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Viscosity iterative algorithm for variational inequality problems and fixed point problems in a real q -uniformly smooth Banach space

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Abstract

The purpose of this paper is to study a viscosity iterative algorithm for finding a common element of the set of solutions of a general variational inequality problem for two inverse strongly accretive operators and the set of fixed points of a δ -strict pseudocontraction in a real q -uniformly smooth Banach space. Some strong convergence theorems are obtained under appropriate conditions. As an application, we prove some strong convergence theorems for fixed point problems and variational inequality problems or equilibrium problems in Hilbert spaces. These results improve and extend the corresponding results announced by many others.

MSC: 49J30; 47H10; 47H17; 49M05

Keywords: strong convergence; variational inequality; fixed point; strict pseudocontraction; q -uniformly smooth Banach space

1 Introduction

Let C be a subset of a real Banach space X and T be a mapping from C into itself. In what follows, we use $F(T)$ to denote the set of fixed points of T . Let X^* be a dual space of X and $q > 1$ be a real number. We recall that the generalized duality mapping $J_q : X \rightarrow 2^{X^*}$ is defined by

$$J_q(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^q, \|x^*\| = \|x\|^{q-1}\}, \quad \forall x \in X.$$

In particular, $J = J_2$ is called a normalized duality mapping and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. We know that J_q is single-valued if X is smooth, which is denoted by j_q . Now we recall the following definitions.

A mapping $T : C \rightarrow C$ is said to be L -Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in C. \quad (1.1)$$

A mapping $T : C \rightarrow C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \quad (1.2)$$

A mapping $f : C \rightarrow C$ is said to be a contraction if there exists a constant $\alpha \in (0, 1)$ such that

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \quad \forall x, y \in C. \tag{1.3}$$

We use Π_C to denote the collection of all contractions on C .

A mapping $T : C \rightarrow C$ is called a λ -strict pseudocontraction if there exists a constant $\lambda \in (0, 1)$ such that

$$\langle Tx - Ty, j_q(x - y) \rangle \leq \|x - y\|^q - \lambda \|(I - T)x - (I - T)y\|^q \tag{1.4}$$

for every $x, y \in C$ and for some $j(x - y) \in J(x - y)$.

A mapping $A : C \rightarrow X$ is said to be α -inverse-strongly accretive if there exist $j_q(x - y) \in J_q(x - y)$ and a constant $\alpha > 0$ such that

$$\langle Ax - Ay, j_q(x - y) \rangle \geq \alpha \|Ax - Ay\|^q, \quad \forall x, y \in C. \tag{1.5}$$

In recent years, a variational inequality problem in Hilbert spaces and Banach spaces has been studied by many authors, see [1–15] and the references therein.

Let C be a nonempty closed convex subset of a real Hilbert space H . The classical variational inequality problem is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.6}$$

Recently, Ceng *et al.* [4] considered the following general variational inequality problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, x - x^* \rangle \geq 0, & \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, x - y^* \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.7}$$

where $\lambda > 0$ and $\mu > 0$ are two constants and $A, B : C \rightarrow H$ are two operators. In particular, if $A = B$ and $x^* = y^*$, then problem (1.7) reduces to the classical variational inequality problem (1.6).

Let C be a nonempty closed convex subset of a smooth Banach space X and $A : C \rightarrow X$ be an accretive operator. Aoyama *et al.* [3] first considered the following generalized variational inequality problem in Banach spaces which is finding a point $x^* \in C$ such that

$$\langle Ax^*, j(x - x^*) \rangle \geq 0, \quad \forall x \in C. \tag{1.8}$$

Very recently, Yao *et al.* [8] considered the following problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle Ay^* + x^* - y^*, j(x - x^*) \rangle \geq 0, & \forall x \in C, \\ \langle Bx^* + y^* - x^*, j(x - y^*) \rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.9}$$

which is called the system of general variational inequalities in a real Banach space, where $A, B : C \rightarrow X$ are two operators.

For finding a common element of the set of solutions of problem (1.7) and the set of fixed points of a nonexpansive mapping T , Ceng *et al.* [4] introduced the following iterative algorithm:

$$\begin{cases} x_1 = u \in C, \\ y_n = Q_C(x_n - \mu Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n TP_C(y_n - \lambda Ay_n), \quad n \geq 0, \end{cases} \tag{1.10}$$

and proved a strong convergence theorem under some suitable conditions.

Yao *et al.* [8] studied the following iterative algorithm:

$$\begin{cases} y_n = Q_C(x_n - Bx_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + \gamma_n Q_C(y_n - Ay_n), \quad n \geq 0, \end{cases} \tag{1.11}$$

and proved that the sequence $\{x_n\}$ converges strongly to an element of the set of solutions of problem (1.9) under appropriate conditions.

Let C be a nonempty closed convex subset of a real Banach space X . For given two operators $A, B : C \rightarrow X$, we consider the problem of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \lambda Ay^* + x^* - y^*, j_q(x - x^*) \rangle \geq 0, \quad \forall x \in C, \\ \langle \mu Bx^* + y^* - x^*, j_q(x - y^*) \rangle \geq 0, \quad \forall x \in C, \end{cases} \tag{1.12}$$

where $\lambda > 0$ and $\mu > 0$ are two constants. When $\lambda = \mu = 1$ and $q = 2$, problem (1.12) reduces to problem (1.9). When X is a Hilbert space, problem (1.12) becomes problem (1.7). Therefore problem (1.12) contains (1.7) or (1.9) as a special case. We also note that problem (1.12) was studied by Cai and Bu [9] when $q = 2$.

In this paper, we introduce a viscosity iterative algorithm for finding a common element of the set of solutions of a general variational inequality (1.12) and the set of fixed points of a δ -strict pseudocontraction in a real q -uniformly smooth Banach space. Then we prove some strong convergence theorems under suitable conditions. The results obtained in this paper extend and improve the results of Ceng *et al.* [4], Yao *et al.* [8] and many others.

2 Preliminaries

A Banach space X is called uniformly smooth if $\frac{\rho_X(t)}{t} \rightarrow 0$ as $t \rightarrow 0$, where $\rho_X : [0, \infty) \rightarrow [0, \infty)$ is the modulus of smoothness of X which is defined by

$$\rho_X(t) = \sup \left\{ \frac{1}{2} (\|x + y\| + \|x - y\|) - 1 : x \in S(X), \|y\| \leq t \right\}.$$

A Banach space X is said to be q -uniformly smooth if there exists a constant $c > 0$ such that $\rho_X(t) \leq ct^q$. If X is q -uniformly smooth, then $q \leq 2$ and X is uniformly smooth.

Let C and D be two nonempty subsets of X such that C is nonempty closed convex and $D \subset C$. We say that a mapping $Q : C \rightarrow D$ is sunny if $Q(Qx + t(x - Qx)) = Qx$, whenever $Qx + t(x - Qx) \in C$ for $x \in C$ and $t \geq 0$. A mapping $Q : C \rightarrow D$ is said to be a retraction if $Qx = x$ for any $x \in D$. Q is called a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D and Q is sunny and nonexpansive. A retraction Q is said to be orthogonal if for each $x, x - Q(x)$ is normal to D in the sense of James [16].

We know that a projection mapping is a sunny nonexpansive retraction Q of X onto C (see Bruck [17]). If X is a real smooth Banach space, then Q is an orthogonal projection of X onto C if and only if

$$Q(x) \in C \quad \text{and} \quad \langle Q(x) - x, j_q(Q(x) - y) \rangle \leq 0, \quad \forall y \in C. \tag{2.1}$$

In order to prove our main results, we need the following lemmas.

Lemma 2.1 ([18], p.63) *Let $q > 1$, then the following inequality holds:*

$$ab \leq \frac{1}{q}a^q + \frac{q-1}{q}b^{\frac{q}{q-1}}$$

for arbitrary positive real numbers a, b .

Lemma 2.2 ([19]) *Let X be a real q -uniformly smooth Banach space, then there exists a constant $C_q > 0$ such that*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, j_q x \rangle + C_q \|y\|^q$$

for all $x, y \in X$. In particular, if X is a real 2-uniformly smooth Banach space, then there exists a best smooth constant $K > 0$ such that

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j_2 x \rangle + 2\|Ky\|^2$$

for all $x, y \in X$.

Lemma 2.3 ([20]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1 - \alpha_n)a_n + \delta_n, n \geq 0$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in \mathbb{R} such that*

- (i) $\sum_{n=0}^\infty \alpha_n = \infty$;
- (ii) $\limsup_{n \rightarrow \infty} \frac{\delta_n}{\alpha_n} \leq 0$ or $\sum_{n=0}^\infty |\delta_n| < \infty$.

Then $\lim_{n \rightarrow \infty} a_n = 0$.

Lemma 2.4 ([21]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X , and let $\{\beta_n\}$ be a sequence in $[0, 1]$ which satisfies the condition $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n, n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n \rightarrow \infty} \|z_n - x_n\| = 0$.*

Lemma 2.5 ([22]) *Let C be a nonempty convex subset of a real q -uniformly smooth Banach space X and $T : C \rightarrow C$ be a λ -strict pseudocontraction. For $\alpha \in (0, 1)$, we define $T_\alpha x = (1 - \alpha)x + \alpha Tx$. Then, as $\alpha \in (0, \mu], \mu = \min\{1, \{\frac{q\lambda}{C_q}\}^{\frac{1}{q-1}}\}$, $T_\alpha : C \rightarrow C$ is nonexpansive such that $F(T_\alpha) = F(T)$.*

Lemma 2.6 ([23]) *Let X be a q -uniformly smooth Banach space, C be a closed convex subset of $X, T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$ with contractive*

constant $\alpha \in (0, 1)$. Then $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ for $t \in (0, 1)$ converges strongly to a point in $F(T)$. If we define $Q : \Pi_C \rightarrow F(T)$ by

$$Q(f) := \lim_{t \rightarrow 0} x_t, \quad f \in \Pi_C,$$

then $Q(f)$ solves the following variational inequality:

$$\langle (I-f)Q(f), j_q(Q(f) - p) \rangle \leq 0, \quad f \in \Pi_C, p \in F(T).$$

Lemma 2.7 ([23]) *Let C be a closed convex subset of a real q -uniformly smooth Banach space X , and let $T : C \rightarrow C$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Assume that $\{x_n\}$ is a bounded sequence such that $x_n - Tx_n \rightarrow 0$ as $n \rightarrow \infty$. Let $x_t = tf(x_t) + (1-t)Tx_t, \forall t \in (0, 1)$, where $f \in \Pi_C$ with contractive constant $\alpha \in (0, 1)$. Assume that $Q(f) := \lim_{t \rightarrow 0} x_t$ exists. Then*

$$\limsup_{n \rightarrow \infty} \langle (f - I)Q(f), j_q(x_n - Q(f)) \rangle \leq 0.$$

Lemma 2.8 *Let X be a q -uniformly smooth Banach space. Let C be a nonempty closed convex subset of X , and let $S : C \rightarrow C$ be a nonexpansive mapping and $T : C \rightarrow C$ be a δ -strict pseudocontraction such that $F(S) \cap F(T) \neq \emptyset$. Let W be a mapping from C into itself defined by $Wx = [(1-\alpha)I + \alpha T]Sx$ for any $x \in C$, where $\alpha \in (0, \mu), \mu = \min\{1, \{\frac{q\delta}{C_q}\}^{\frac{1}{q-1}}\}$. Then $F(W) = F(S) \cap F(T)$.*

Proof First we show that $F(S) \cap F(T) \subseteq F(W)$. Indeed, for any $x \in F(S) \cap F(T)$, we have

$$[(1-\alpha)I + \alpha T]Sx = [(1-\alpha)I + \alpha T]x = (1-\alpha)x + \alpha x = x,$$

which implies that $x \in F(W)$. Hence $F(S) \cap F(T) \subseteq F(W)$ holds. Next we prove that $F(W) \subseteq F(S) \cap F(T)$. For any $x \in F(W)$ and $y \in F(S) \cap F(T)$, it follows from Lemma 2.2 that

$$\begin{aligned} & \|x - y\|^q \\ &= \left\| [(1-\alpha)I + \alpha T]Sx - y \right\|^q \\ &= \|Sx - y + \alpha(TSx - Sx)\|^q \\ &\leq \|Sx - y\|^q + q\alpha \langle TSx - Sx, j_q(Sx - y) \rangle + C_q \alpha^q \|TSx - Sx\|^q \\ &= \|Sx - y\|^q + q\alpha \langle TSx - y, j_q(Sx - y) \rangle + q\alpha \langle y - Sx, j_q(Sx - y) \rangle + C_q \alpha^q \|TSx - Sx\|^q \\ &\leq \|x - y\|^q + q\alpha (\|Sx - y\|^q - \delta \|TSx - Sx\|^q) - q\alpha \|Sx - y\|^q + C_q \alpha^q \|TSx - Sx\|^q \\ &= \|x - y\|^q - (q\alpha\delta - C_q \alpha^q) \|TSx - Sx\|^q, \end{aligned}$$

which implies that

$$(q\alpha\delta - C_q \alpha^q) \|TSx - Sx\|^q \leq 0.$$

Therefore we obtain

$$TSx = Sx. \tag{2.2}$$

It follows that

$$x = [(1 - \alpha)I + \alpha T]Sx = (1 - \alpha)Sx + \alpha Sx = Sx. \tag{2.3}$$

This implies that $x \in F(S)$. By (2.2) and (2.3), we have $x = Sx = TSx = Tx$, and hence $x \in F(T)$. So $x \in F(S) \cap F(T)$. Consequently, $F(W) \subseteq F(S) \cap F(T)$ also holds. This proof is complete. \square

Lemma 2.9 *Let C be a nonempty closed convex subset of a real q -uniformly smooth Banach space X . Let the mapping $A : C \rightarrow X$ be α -inverse-strongly accretive. Then we have*

$$\|(I - \lambda A)x - (I - \lambda A)y\|^q \leq \|x - y\|^q + (C_q \lambda^{q-1} - q\alpha)\lambda \|Ax - Ay\|^q,$$

where $\lambda > 0$. In particular, if $\lambda \leq (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$, then $I - \lambda A$ is nonexpansive.

Proof For all $x, y \in C$, we have by Lemma 2.2

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^q &= \|x - y - \lambda(Ax - Ay)\|^q \\ &\leq \|x - y\|^q - q\lambda \langle Ax - Ay, j_q(x - y) \rangle + C_q \lambda^q \|Ax - Ay\|^q \\ &\leq \|x - y\|^q - q\lambda \alpha \|Ax - Ay\|^q + C_q \lambda^q \|Ax - Ay\|^q \\ &= \|x - y\|^q + (C_q \lambda^{q-1} - q\alpha)\lambda \|Ax - Ay\|^q. \end{aligned}$$

Therefore when $\lambda \leq (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$, we have that $I - \lambda A$ is nonexpansive. \square

Lemma 2.10 *Let C be a nonempty closed convex subset of a real q -uniformly smooth Banach space X . Let P_C be a sunny nonexpansive retraction from X onto C . Let the mapping $A : C \rightarrow X$ be α -inverse-strongly accretive, and let $B : C \rightarrow X$ be β -inverse-strongly accretive. Let $G : C \rightarrow C$ be a mapping defined by*

$$G(x) = P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)], \quad \forall x \in C.$$

If $0 < \lambda \leq (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$ and $0 < \mu \leq (\frac{q\beta}{C_q})^{\frac{1}{q-1}}$, then $G : C \rightarrow C$ is nonexpansive.

Proof For all $x, y \in C$, it follows from Lemma 2.9 that

$$\begin{aligned} &\|G(x) - G(y)\| \\ &= \|P_C[P_C(x - \mu Bx) - \lambda A P_C(x - \mu Bx)] - P_C[P_C(y - \mu By) - \lambda A P_C(y - \mu By)]\| \\ &\leq \|(I - \lambda A)P_C(I - \mu B)x - (I - \lambda A)P_C(I - \mu B)y\| \\ &\leq \|P_C(I - \mu B)x - P_C(I - \mu B)y\| \end{aligned}$$

$$\begin{aligned} &\leq \|(I - \mu B)x - (I - \mu B)y\| \\ &\leq \|x - y\|, \end{aligned}$$

which implies that G is nonexpansive. □

Lemma 2.11 ([23]) *Let C be a nonempty closed convex subset of a real q -uniformly smooth Banach space X . Let P_C be a sunny nonexpansive retraction from X onto C . Let $A, B : C \rightarrow X$ be two nonlinear mappings. For given $x^*, y^* \in C$, (x^*, y^*) is a solution of problem (1.12) if and only if $x^* = P_C(y^* - \lambda Ay^*)$, where $y^* = P_C(x^* - \mu Bx^*)$.*

3 Main results

Theorem 3.1 *Let C be a closed convex subset of a real q -uniformly smooth Banach space X ($q > 1$) which is also a sunny nonexpansive retraction of X . Let the mapping $A : C \rightarrow X$ be α -inverse-strongly accretive, and let $B : C \rightarrow X$ be β -inverse-strongly accretive. Let $f \in \Pi_C$ with the coefficient $0 < \eta < 1$ and $T : C \rightarrow C$ be a δ -strict pseudocontraction such that $F := F(G) \cap F(T) \neq \emptyset$, where G is defined by Lemma 2.10. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = Q_C(x_n - \mu Bx_n), \\ z_n = Q_C(y_n - \lambda Ay_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_\theta z_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where Q_C is a sunny nonexpansive retraction of X onto C , $0 < \lambda \leq (\frac{q\alpha}{C_q})^{\frac{1}{q-1}}$, $0 < \mu \leq (\frac{q\beta}{C_q})^{\frac{1}{q-1}}$ and $T_\theta : C \rightarrow C$ is a mapping defined by $T_\theta x = (1 - \theta)x + \theta Tx$, where $\theta \in (0, \rho)$, $\rho = \min\{1, \{\frac{q\delta}{C_q}\}^{\frac{1}{q-1}}\}$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\alpha_n + \beta_n + \gamma_n = 1$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ solves the following variational inequality:

$$\langle (I - f)Q(f), j_q(Q(f) - p) \rangle \leq 0, \quad f \in \Pi_C, p \in F.$$

Proof First we prove that $\{x_n\}$ is bounded. Let $W : C \rightarrow C$ be a mapping defined by $Wx = T_\theta Gx$ for all $x \in C$. By Lemma 2.8, we have $F(W) = F(G) \cap F(T)$. It follows from Lemma 2.5 that T_α is nonexpansive, then W is also nonexpansive. We can rewrite (3.1) as

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n Wx_n. \tag{3.2}$$

Take $p \in F$, by (3.2), we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(Wx_n - p)\| \\ &\leq \alpha_n \|f(x_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|Wx_n - p\| \\ &\leq \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|Wx_n - p\| \end{aligned}$$

$$\begin{aligned} &\leq \alpha_n \eta \|x_n - p\| + \alpha_n \|f(p) - p\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| \\ &= [1 - \alpha_n(1 - \eta)] \|x_n - p\| + \alpha_n(1 - \eta) \frac{\|f(p) - p\|}{1 - \eta} \\ &\leq \max \left\{ \|x_1 - p\|, \frac{\|f(p) - p\|}{1 - \eta} \right\} \end{aligned}$$

by induction. This implies that $\{x_n\}$ is bounded.

Next we show that $\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = 0$. Put $x_{n+1} = \beta_n x_n + (1 - \beta_n)l_n$, then we have

$$\begin{aligned} l_{n+1} - l_n &= \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}Wx_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n Wx_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (f(x_{n+1}) - Wx_{n+1}) - \frac{\alpha_n}{1 - \beta_n} (f(x_n) - Wx_n) + Wx_{n+1} - Wx_n, \end{aligned}$$

which implies

$$\begin{aligned} \|l_{n+1} - l_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \|f(x_{n+1}) - Wx_{n+1}\| \\ &\quad + \frac{\alpha_n}{1 - \beta_n} \|f(x_n) - Wx_n\| + \|Wx_{n+1} - Wx_n\| \\ &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\|f(x_{n+1})\| + \|Wx_{n+1}\|) \\ &\quad + \frac{\alpha_n}{1 - \beta_n} (\|f(x_n)\| + \|Wx_n\|) + \|x_{n+1} - x_n\|. \end{aligned}$$

It follows that

$$\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\| \leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (\|f(x_{n+1})\| + \|Wx_{n+1}\|) + \frac{\alpha_n}{1 - \beta_n} (\|f(x_n)\| + \|Wx_n\|).$$

By condition (ii), we have that

$$\limsup_{n \rightarrow \infty} (\|l_{n+1} - l_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

By Lemma 2.4, we obtain $\lim_{n \rightarrow \infty} \|l_n - x_n\| = 0$. Therefore,

$$\lim_{n \rightarrow \infty} \|x_{n+1} - x_n\| = \lim_{n \rightarrow \infty} (1 - \beta_n) \|l_n - x_n\| = 0. \tag{3.3}$$

Again using (3.2), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|\alpha_n (f(x_n) - x_n) + \gamma_n (Wx_n - x_n)\| \\ &\geq \gamma_n \|Wx_n - x_n\| - \alpha_n \|f(x_n) - x_n\|, \end{aligned}$$

which implies

$$\|Wx_n - x_n\| \leq \frac{1}{\gamma_n} [\alpha_n \|f(x_n) - x_n\| + \|x_{n+1} - x_n\|].$$

By conditions (i)-(iii) and (3.3), we have

$$\lim_{n \rightarrow \infty} \|Wx_n - x_n\| = 0. \tag{3.4}$$

Let $Q(f) = \lim_{n \rightarrow \infty} x_n$ and x_t be the unique fixed point of the contraction $T_t : C \rightarrow C$ given by

$$T_t x = t f(x) + (1 - t) Wx, \quad t \in (0, 1).$$

In view of Lemma 2.6, we obtain $Q(f) \in F(W) = F$ which solves the following variational inequality:

$$\langle (I - f)Q(f), j_q(Q(f) - p) \rangle \leq 0, \quad \forall p \in F.$$

By Lemma 2.7 and (3.4), we get

$$\limsup_{n \rightarrow \infty} \langle f(z) - z, j_q(x_n - z) \rangle \leq 0, \tag{3.5}$$

where $z = Q(f)$.

Finally we prove that $x_n \rightarrow z$ as $n \rightarrow \infty$. In fact, by Lemma 2.1, we have

$$\begin{aligned} & \|x_{n+1} - z\|^q \\ &= \alpha_n \langle f(x_n) - z, j_q(x_{n+1} - z) \rangle + \beta_n \langle x_n - z, j_q(x_{n+1} - z) \rangle + \gamma_n \langle Wx_n - z, j_q(x_{n+1} - z) \rangle \\ &= \alpha_n \langle f(x_n) - f(z), j_q(x_{n+1} - z) \rangle + \alpha_n \langle f(z) - z, j_q(x_{n+1} - z) \rangle + \beta_n \langle x_n - z, j_q(x_{n+1} - z) \rangle \\ &\quad + \gamma_n \langle Wx_n - z, j_q(x_{n+1} - z) \rangle \\ &\leq \alpha_n \eta \|x_n - z\| \|x_{n+1} - z\|^{q-1} + \beta_n \|x_n - z\| \|x_{n+1} - z\|^{q-1} + \gamma_n \|x_n - z\| \|x_{n+1} - z\|^{q-1} \\ &\quad + \alpha_n \langle f(z) - z, j_q(x_{n+1} - z) \rangle \\ &= [1 - \alpha_n(1 - \eta)] \|x_n - z\| \|x_{n+1} - z\|^{q-1} + \alpha_n \langle f(z) - z, j_q(x_{n+1} - z) \rangle \\ &\leq [1 - \alpha_n(1 - \eta)] \left(\frac{1}{q} \|x_n - z\|^q + \frac{q-1}{q} \|x_{n+1} - z\|^q \right) + \alpha_n \langle f(z) - z, j_q(x_{n+1} - z) \rangle, \end{aligned}$$

which implies

$$\|x_{n+1} - z\|^q \leq [1 - \alpha_n(1 - \eta)] \|x_n - z\|^q + \alpha_n(1 - \eta) \frac{q \langle f(z) - z, j_q(x_{n+1} - z) \rangle}{1 - \eta}. \tag{3.6}$$

Applying Lemma 2.3 to (3.6), we obtain that $x_n \rightarrow z$ as $n \rightarrow \infty$. This completes the proof. \square

Corollary 3.2 *Let C be a closed convex subset of a real 2-uniformly smooth Banach space X which is also a sunny nonexpansive retraction of X . Let the mapping $A : C \rightarrow X$ be α -inverse-strongly accretive, and let $B : C \rightarrow X$ be β -inverse-strongly accretive. Let $f \in \Pi_C$ with the coefficient $0 < \eta < 1$ and $T : C \rightarrow C$ be a δ -strict pseudocontraction such that*

$F := F(G) \cap F(T) \neq \emptyset$, where G is defined by Lemma 2.10. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = Q_C(x_n - \mu Bx_n), \\ z_n = Q_C(y_n - \lambda Ay_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_\theta z_n, \quad n \geq 1, \end{cases} \tag{3.7}$$

where Q_C is a sunny nonexpansive retraction of X onto C , $0 < \lambda \leq \frac{\alpha}{K^2}$, $0 < \mu \leq \frac{\beta}{K^2}$ and $T_\theta : C \rightarrow C$ is a mapping defined by $T_\theta x = (1 - \theta)x + \theta Tx$, $\theta \in (0, \rho)$, $\rho = \min \{1, \frac{\delta}{K^2}\}$, where K is the 2-uniformly smooth constant. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\alpha_n + \beta_n + \gamma_n = 1$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ solves the variational inequality

$$\langle (I - f)Q(f), j(Q(f) - p) \rangle \leq 0, \quad f \in \Pi_C, p \in F.$$

Proof Take $q = 2$ in Theorem 3.1, we obtain the desired result by Theorem 3.1. □

Corollary 3.3 *Let C be a closed convex subset of a real Hilbert space H . Let the mapping $A : C \rightarrow H$ be α -inverse-strongly accretive, and let $B : C \rightarrow H$ be β -inverse-strongly accretive. Let $f \in \Pi_C$ with the coefficient $0 < \eta < 1$ and $T : C \rightarrow C$ be a δ -strict pseudocontraction such that $F := F(G) \cap F(T) \neq \emptyset$, where G is defined by Lemma 2.10. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = P_C(x_n - \mu Bx_n), \\ z_n = P_C(y_n - \lambda Ay_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_\theta z_n, \quad n \geq 1, \end{cases} \tag{3.8}$$

where $0 < \lambda \leq 2\alpha$, $0 < \mu \leq 2\beta$ and $T_\theta : C \rightarrow C$ is a mapping defined by $T_\theta x = (1 - \theta)x + \theta Tx$, where $\theta \in (0, \rho)$, $\rho = \min \{1, 2\delta\}$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\alpha_n + \beta_n + \gamma_n = 1$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ solves the variational inequality

$$\langle (I - f)Q(f), Q(f) - p \rangle \leq 0, \quad f \in \Pi_C, p \in F.$$

Proof We note that if X is a Hilbert space, then H is 2-uniformly smooth and $C_q = 1$, therefore we obtain the desired result by Theorem 3.1. □

Remark 3.4 Theorem 3.1 extends and improves Theorem 3.1 of Ceng *et al.* [4] in the following aspects.

- (i) From a Hilbert space to a more general q -uniformly smooth Banach space.
- (ii) From a nonexpansive mapping to a more general strict pseudocontraction.

- (iii) From variational inequality problem (1.7) to more general variational inequality problem (1.12).
- (iv) u is replaced by $f(x_n)$, where f is a contractive mapping.
- (v) The proof method of Theorem 3.1 is more simple than the ones of Ceng *et al.* [4] because we do not need to use the relaxed extragradient method. In fact, in the course of proof of Theorem 3.1, Lemma 2.8 plays an important role. By using Lemma 2.8, we can transform our iterative algorithm (3.1) into more uncomplicated form (3.2). In this way, we simplify the proof of Theorem 3.1.

4 Applications

(I) Application to variational inequality problems for two strict pseudocontractive mappings in Hilbert space.

Definition 4.1 A mapping $T : C \rightarrow C$ is said to be a k -strict pseudocontractive mapping if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{4.1}$$

Let $T : C \rightarrow C$ be k -strict pseudocontractive, we define a mapping $A = I - T : C \rightarrow H$, then A is a $\frac{1-k}{2}$ -inverse-strongly accretive mapping. In fact, from (4.1) we have

$$\|(I - T)x - (I - T)y\|^2 \leq \|x - y\|^2 + k\|Ax - Ay\|^2.$$

On the other hand,

$$\|(I - T)x - (I - T)y\|^2 = \|x - y\|^2 - 2\langle x - y, Ax - Ay \rangle + \|Ax - Ay\|^2.$$

Hence we have

$$\langle x - y, Ax - Ay \rangle \geq \frac{1-k}{2} \|Ax - Ay\|^2. \tag{4.2}$$

This shows that A is a $\frac{1-k}{2}$ -inverse-strongly accretive mapping.

Theorem 4.2 Let C be a closed convex subset of a real Hilbert space H . Let $T_1, T_2 : C \rightarrow C$ be a k_1 -strict pseudocontractive mapping and a k_2 -strict pseudocontractive mapping, respectively. Let $f \in \Pi_C$ with the coefficient $0 < \eta < 1$ and $T : C \rightarrow C$ be a δ -strict pseudocontraction such that $F := F(G) \cap F(T) \neq \emptyset$, where G is defined by Lemma 2.10. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} y_n = (1 - \mu)x_n + \mu T_2 x_n, \\ z_n = (1 - \lambda)y_n + \lambda T_1 y_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_\theta z_n, \quad n \geq 1, \end{cases} \tag{4.3}$$

where $0 < \lambda \leq 1 - k_1$, $0 < \mu \leq 1 - k_2$ and $T_\theta : C \rightarrow C$ is a mapping defined by $T_\theta x = (1 - \theta)x + \theta Tx$, where $\theta \in (0, \rho)$, $\rho = \min\{1, 2\delta\}$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\alpha_n + \beta_n + \gamma_n = 1$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ solves the following variational inequality:

$$\langle (I - f)Q(f), Q(f) - p \rangle \leq 0, \quad f \in \Pi_C, p \in F.$$

Proof Taking $A = I - T_1 : C \rightarrow H$ and $B = I - T_2 : C \rightarrow H$, from (4.2) we know that $A : C \rightarrow H$ is α -inverse-strongly accretive with $\alpha = \frac{1-k_1}{2}$ and $B : C \rightarrow H$ is β -inverse-strongly monotone with $\beta = \frac{1-k_2}{2}$. On the other hand, we have

$$z_n = P_C(y_n - \lambda Ay_n) = P_C((1 - \lambda)y_n + \lambda T_1 y_n) = (1 - \lambda)y_n + \lambda T_1 y_n \in C$$

and

$$y_n = P_C(x_n - \mu Bx_n) = P_C((1 - \mu)x_n + \mu T_2 x_n) = (1 - \mu)x_n + \mu T_2 x_n \in C.$$

The conclusion of Theorem 4.2 can be obtained from Theorem 3.1 immediately. □

(II) Application to equilibrium problems.

Let $\phi : C \times C \rightarrow \mathbb{R}$ be a bifunction, where \mathbb{R} is a set of real numbers. The equilibrium problem for the function ϕ is to find a point $x \in C$ such that

$$\phi(x, y) \geq 0 \quad \text{for all } y \in C. \tag{4.4}$$

The set of solutions of (4.4) is denoted by $EP(\phi)$.

For solving the equilibrium problem, we assume that the bifunction ϕ satisfies the following conditions (see [24]):

- (A1) $\phi(x, x) = 0$ for all $x \in C$;
- (A2) ϕ is monotone, i.e., $\phi(x, y) + \phi(y, x) \leq 0$ for any $x, y \in C$;
- (A3) ϕ is upper-hemicontinuous, i.e., for each $x, y, z \in C$,

$$\limsup_{t \rightarrow 0^+} \phi(tz + (1 - t)x, y) \leq \phi(x, y);$$

- (A4) $\phi(x, \cdot)$ is convex and weakly lower semicontinuous for each $x \in C$.

Lemma 4.1 ([24]) *Let C be a nonempty closed convex subset of H , and let ϕ be a bifunction of $C \times C$ into \mathbb{R} satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$\phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \text{for all } y \in C.$$

Lemma 4.2 ([25]) *Assume that $\phi : C \times C \rightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \rightarrow C$ as follows:*

$$T_r(x) = \left\{ z \in C : \phi(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

for all $z \in H$. Then the following hold:

- (1) T_r is single-valued;
- (2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle$.
This implies that $\|T_r x - T_r y\| \leq \|x - y\|, \forall x, y \in H$, i.e., T_r is nonexpansive;
- (3) $F(T_r) = \text{EP}(\phi), \forall r > 0$;
- (4) $\text{EP}(\phi)$ is a closed and convex set.

Combining Lemma 2.8 and the proof of Theorem 3.1, we obtain the following result.

Theorem 4.3 *Let C be a closed convex subset of a real q -uniformly smooth Banach space X ($q > 1$) which is also a sunny nonexpansive retraction of X . Let $f \in \Pi_C$ with the coefficient $0 < \eta < 1$ and $T, S : C \rightarrow C$ be a δ -strict pseudocontraction and a nonexpansive mapping, respectively, such that $F := F(S) \cap F(T) \neq \emptyset$. For given $x_1 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_\theta S x_n, \quad n \geq 1, \tag{4.5}$$

where $T_\theta : C \rightarrow C$ is a mapping defined by $T_\theta x = (1 - \theta)x + \theta T x$, where $\theta \in (0, \rho)$, $\rho = \min\{1, \{\frac{q\delta}{C}\}^{\frac{1}{q-1}}\}$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$ satisfying the following conditions:

- (i) $\alpha_n + \beta_n + \gamma_n = 1$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

Then $\{x_n\}$ converges strongly to $Q(f)$, where $Q(f) \in F$ solves the following variational inequality:

$$\langle (I - f)Q(f), j_q(Q(f) - p) \rangle \leq 0, \quad f \in \Pi_C, p \in F.$$

Using Theorem 4.3, we can obtain the following strong convergence theorem for the fixed point problem of a strict pseudocontraction and the equilibrium problem in a Hilbert space.

Theorem 4.4 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $\Phi : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying conditions (A1)-(A4). Let $T : C \rightarrow C$ a δ -strict pseudocontraction such that $F = F(T) \cap \text{EP}(\Phi) \neq \emptyset$. Let $f : C \rightarrow C$ be an η -contraction with $\eta \in (0, 1)$. Suppose that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $(0, 1)$ satisfying the following conditions:*

- (i) $\alpha_n + \beta_n + \gamma_n = 1$;
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0, \sum_{n=1}^\infty \alpha_n = \infty$;
- (iii) $0 < \liminf_{n \rightarrow \infty} \beta_n \leq \limsup_{n \rightarrow \infty} \beta_n < 1$.

For any $x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n \in C \text{ such that } \Phi(u_n, y) + \frac{1}{r}(y - u_n, u_n - x_n) \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_\theta u_n, \quad n \geq 1, \end{cases} \tag{4.6}$$

where $T_\theta : C \rightarrow C$ is a mapping defined by $T_\theta x = (1 - \theta)x + \theta T x$, where $\theta \in (0, \rho)$, $\rho = \min\{1, 2\delta\}$. Then $\{x_n\}$ converges strongly to z , where $z \in F$ solves the following variational

inequality:

$$\langle (I-f)z, j_q(z-p) \rangle \leq 0, \quad f \in \Pi_C, p \in F.$$

Proof By Lemma 4.2, we know that T_r is nonexpansive and $F(T_r) = \text{EP}(\phi)$. Hence we obtain the desired result by Theorem 4.3. This proof is complete. \square

Competing interests

The author declares that they have no competing interests.

Acknowledgements

This work was supported by the NSF of China (Nos. 11171172, 11401063), the Specialized Research Fund for the Doctoral Program of Higher Education (20120002110044) and the Natural Science Foundation of Chongqing (cstc2014jcyjA00016).

Received: 29 January 2015 Accepted: 22 April 2015 Published online: 07 May 2015

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