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A new iterative process for a hybrid pair of generalized asymptotically nonexpansive single-valued and generalized nonexpansive multi-valued mappings in Banach spaces

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Abstract

In this paper, we construct an iterative process involving a hybrid pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of generalized nonexpansive multi-valued mappings and prove weak and strong convergence theorems of the proposed iterative process in Banach spaces. Our main results extend and generalize many results in the literature.

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Keywords: fixed point; generalized asymptotically nonexpansive mapping; nonexpansive mapping; Banach space

1 Introduction

Throughout this paper we denote by $\mathbb{N}$ the set of all positive integers. Let X be a Banach space and let D be a nonempty subset of X. Let $CB(D)$ and $KC(D)$ denote the families of nonempty, closed, and bounded subsets and nonempty, compact, and convex subsets of D, respectively. The *Hausdorff metric* on $CB(D)$ is defined by

$$H(A,B) = \max\Bigl\{\sup_{x\in A}\operatorname{dist}(x,B), \sup_{y\in B}\operatorname{dist}(y,A)\Bigr\} \quad \text{for } A,B \in CB(D),$$

where $\operatorname{dist}(x,D) = \inf\{\|x-y\| : y \in D\}$ is the distance from a point x to a subset D. Let t be a single-valued mapping of D into D and T be a multi-valued mapping of D into $CB(D)$. The set of fixed points of t and T will be denoted by $F(t) = \{x \in D : x = tx\}$ and $F(T) = \{x \in D : x \in Tx\}$, respectively. A point x is called a *common fixed point* of t and T if $x = tx \in Tx$.

Definition 1.1 A single-valued mapping $t : D \to D$ is said to be *generalized asymptotically nonexpansive* if there exist sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} s_n = 0$ such that

$$\|t^n x - t^n y\| \le k_n\|x-y\| + s_n,$$

for all $x, y \in D$ and $n \in \mathbb{N}$.

In the case of $s_n = 0$, for all $n \in \mathbb{N}$, a single-valued mapping t is called *an asymptotically nonexpansive mapping*. In particular, if $k_n = 1$ and $s_n = 0$, for all $n \in \mathbb{N}$, a single-valued mapping t reduce to a *nonexpansive mapping*. The fixed point property for generalized asymptotically nonexpansive single-valued mappings can be found in [1]. The following example shows that the fixed point set of a generalized asymptotically nonexpansive mapping is not necessarily closed; see also [2].

Example 1.2 ([1]) Define a single-valued mapping $t : [-\frac{2}{3}, \frac{2}{3}] \to [-\frac{2}{3}, \frac{2}{3}]$ by

$$tx = \begin{cases} x, & \text{if } x \in [-\frac{2}{3}, 0), \\ \frac{16}{81}, & \text{if } x = 0, \\ x^4, & \text{if } x \in (0, \frac{2}{3}]. \end{cases}$$

Then t is generalized asymptotically nonexpansive and $F(t) = [-\frac{2}{3}, 0)$ which is not closed.

Definition 1.3 A multi-valued mapping $T : D \to CB(D)$ is said to be

(i) *nonexpansive* if $H(Tx, Ty) \leq \|x - y\|$, for all $x, y \in D$;

(ii) *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq \|x - p\|$, for all $x \in D$ and $p \in F(T)$.

The study of fixed points for nonexpansive multi-valued mappings using the Hausdorff metric was initiated by Markin [3]. Different iterative processes have been used to approximate fixed points of nonexpansive and quasi-nonexpansive multi-valued mappings; in particular, Sastry and Babu [4] considered Mann and Ishikawa iterates for a multi-valued mapping T with a fixed point p and proved that these iterates converge to a fixed point q of T under certain conditions. Moreover, they illustrated that the fixed point q may be different from p. Later in 2007, Panyanak [5] generalized results of Sastry and Babu [4] to uniformly convex Banach spaces and proved a convergence theorem of Mann iterates for a mapping defined on a noncompact domain. In 2009, Shahzad and Zegeye [6] proved strong convergence theorems for the Ishikawa iteration scheme involving quasi-nonexpansive multi-valued mappings. They constructed an iterative process which removes the restriction of T, namely *end-point condition, i.e.*, $Tp = \{p\}$ for any $p \in F(T)$; see also [7, 8].

In 2011, Garcia-Falset *et al.* [9] introduced a new condition on single-valued mappings, called *condition* (E), which is weaker than nonexpansiveness. Later, Abkar and Eslamian [10] used a modified condition for multi-valued mappings as follows.

Definition 1.4 A multi-valued mapping $T : D \to CB(D)$ is said to satisfy *condition* (E_μ) where $\mu \geq 0$ if for each $x, y \in D$,

$$\operatorname{dist}(x, Ty) \leq \mu \operatorname{dist}(x, Tx) + \|x - y\|.$$

We say that T satisfies *condition* (E) whenever T satisfies (E_μ) for some $\mu \geq 1$.

Remark 1.5 From the above definitions, it is clear that if T is nonexpansive, then T satisfies the condition (E_1).

In 2011, Sokhuma and Kaewkhao [11] introduced the following iterative process for approximating a common fixed point of a pair of a nonexpansive single-valued mapping t and a nonexpansive multi-valued mapping T:

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n z_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n t y_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $x_1 \in D$, $z_n \in Tx_n$, and $0 < a \leq \alpha_n, \beta_n \leq b < 1$. They also proved a strong convergence theorem for the iterative process (1.1) in uniformly convex Banach spaces.

In 2013, Eslamian [12] extended the results of [11, 13] in uniformly convex Banach spaces. He used the following iterative process for a pair of a finite family of asymptotically nonexpansive single-valued mappings $\{t_i\}_{i=1}^N$ and a finite family of quasi-nonexpansive multi-valued mapping $\{T_i\}_{i=1}^N$:

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $x_1 \in D$, $z_n^{(i)} \in T_i x_n$, and $\{\alpha_n^{(i)}\}$, $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, \ldots, N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = \sum_{i=0}^N \beta_n^{(i)} = 1$.

In this paper, motivated by the above results, we propose an iterative process for approximating a common fixed point of a pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of quasi-nonexpansive multi-valued mappings and prove weak and strong convergence theorems of the proposed iterative process in Banach spaces.

2 Preliminaries

A Banach space X is called *uniformly convex* if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for $x, y \in X$ with $\|x\| \leq 1$, $\|y\| \leq 1$, and $\|x-y\| \geq \varepsilon$, $\|x+y\| \leq 2(1-\delta)$ holds. The following result was proved by Xu [14].

Proposition 2.1 *Let X be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\bigl\|\lambda x + (1-\lambda)y\bigr\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g\bigl(\|x-y\|\bigr)$$

for all $x, y \in B_r = \{z \in X : \|z\| \leq r\}$ and $\lambda \in [0,1]$.

A Banach space X is said to satisfy the *Opial property* (see [15]) if it is given that whenever $\{x_n\}$ converges weakly to $x \in X$,

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for each $y \in X$ with $y \neq x$. The examples of Banach spaces which satisfy the Opial property are Hilbert spaces and all $L^p[0, 2\pi]$ with $1 < p \neq 2$ fail to satisfy the Opial property.

The following results are needed for proving our results.

Definition 2.2 (see [2]) Let F be a nonempty subset of a Banach space X and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ is of *monotone type (I) with respect to F* if there exist sequences $\{\delta_n\}$ and $\{\varepsilon_n\}$ of nonnegative real numbers such that $\sum_{n=1}^{\infty}\delta_n < \infty$, $\sum_{n=1}^{\infty}\varepsilon_n < \infty$, and $\|x_{n+1} - p\| \leq (1+\delta_n)\|x_n - p\| + \varepsilon_n$ for all $n \in \mathbb{N}$ and $p \in F$.

Proposition 2.3 (see [2]) *Let F be a nonempty subset of a Banach space X and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is of monotone type (I) with respect to F and $\liminf_{n\to\infty} \operatorname{dist}(x_n, F) = 0$, then $\lim_{n\to\infty} x_n = p$ for some $p \in X$ satisfying $\operatorname{dist}(p, F) = 0$. In particular, if F is closed, then $p \in F$.*

Lemma 2.4 (see [16]) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of nonnegative real numbers satisfy*

$$a_{n+1} \leq (1+c_n)a_n + b_n, \quad \textit{for all } n \in \mathbb{N},$$

where $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$. Then:
(i) $\lim_{n\to\infty} a_n$ *exists.*
(ii) *If* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5 (see [17]) *Let X be a uniformly convex Banach space, let $\{\lambda_n\}$ be a sequence of real numbers such that $0 < a \leq \lambda_n \leq b < 1$, for all $n \in \mathbb{N}$, and let $\{x_n\}$ and $\{y_n\}$ be sequences of X satisfying, for some $r \geq 0$,*
(i) $\limsup_{n\to\infty} \|x_n\| \leq r$,
(ii) $\limsup_{n\to\infty} \|y_n\| \leq r$ *and*
(iii) $\lim_{n\to\infty} \|\lambda_n x_n + (1-\lambda_n)y_n\| = r$.
Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.6 (see [18]) *Let X be a Banach space which satisfies the Opial property and $\{x_n\}$ be a sequence in X. Let $u, v \in X$ be such that $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - v\|$ exist. If $\{x_{n_i}\}$ and $\{x_{n_j}\}$ are subsequences of $\{x_n\}$ which converge weakly to u and v, respectively, then $u = v$.*

3 Main results

In this section, we prove weak and strong convergence theorems of the proposed iterative process in Banach spaces. We first note that if $\{t_i\}_{i=1}^{N}$ is a finite family of generalized asymptotically nonexpansive single-valued mappings of D into itself, where D is a nonempty convex subset of a Banach space X. Then we have $\|t_i^n x - t_i^n y\| \leq k_n^{(i)}\|x - y\| + s_n^{(i)}$, for all $x, y \in D$ and all $i = 1, 2, \ldots, N$, where $\{k_n^{(i)}\} \subset [1, \infty)$ and $\{s_n^{(i)}\} \subset [0, \infty)$ with $\lim_{n\to\infty} k_n^{(i)} = 1$ and $\lim_{n\to\infty} s_n^{(i)} = 0$. Put $k_n = \max_{1\leq i\leq N}\{k_n^{(i)}\}$ and $s_n = \max_{1\leq i\leq N}\{s_n^{(i)}\}$. It is clear that $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$ and

$$\left\| t_i^n x - t_i^n y \right\| \leq k_n \|x - y\| + s_n$$

for all $x, y \in D$, $i = 1, 2, \ldots, N$, and all $n \in \mathbb{N}$.

In order to prove our main results, the following lemma is needed.

Lemma 3.1 *Let D be a nonempty, closed, and convex subset of a Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^\infty (k_n - 1) < \infty$ and $\sum_{n=1}^\infty s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \ldots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, \ldots, N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$.

Proof Let $p \in \mathcal{F}$, for $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} \left\| t_i^n y_n - p \right\| \\
&\leq \alpha_n^{(0)} \|x_n - p\| + \sum_{i=1}^N \alpha_n^{(i)} \bigl(k_n \|y_n - p\| + s_n \bigr) \\
&= \alpha_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \|y_n - p\| + s_n \sum_{i=1}^N \alpha_n^{(i)} \\
&\leq \alpha_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \|y_n - p\| + s_n \\
&\leq \alpha_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \left(\beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^N \beta_n^{(i)} \left\| z_n^{(i)} - p \right\| \right) + s_n \\
&= \left(\alpha_n^{(0)} + k_n \beta_n^{(0)} \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \sum_{i=1}^N \beta_n^{(i)} \left\| z_n^{(i)} - p \right\| + s_n \\
&= \left(\alpha_n^{(0)} + k_n \beta_n^{(0)} \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \sum_{i=1}^N \beta_n^{(i)} \operatorname{dist}\bigl(z_n^{(i)}, T_i p \bigr) + s_n \\
&\leq \left(\alpha_n^{(0)} + k_n \beta_n^{(0)} \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \sum_{i=1}^N \beta_n^{(i)} H(T_i x_n, T_i p) + s_n \\
&\leq \left(\alpha_n^{(0)} + k_n \beta_n^{(0)} \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + k_n \sum_{i=1}^N \alpha_n^{(i)} \sum_{i=1}^N \beta_n^{(i)} \|x_n - p\| + s_n \\
&= \left(\alpha_n^{(0)} + k_n \sum_{i=1}^N \alpha_n^{(i)} \right) \|x_n - p\| + s_n \\
&\leq k_n \|x_n - p\| + s_n \\
&= \bigl(1 + (k_n - 1) \bigr) \|x_n - p\| + s_n.
\end{aligned}$$

By Lemma 2.4, $\sum_{n=1}^\infty (k_n - 1) < \infty$ and $\sum_{n=1}^\infty s_n < \infty$, we conclude that $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in \mathcal{F}$. □

Theorem 3.2 *Let D be a nonempty, closed, and convex subset of a Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty closed and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1,2,\ldots,N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1,2,\ldots,N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0$.

Proof The necessity is obvious and thus we prove only the sufficiency. Suppose that $\liminf_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0$. In the proof of Lemma 3.1, we see that the sequence $\{x_n\}$ is of monotone type (I) with respect to $\mathcal{F}$. It follows by Proposition 2.3 that $\{x_n\}$ converges to a point in $\mathcal{F}$. □

The closedness of $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ can be dropped if t_i is asymptotically nonexpansive for all $i = 1,2,\ldots,N$. Then the following corollary is obtained directly from Theorem 3.2.

Corollary 3.3 *Let D be a nonempty, closed, and convex subset of a Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of asymptotically nonexpansive single-valued mappings of D into itself with a sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1,2,\ldots,N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1,2,\ldots,N$ such that $\sum_{i=0}^N \alpha_n^{(i)} = 1$ and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$ if and only if $\liminf_{n\to\infty} \operatorname{dist}(x_n, \mathcal{F}) = 0$.

Recall that a mapping $t : D \to D$ is called *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that $\|t^n x - t^n y\| \leq L\|x - y\|$ for all $x, y \in D$ and $n \in \mathbb{N}$. Next, we prove a strong convergence theorem in a uniformly convex Banach space.

Lemma 3.4 *Let D be a nonempty, closed, and convex subset of a uniformly convex Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$. Assume that*

$\mathcal{F} = \bigcap_{i=1}^{N} F(t_i) \cap \bigcap_{i=1}^{N} F(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \dots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^{N} \beta_n^{(i)} z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^{N} \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, \dots, N$ such that $0 < a \le \alpha_n^{(i)}, \beta_n^{(i)} \le b < 1$, $\sum_{i=0}^{N} \alpha_n^{(i)} = 1$, and $\sum_{i=0}^{N} \beta_n^{(i)} = 1$. Then we have the following:

(i) $\lim_{n\to\infty} \|x_n - z_n^{(i)}\| = 0$ *for all* $i = 1, 2, \dots, N$;

(ii) $\lim_{n\to\infty} \|x_n - t_i x_n\| = 0$ *for all* $i = 1, 2, \dots, N$.

Proof (i) By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists. Put $\lim_{n\to\infty} \|x_n - p\| = c$. By the definition of $\{x_n\}$, we have

$$\begin{aligned} \|t_i^n y_n - p\| &\le k_n \|y_n - p\| + s_n \\ &\le k_n \left(\beta_n^{(0)} \|x_n - p\| + \sum_{i=1}^{N} \beta_n^{(i)} \|z_n^{(i)} - p\| \right) + s_n \\ &= k_n \beta_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^{N} \beta_n^{(i)} \|z_n^{(i)} - p\| + s_n \\ &= k_n \beta_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^{N} \beta_n^{(i)} \operatorname{dist}(z_n^{(i)}, T_i p) + s_n \\ &\le k_n \beta_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^{N} \beta_n^{(i)} H(T_i x_n, T_i p) + s_n \\ &\le k_n \beta_n^{(0)} \|x_n - p\| + k_n \sum_{i=1}^{N} \beta_n^{(i)} \|x_n - p\| + s_n \\ &= k_n \left(\beta_n^{(0)} + \sum_{i=1}^{N} \beta_n^{(i)} \right) \|x_n - p\| + s_n \\ &= k_n \|x_n - p\| + s_n. \end{aligned}$$

Then we have

$$\limsup_{n\to\infty} \|t_i^n y_n - p\| \le \limsup_{n\to\infty} \bigl(k_n \|y_n - p\| + s_n\bigr) \le \limsup_{n\to\infty} \bigl(k_n \|x_n - p\| + s_n\bigr).$$

By $\lim_{n\to\infty} k_n = 1$ and $\lim_{n\to\infty} s_n = 0$, we have

$$\limsup_{n\to\infty} \|t_i^n y_n - p\| \le \limsup_{n\to\infty} \|y_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = c. \tag{3.1}$$

Since $c = \lim_{n\to\infty} \|x_{n+1} - p\| = \lim_{n\to\infty} \|\alpha_n^{(0)}(x_n - p) + \sum_{i=1}^{N} \alpha_n^{(i)}(t_i^n y_n - p)\|$, it follows by Lemma 2.5 that

$$\lim_{n\to\infty} \|x_n - t_i^n y_n\| = 0 \quad \text{for all } i = 1, 2, \dots, N. \tag{3.2}$$

Consider

$$\begin{aligned}\|x_{n+1}-p\| &\leq \alpha_n^{(0)}\|x_n-p\| + \sum_{i=1}^{N}\alpha_n^{(i)}\|t_i^n y_n - p\| \\ &= \left(1-\sum_{i=1}^{N}\alpha_n^{(i)}\right)\|x_n-p\| + \sum_{i=1}^{N}\alpha_n^{(i)}\|t_i^n y_n - p\| \\ &\leq \left(1-\sum_{i=1}^{N}\alpha_n^{(i)}\right)\|x_n-p\| + \sum_{i=1}^{N}\alpha_n^{(i)}\big(k_n\|y_n-p\|+s_n\big).\end{aligned}$$

This implies that

$$\|x_{n+1}-p\| - \|x_n-p\| \leq \sum_{i=1}^{N}\alpha_n^{(i)}\big(k_n\|y_n-p\| - \|x_n-p\| + s_n\big).$$

Therefore,

$$\begin{aligned}\frac{\|x_{n+1}-p\|-\|x_n-p\|}{bN} + \|x_n-p\| &\leq \frac{\|x_{n+1}-p\|-\|x_n-p\|}{\sum_{i=1}^{N}\alpha_n^{(i)}} + \|x_n-p\| \\ &\leq k_n\|y_n-p\| + s_n.\end{aligned}$$

By (3.1), we obtain

$$\begin{aligned}c &= \liminf_{n\to\infty}\left(\frac{\|x_{n+1}-p\|-\|x_n-p\|}{bN} + \|x_n-p\|\right) \\ &\leq \liminf_{n\to\infty}\big(k_n\|y_n-p\|+s_n\big) \\ &= \liminf_{n\to\infty}\|y_n-p\| \\ &\leq \limsup_{n\to\infty}\|y_n-p\| \leq c.\end{aligned}$$

Thus,

$$c = \lim_{n\to\infty}\|y_n-p\| = \lim_{n\to\infty}\left\|\beta_n^{(0)}(x_n-p) + \sum_{i=1}^{N}\beta_n^{(i)}\big(z_n^{(i)}-p\big)\right\|.$$

Since

$$\big\|z_n^{(i)}-p\big\| = \operatorname{dist}\big(z_n^{(i)}, T_i p\big) \leq H(T_i x_n, T_i p) \leq \|x_n-p\|,$$

it implies that

$$\limsup_{n\to\infty}\big\|z_n^{(i)}-p\big\| \leq \limsup_{n\to\infty}\|x_n-p\| = c.$$

Hence, by Lemma 2.5, we have

$$\lim_{n\to\infty}\big\|x_n - z_n^{(i)}\big\| = 0 \quad \text{for all } i = 1,2,\ldots,N.$$

(ii) Since t_i is generalized asymptotically nonexpansive, for all $i = 1, 2, \dots, N$, we get

$$\left\| t_i^n x_n - x_n \right\| \le \left\| t_i^n x_n - t_i^n y_n \right\| + \left\| t_i^n y_n - x_n \right\| \le k_n \|x_n - y_n\| + s_n + \left\| t_i^n y_n - x_n \right\|.$$

By the definition of $\{x_n\}$, we have $y_n - x_n = \sum_{i=1}^N \beta_n^{(i)}(z_n^{(i)} - x_n)$. This implies that

$$\begin{aligned} \left\| t_i^n x_n - x_n \right\| &\le k_n \sum_{i=1}^N \beta_n^{(i)} \left\| z_n^{(i)} - x_n \right\| + \left\| t_i^n y_n - x_n \right\| + s_n \\ &\le k_n \left\| z_n^{(i)} - x_n \right\| + \left\| t_i^n y_n - x_n \right\| + s_n. \end{aligned}$$

Then, by (i) and (3.2), we get

$$\lim_{n\to\infty} \left\| x_n - t_i^n x_n \right\| = 0 \quad \text{for all } i = 1, 2, \dots, N. \tag{3.3}$$

For $i = 1, 2, \dots, N$, we have

$$\begin{aligned} \|x_n - t_i x_n\| &\le \|x_n - x_{n+1}\| + \left\| x_{n+1} - t_i^{n+1} x_{n+1} \right\| + \left\| t_i^{n+1} x_{n+1} - t_i^{n+1} x_n \right\| + \left\| t_i^{n+1} x_n - t_i x_n \right\| \\ &\le (1 + L)\|x_n - x_{n+1}\| + \left\| x_{n+1} - t_i^{n+1} x_{n+1} \right\| + L \left\| t_i^n x_n - x_n \right\| \\ &\le (1 + L) \sum_{i=1}^N \alpha_n^{(i)} \left\| x_n - t_i^n y_n \right\| + \left\| x_{n+1} - t_i^{n+1} x_{n+1} \right\| + L \left\| t_i^n x_n - x_n \right\|. \end{aligned}$$

By (3.2) and (3.3), we conclude that $\lim_{n\to\infty} \|x_n - t_i x_n\| = 0$ for all $i = 1, 2, \dots, N$. □

Theorem 3.5 *Let D be a nonempty, compact, and convex subset of a uniformly convex Banach space X. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1, \infty)$ and $\{s_n\} \subset [0, \infty)$ such that $\sum_{n=1}^\infty (k_n - 1) < \infty$ and $\sum_{n=1}^\infty s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $CB(D)$ satisfying condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \dots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, \quad z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, \dots, N$ such that $0 < a \le \alpha_n^{(i)}, \beta_n^{(i)} \le b < 1$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$, and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges strongly to a point in $\mathcal{F}$.

Proof By Lemma 3.1, we have $\{x_n\}$ is bounded. Since D is compact, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging strongly to $p \in D$. By condition (E), there exists $\mu \ge 1$ such that for $i = 1, 2, \dots, N$,

$$\begin{aligned} \operatorname{dist}(p, T_i p) &\le \|p - x_{n_j}\| + \operatorname{dist}(x_{n_j}, T_i p) \\ &\le \|x_{n_j} - p\| + \mu \operatorname{dist}(x_{n_j}, T_i x_{n_j}) + \|x_{n_j} - p\| \end{aligned}$$

$$= 2\|x_{n_j} - p\| + \mu \operatorname{dist}(x_{n_j}, T_i x_{n_j})$$
$$\leq 2\|x_{n_j} - p\| + \mu \left\|x_{n_j} - z_{n_j}^{(i)}\right\|.$$

Then, by Lemma 3.4(i), we have $p \in T_i p$ for all $i = 1, 2, \ldots, N$. So $p \in \bigcap_{i=1}^{N} F(T_i)$.

Since t_i is uniformly L-Lipschitzian, for all $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}\|t_i p - p\| &\leq \|t_i p - t_i x_{n_j}\| + \|t_i x_{n_j} - x_{n_j}\| + \|x_{n_j} - p\| \\ &\leq (L+1)\|x_{n_j} - p\| + \|t_i x_{n_j} - x_{n_j}\|.\end{aligned}$$

By Lemma 3.4(ii), it implies that $t_i p = p$ for all $i = 1, 2, \ldots, N$. Thus, $p \in \bigcap_{i=1}^{N} F(t_i)$. Therefore, $p \in \mathcal{F}$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we get $\lim_{n\to\infty} \|x_n - p\| = \lim_{j\to\infty} \|x_{n_j} - p\| = 0$. This shows that $\{x_n\}$ converges strongly to a point in $\mathcal{F}$. □

Next, we give a numerical example to support Theorem 3.5.

Example 3.6 Let $\mathbb{R}$ be the real line with the usual norm $|\cdot|$ and let $D = [0, 3]$. Define two single-valued mappings t_1 and t_2 on D as follows:

$$t_1 x = \sin x, \qquad t_2 x = x.$$

Also we define two multi-valued mappings T_1 and T_2 on D as follows:

$$T_1 x = \begin{cases} [0, \frac{x}{3}], & x \neq 3; \\ \{1\}, & x = 3; \end{cases} \qquad T_2 x = \left[\frac{x}{4}, \frac{x}{2}\right].$$

Let $\{x_n\}$ and $\{y_n\}$ be generated by

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^{2} \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^{2} \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases} \tag{3.4}$$

where $\alpha_n^{(0)} = \frac{3n+4}{10n}$, $\alpha_n^{(1)} = \frac{2n-1}{5n}$, $\alpha_n^{(2)} = \frac{3n-2}{10n}$, $\beta_n^{(0)} = \frac{15n+7}{60n}$, $\beta_n^{(1)} = \frac{5n-1}{20n}$, $\beta_n^{(2)} = \frac{15n-2}{30n}$, for all $n \in \mathbb{N}$. Then the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to 0, where $\{0\} = \bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(T_i)$.

Solution *It is shown in* [19] *that both t_1 and t_2 are generalized asymptotically nonexpansive single-valued mappings. Moreover, they are uniformly L-Lipschitzian mappings and $\bigcap_{i=1}^{2} F(t_i) = \{0\}$. It is easy to see that both T_1 and T_2 are quasi-nonexpansive multi-valued mappings satisfying condition* (E) *and $\bigcap_{i=1}^{2} F(T_i) = \{0\}$. Thus, $\bigcap_{i=1}^{2} F(t_i) \cap \bigcap_{i=1}^{2} F(T_i) = \{0\}$. For every $n \in \mathbb{N}$, $\alpha_n^{(0)} = \frac{3n+4}{10n}$, $\alpha_n^{(1)} = \frac{2n-1}{5n}$, $\alpha_n^{(2)} = \frac{3n-2}{10n}$, $\beta_n^{(0)} = \frac{15n+7}{60n}$, $\beta_n^{(1)} = \frac{5n-1}{20n}$, $\beta_n^{(2)} = \frac{15n-2}{30n}$. Then the sequences $\{\alpha_n^{(0)}\}$, $\{\alpha_n^{(1)}\}$, $\{\alpha_n^{(2)}\}$, $\{\beta_n^{(0)}\}$, $\{\beta_n^{(1)}\}$, and $\{\beta_n^{(2)}\}$ satisfy all the conditions of Theorem* 3.5. *Put $z_n^{(1)} = \frac{x_n}{2}$ and $z_n^{(2)} = \frac{x_n}{3}$ for all $n \in \mathbb{N}$. Then the algorithm* (3.4) *becomes*

$$\begin{cases} y_n = (\frac{13}{24} + \frac{5}{72n}) x_n, \\ x_{n+1} = (\frac{37}{80} + \frac{5}{16n} - \frac{1}{72n^2}) x_n + (\frac{2n-1}{5n}) t_1^n y_n, & n \in \mathbb{N}. \end{cases} \tag{3.5}$$

Using the algorithm (3.5) *with the initial point $x_1 = 2.5$, we have numerical results in Table* 1.

Table 1 The values of the sequences $\{x_n\}$ and $\{y_n\}$ in Example 3.6

n	x_n	y_n
1	2.5000000	1.5277778
2	2.1025927	1.2119111
3	1.5352877	0.8671533
4	1.0799923	0.6037457
5	0.7544605	0.4191447
⋮	⋮	⋮
21	0.0023377	0.0012740
⋮	⋮	⋮
38	0.0000040	0.0000022
39	0.0000027	0.0000015
40	0.0000019	0.0000010
41	0.0000013	0.0000007
42	0.0000009	0.0000005

Finally, we prove a weak convergence theorem in uniformly convex Banach spaces.

Theorem 3.7 *Let D be a nonempty, closed, and convex subset of a uniformly convex Banach space X with the Opial property. Let $\{t_i\}_{i=1}^N$ be a finite family of uniformly L-Lipschitzian and generalized asymptotically nonexpansive single-valued mappings of D into itself with sequences $\{k_n\} \subset [1,\infty)$ and $\{s_n\} \subset [0,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$. Let $\{T_i\}_{i=1}^N$ be a finite family of quasi-nonexpansive multi-valued mappings of D into $KC(D)$ satisfying the condition (E). Assume that $\mathcal{F} = \bigcap_{i=1}^N F(t_i) \cap \bigcap_{i=1}^N F(T_i)$ is nonempty and $T_i p = \{p\}$ for all $p \in \mathcal{F}$ and $i = 1, 2, \ldots, N$. Let $x_1 \in D$ and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = \beta_n^{(0)} x_n + \sum_{i=1}^N \beta_n^{(i)} z_n^{(i)}, & z_n^{(i)} \in T_i x_n, \\ x_{n+1} = \alpha_n^{(0)} x_n + \sum_{i=1}^N \alpha_n^{(i)} t_i^n y_n, & n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ are sequences in $[0,1]$ for all $i = 1, 2, \ldots, N$ such that $0 < a \le \alpha_n^{(i)}, \beta_n^{(i)} \le b < 1$, $\sum_{i=0}^N \alpha_n^{(i)} = 1$, and $\sum_{i=0}^N \beta_n^{(i)} = 1$. Then the sequence $\{x_n\}$ converges weakly to a point in $\mathcal{F}$.

Proof By Lemma 3.1, $\{x_n\}$ is bounded. Since X is uniformly convex, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ converging weakly to $p \in D$. By Lemma 3.4, we have $\lim_{j\to\infty} \|x_{n_j} - z_{n_j}^{(i)}\| = 0$ and $\lim_{j\to\infty} \|x_{n_j} - t_i x_{n_j}\| = 0$ for all $i = 1, 2, \ldots, N$. We will show that $p \in \mathcal{F}$. Since $T_1 p$ is compact, for all $j \in \mathbb{N}$, we can choose $w_{n_j} \in Tp$ such that $\|x_{n_j} - w_{n_j}\| = \operatorname{dist}(x_{n_j}, T_1 p)$ and the sequence $\{w_{n_j}\}$ has a convergent subsequence $\{w_{n_k}\}$ with $\lim_{k\to\infty} w_{n_k} = w \in T_1 p$. By condition (E), we have

$$\operatorname{dist}(x_{n_k}, T_1 p) \le \mu \operatorname{dist}(x_{n_k}, T_1 x_{n_k}) + \|x_{n_k} - p\|.$$

Then we have

$$\begin{aligned} \|x_{n_k} - w\| &\le \|x_{n_k} - w_{n_k}\| + \|w_{n_k} - w\| \\ &= \operatorname{dist}(x_{n_k}, T_1 p) + \|w_{n_k} - w\| \\ &\le \mu \operatorname{dist}(x_{n_k}, T_1 x_{n_k}) + \|x_{n_k} - p\| + \|w_{n_k} - w\| \\ &\le \mu \left\| x_{n_k} - z_{n_k}^{(i)} \right\| + \|x_{n_k} - p\| + \|w_{n_k} - w\|. \end{aligned}$$

This implies that

$$\limsup_{k\to\infty}\|x_{n_k}-w\|\leq\limsup_{k\to\infty}\|x_{n_k}-p\|.$$

From the Opial property, we have $p = w \in T_1p$. Similarly, it can be shown that $p \in T_ip$ for all $i = 2,\dots,N$. Thus, $p \in \bigcap_{i=1}^N F(T_i)$.

Next, by mathematical induction, we can prove that, for $i = 1,2,\dots,N$,

$$\lim_{j\to\infty}\left\|x_{n_j}-t_i^m x_{n_j}\right\| = 0 \quad \text{for each } m\in\mathbb{N}. \tag{3.6}$$

Indeed, it is obvious that the conclusion it true for $m = 1$. Suppose the conclusion holds for $m \geq 1$. Since t_i is uniformly L-Lipschitzian, we have

$$\begin{aligned}\left\|x_{n_j}-t_i^{m+1}x_{n_j}\right\| &\leq \left\|x_{n_j}-t_i^m x_{n_j}\right\| + \left\|t_i^m x_{n_j}-t_i^{m+1}x_{n_j}\right\| \\ &\leq \left\|x_{n_j}-t_i^m x_{n_j}\right\| + L\|x_{n_j}-t_i x_{n_j}\|.\end{aligned}$$

This shows that $\lim_{j\to\infty}\|x_{n_j}-t_i^{m+1}x_{n_j}\| = 0$ for all $i = 1,2,\dots,N$. Hence, (3.6) holds.

From (3.6), we have for each $x \in D$, $m \in \mathbb{N}$ and $i = 1,2,\dots,N$,

$$\limsup_{j\to\infty}\|x_{n_j}-x\| = \limsup_{j\to\infty}\left\|t_i^m x_{n_j}-x\right\|. \tag{3.7}$$

Since t_i is generalized asymptotically nonexpansive, we get

$$\limsup_{j\to\infty}\left\|t_i^m x_n - t_i^m p\right\| \leq \limsup_{j\to\infty}\bigl(k_m\|x_{n_j}-p\| + s_m\bigr).$$

Then we have

$$\limsup_{m\to\infty}\Bigl(\limsup_{j\to\infty}\left\|t_i^m x_{n_j}-t_i^m p\right\|\Bigr) \leq \limsup_{j\to\infty}\|x_{n_j}-p\|. \tag{3.8}$$

By Proposition 2.1, we have

$$\begin{aligned}\left\|x_{n_j}-\frac{p+t_i^m p}{2}\right\|^2 &= \left\|\frac{1}{2}(x_{n_j}-p)+\frac{1}{2}\bigl(x_{n_j}-t_i^m p\bigr)\right\|^2 \\ &\leq \frac{1}{2}\|x_{n_j}-p\|^2 + \frac{1}{2}\left\|x_{n_j}-t_i^m p\right\|^2 - \frac{1}{4}g\bigl(\left\|p-t_i^m p\right\|\bigr).\end{aligned}$$

It implies that

$$\begin{aligned}\limsup_{j\to\infty}\left\|x_{n_j}-\frac{p+t_i^m p}{2}\right\|^2 &\leq \frac{1}{2}\limsup_{j\to\infty}\|x_{n_j}-p\|^2 + \frac{1}{2}\limsup_{j\to\infty}\left\|x_{n_j}-t_i^m p\right\|^2 \\ &\quad - \frac{1}{4}g\bigl(\left\|p-t_i^m p\right\|\bigr).\end{aligned} \tag{3.9}$$

By the Opial property and $\{x_{n_j}\}$ converging weakly to p, we obtain

$$\limsup_{j\to\infty}\|x_{n_j}-p\|^2 \leq \limsup_{j\to\infty}\left\|x_{n_j}-\frac{p+t_i^m p}{2}\right\|^2.$$

Then, by (3.9), we have

$$g\big(\|p - t^m p\|\big) \leq 2\limsup_{j\to\infty}\big\|x_{n_j} - t_i^m p\big\|^2 - 2\limsup_{j\to\infty}\|x_{n_j} - p\|^2. \tag{3.10}$$

It implies by (3.7), (3.8), and (3.10) that

$$\begin{aligned}\limsup_{m\to\infty} g\big(\|p - t_i^m p\|\big) &\leq 2\limsup_{m\to\infty}\Big(\limsup_{j\to\infty}\big\|x_{n_j} - t_i^m p\big\|^2\Big) - 2\limsup_{j\to\infty}\|x_{n_j} - p\|^2 \\ &\leq 0.\end{aligned}$$

This shows that $\lim_{m\to\infty} g(\|p - t_i^m p\|) = 0$ for all $i = 1, 2, \ldots, N$. Then the properties of g yield $\lim_{m\to\infty}\|p - t_i^m p\| = 0$ for all $i = 1, 2, \ldots, N$. So we have

$$\begin{aligned}\|t_i p - p\| &\leq \big\|t_i p - t_i^{m+1} p\big\| + \big\|t_i^{m+1} p - p\big\| \\ &\leq L\big\|p - t_i^m p\big\| + \big\|t_i^{m+1} p - p\big\| \to 0 \quad \text{as } m \to \infty.\end{aligned}$$

This implies that $t_i p = p$ for all $i = 1, 2, \ldots, N$. Thus, $p \in \bigcap_{i=1}^N F(t_i)$.

Hence, we obtain $p \in \mathcal{F}$.

Finally, we show that $\{x_n\}$ converges weakly to p. To show this, suppose not. Then there exists a subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $\{x_{n_l}\}$ converges weakly to $q \in D$ and $q \neq p$. By the same method as given above, we can prove that $q \in \mathcal{F}$. By Lemma 3.1, $\lim_{n\to\infty}\|x_n - p\|$ and $\lim_{n\to\infty}\|x_n - q\|$ exist. It follows by Lemma 2.6 that $q = p$. Thus, $\{x_n\}$ converges weakly to a point in $\mathcal{F}$. □

Remark 3.8 Theorem 3.5 extends and generalizes the results of Sokhuma and Kaewkhao [11] to a pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of quasi-nonexpansive multi-valued mappings satisfying condition (E). Theorems 3.5 and 3.7 extend and generalize the results of Eslamian [12] and Eslamian and Abkar [13] to a pair of a finite family of generalized asymptotically nonexpansive single-valued mappings and a finite family of quasi-nonexpansive multi-valued mappings satisfying condition (E).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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