a-2e)d(p,q) \leq 0$, thus $d(p,q) = 0$, So, $p = q$.

Step 3. We finally show the formula for the error estimate of a successive sequence. From a chain of inequalities presented before we see

$$d(x_{m+n}, x_n) \le \frac{b^n}{1-b} d(Tx_0, x_0). \tag{3.3}$$

Letting $m \to \infty$ in (3.3), we get

$$d(x_n, p) \le \frac{b^n}{1-b} d(Tx_0, x_0), \tag{3.4}$$

which completes the sketch of the proof of Proposition 3.1.

By means of Proposition 3.1, we easily get the following corollary, which is a Banach-like fixed point theorem.

Corollary 3.1 *Let (X, d) be a complete metric space. Suppose that the mapping $T : X \to X$ is a Reich contraction, i.e., there exist constants L, M with $L, M > 0$, $L + 2M < 1$ such that*

$$d(Tx, Ty) \le Ld(x, y) + M\big[d(x, Tx) + d(y, Ty)\big]$$

for all $x, y \in X$. Then T has a unique fixed point p in X. Moreover, for any $x_0 \in X$, the Picard iterate $\{x_n\}$, where $x_n = Tx_{n-1}$ $(n = 1, 2, \ldots)$, converges to the fixed point p and satisfies $d(x_n, p) \le \frac{b^n}{1-b} d(Tx_0, x_0)$, where $b = \frac{L+M}{1-M}$.

By the argument in the proof of Proposition 3.1, we easily obtain the following result.

Proposition 3.2 *Let (X, d) be a complete metric space. Suppose that the mapping $T : X \to X$ is a generalized contractive mapping satisfying* (3.2). *Let $x^* \in X$ be the fixed point of T. Then we have*

$$d\big(x_0, x^*\big) \le \frac{1-c-e}{1-a-2c-2e} d(x_0, Tx_0), \quad \forall x_0 \in X. \tag{3.5}$$

Proof Setting $n = 0$ in (3.4) of Step 3 above, we obtain (3.5) immediately. □

Remark 3.3 We note that Proposition 3.1 can be seen from [10] and [11], but the proof of Proposition 3.1 is somewhat different from that of [10], Theorem 1. In fact, as is indicated in [11], the idea of the proof of [10], Theorem 1 is to prove that f is a contraction on the graphic of the operator, *i.e.*, there exists a positive constant $\beta < 1$ such that

$$d\big(Tx, T^2x\big) \le \beta d(x, Tx) \quad \text{for all } x \in X.$$

However, ours need not do so since we directly prove the Picard iterated sequence is Cauchy and converges to the unique fixed point, which is a valuable addition to [10].

Remark 3.4 Proposition 3.1 generalizes the famous Banach contraction principle.

In order to present the new iterated function systems, we need the following lemmas.

Lemma 3.1 *Let (X,d) be a complete metric space. Suppose that the mapping $T : X \to X$ is a continuous Reich contraction satisfying*

$$d(Tx,Ty) \le Ld(x,y) + M\bigl[d(x,Tx) + d(y,Ty)\bigr]$$

for all $x,y \in X$, where $L,M \ge 0$, $L + 2M < 1$. Then $T : H(X) \to H(X)$ defined by $T(B) = \{T(x) : x \in B\}$ is also a Reich contraction satisfying

$$h\bigl(T(B),T(C)\bigr) \le Lh(B,C) + M\bigl[h\bigl(B,T(B)\bigr) + h\bigl(C,T(C)\bigr)\bigr] \tag{3.6}$$

for each $B,C \in H(X)$.

Proof Let us first prove that T maps $H(X)$ into itself. In fact, if $S \in H(X)$, then S is nonempty and compact in X. It is obvious that $T(S)$ is nonempty. Now we prove that $T(S)$ is compact in X. Let $\{y_n\} \subset T(S)$ be any sequence. Then there is a sequence $\{x_n\} \subset S$ such that $y_n = Tx_n$ $(n = 1,2,\ldots)$. Since S is compact, there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x \in S$. By the continuity of T we see $y_{n_k} = Tx_{n_k} \to Tx \in T(S)$, so $T(S)$ is compact in X. It is easily shown by the definition of Reich contraction that T satisfies

$$d\bigl(T(B),T(C)\bigr) \le Ld(B,C) + M\bigl[d\bigl(B,T(B)\bigr) + d\bigl(C,T(C)\bigr)\bigr], \quad \forall B,C \in H(X)$$

and

$$d\bigl(T(C),T(B)\bigr) \le Ld(C,B) + M\bigl[d\bigl(C,T(C)\bigr) + d\bigl(B,T(B)\bigr)\bigr], \quad \forall B,C \in H(X).$$

So, for all $B,C \in H(X)$, we have

$$\begin{aligned}
h\bigl(T(B),T(C)\bigr) &= d\bigl(T(B),T(C)\bigr) \vee d\bigl(T(C),T(B)\bigr) \\
&\le \bigl\{Ld(B,C) + M\bigl[d\bigl(B,T(B)\bigr) + d\bigl(C,T(C)\bigr)\bigr]\bigr\} \\
&\quad \vee \bigl\{Ld(C,B) + M\bigl[d\bigl(C,T(C)\bigr) + d\bigl(B,T(B)\bigr)\bigr]\bigr\} \\
&\le L\bigl[d(B,C) \vee d(C,B)\bigr] + M\bigl[d\bigl(B,T(B)\bigr) + d\bigl(C,T(C)\bigr)\bigr] \\
&\le Lh(B,C) + M\bigl[h\bigl(B,T(B)\bigr) + h\bigl(C,T(C)\bigr)\bigr],
\end{aligned}$$

which completes the proof of Lemma 3.1. □

Lemma 3.2 *Let (X,d) be a complete metric space. Let $T_n : n = 1,2,\ldots,N$ be mappings which map $(H(X),h)$ into $(H(X),h)$. Suppose that the mappings T_n satisfy*

$$h\bigl(T_n(B),T_n(C)\bigr) \le L_n h(B,C) + M\bigl[h\bigl(B,T_n(B)\bigr) + h\bigl(C,T_n(C)\bigr)\bigr]$$

for all $B,C \in H(X)$, where $L_n,M_n \ge 0$, $L_n + 2M_n < 1$. Define $T : H(X) \to H(X)$ by $T(B) = T_1(B) \cup T_2(B) \cup \cdots \cup T_N(B) = \bigcup_{n=1}^N T_n(B)$ for each $B \in H(X)$. Then T also satisfies

$$h\bigl(T(B),T(C)\bigr) \le Lh(B,C) + M\bigl[h\bigl(B,T(B)\bigr) + h\bigl(C,T(C)\bigr)\bigr]$$

for all $B,C \in H(X)$, where $L = \max\{L_n : n = 1,2,\ldots,N\}$, $M = \max\{M_n : n = 1,2,\ldots,N\}$.

Proof We shall prove the lemma by using the mathematical induction method. For $N = 1$, the statement is obviously true. Now, for $N = 2$, we see that

$$\begin{aligned}
h\big(T(B), T(C)\big) &= h\big(T_1(B) \cup T_2(B), T_1(C) \cup T_2(C)\big) \\
&\leq h\big(T_1(B), T_1(C)\big) \vee h\big(T_2(B), T_2(C)\big) \\
&\leq \big\{L_1 h(B,C) + M_1\big[h\big(B, T_1(B)\big) + h\big(C, T_1(C)\big)\big]\big\} \\
&\quad \vee \big\{L_2 h(B,C) + M_2\big[h\big(B, T_2(B)\big) + h\big(C, T_2(C)\big)\big]\big\} \\
&\leq \max\{L_1, L_2\} h(B,C) + \max\{M_1, M_2\}\big[h\big(B, T_1(B)\big) \vee h\big(B, T_2(B)\big) \\
&\quad + h\big(C, T_1(C)\big) \vee h\big(C, T_2(C)\big)\big] \\
&= Lh(B,C) + M\big[h\big(B, T_1(B) \cup T_2(B)\big) + h\big(C, T_1(C) \cup T_2(C)\big)\big],
\end{aligned}$$

where $L = \max\{L_1, L_2\}$, $M = \max\{M_1, M_2\}$. Therefore, we see

$$h\big(T(B), T(C)\big) \leq Lh(B,C) + M\big[h\big(B, T(B)\big) + h\big(C, T(C)\big)\big].$$

By induction we see Lemma 3.2 is proved. □

Thus, from all the above results, we are in a position to present the following theorem for the new iterated function system $\{T_n\}_{n=1}^N$ consisting of the continuous Reich contractions defined as

$$d(T_n x, T_n y) \leq L_n d(x,y) + M_n\big[d(x, T_n x) + d(y, T_n y)\big] \tag{3.7}$$

for all $x, y \in X$, where L_n, M_n are constants with $L_n, M_n \geq 0$, $L_n + 2M_n < 1$. Such an iterated function system is called Reich's iterated function system.

Theorem 3.1 *Let (X,d) be a complete metric space. Suppose that the mappings $T_n : X \to X$ are continuous and satisfy the Reich contractive condition as* (3.7). *Then the transformation $T : H(X) \to H(X)$ defined by $T(B) = \bigcup_{n=1}^N T_n(B)$ for all $B \in H(X)$ also satisfies the Reich contractive condition* (3.6). *Its unique fixed point in $(H(X), h(d))$, which is also called an attractor, $A \in H(X)$ obeys $A = T(A) = \bigcup_{n=1}^N T_N(A)$ and is given by $A = \lim_{n\to\infty} T^{\circ n}(B)$ for any $B \in H(X)$.*

Similar to Theorem 3.1, the following Theorem 3.2 can be deduced by the same method, so we omit its proof.

Theorem 3.2 *Let (X,d) be a complete metric space. Suppose that the mappings $T_n : X \to X$ are continuous and satisfy the following condition*:

$$d(T_n x, T_n y) \leq L_n\big(d(x,y) + d(x, T_n x) + d(y, T_n y)\big) \tag{3.8}$$

for all $x, y \in X$, where L_n are constants with $0 < L_n < \frac{1}{3}$. Then the transformation $T : H(X) \to H(X)$ defined by $T(B) = \bigcup_{n=1}^N T_n(B)$ for all $B \in H(X)$ also satisfies the condition

$$h\big(T(B), T(C)\big) \leq L\big(h(B,C) + h\big(B, T(B)\big) + h\big(C, T(C)\big)\big),$$

where $B, C \in H(X)$, $L = \max\{L_n : n = 1, 2, \ldots, N\}$. *Its unique fixed point in* $(H(X), h(d))$, *which is also called an attractor,* $A \in H(X)$ *obeys* $A = T(A) = \bigcup_{n=1}^{N} T_n(A)$ *and is given by* $A = \lim_{n\to\infty} T^{\circ n}(B)$ *for any* $B \in H(X)$.

Based on the above arguments from Lemma 3.1, Lemma 3.2 and Proposition 3.2, we are now in a position to formulate a collage theorem for Reich's iterated function system.

Theorem 3.3 *Let* (X, d) *be a complete metric space. Let* $L \in H(X)$ *be given and* ϵ *be given. Choose Reich's iterated function system* $\{T_n\}_{n=1}^{N}$ *consisting the Reich contractions as* (3.7). *If*

$$h\left(L, \bigcup_{n=1}^{N} T_n(L)\right) \leq \epsilon,$$

then $h(L, A) \leq \frac{(1-M)\epsilon}{1-L-2M}$, *where* A *is the attractor of Reich's IFS. Equivalently, the equality*

$$h(L, A) \leq \frac{1-M}{1-L-2M} h\left(L, \bigcup_{n=1}^{N} T_n(L)\right)$$

holds for all $L \in H(X)$.

Remark 3.5 In Theorem 3.1, we obtain the attractors for Reich's iterated function systems, which generalizes the main results concerning both the famous Hutchinson iterated function systems and the K-iterated function systems in [13] and [25], respectively.

Remark 3.6 Theorem 3.1 gives an affirmative answer to the above-mentioned open question (*i.e.*, Question 2.1) under the condition that the Reich contractions discussed are continuous. Theorem 3.1 and Theorem 3.2 are a valuable addition to the main results of literature [13, 14, 24, 25].

Remark 3.7 Theorem 3.1 shows that the Hutchinson-Barnsley operator for Reich's iterated function system is a Picard operator and its unique 'fixed point' is a single-valued fractal. Such is called single-valued fractal for single-valued Reich contractions. As for multi-valued fractals for self multi-valued operators and nonself multi-valued operators, we refer to [9, 12], respectively.

Remark 3.8 It should be noticed that Theorem 3.1 is a generalization of [22], Theorem 1 and Theorem 3.3 is a generalization of both [22], Theorem 2 and [37], Theorem 1. Hence, our main results in this paper may have potential applications to the complex polynomial hyperbolic NIFS (nonlinear iterated function system) mentioned in [22] and image compression referred to in [37].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed equally and significantly in writing this paper. All three authors read and approved the final manuscript.

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