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Common best proximity points theorem for four mappings in metric-type spaces

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Abstract

In this article, we first give an existence and uniqueness common best proximity points theorem for four mappings in a metric-type space (X, D, K) such that D is not necessarily continuous. An example is also given to support our main result. We also discuss the unique common fixed point existence result of four mappings defined on such a metric space.

Keywords: common best proximity point; metric-type space; common fixed point

1 Introduction and preliminary

Fixed point theory is essential for solving various equations of the form $Tx = x$ for self-mappings T defined on subsets of metric spaces or normed linear spaces. Given non-void subsets A and B of a metric space and a non-self-mapping $T : A \to B$, the equation $Tx = x$ does not necessarily have a solution, which is known as a fixed point of the mapping T. However, in such conditions, it may be considered to determine an element x for which the error $d(x, Tx)$ is minimum, in which case x and Tx are in close proximity to each other. It is remarked that best proximity point theorems are relevant to this end. A best proximity point theorem provides sufficient conditions that confirm the existence of an optimal solution to the problem of globally minimizing the error $d(x, Tx)$, and hence the existence of a complete approximate solution to the equation $Tx = x$. In fact, with respect to the fact that $d(x, Tx) \geq d(A, B)$ for all x, a best proximity point theorem requires the global minimum of the error $d(x, Tx)$ to be the least possible value $d(A, B)$. Eventually, a best proximity point theorem offers sufficient conditions for the existence of an element x, called a best proximity point of the mapping T, satisfying the condition that $d(x, Tx) = d(A, B)$. Moreover, it is interesting to observe that best proximity theorems also appear as a natural generalization of fixed point theorems, for a best proximity point reduces to a fixed point if the mapping under consideration is a self-mapping.

A study of several variants of contractions for the existence of a best proximity point can be found in [1–7]. Best proximity point theorems for multivalued mappings are available in [8–14]. Eldred *et al.* [15] have established a best proximity point theorem for relatively non-expansive mappings. Further, Anuradha and Veeramani have investigated best proximity point theorems for proximal pointwise contraction mappings [16].

On the other hand, Khamsi and Hussain [17] generalized the definition of a metric and defined the metric-type as follows.

Definition 1.1 [17] Let X be a non-empty set, $K \geq 1$ be a real number, and let the function $D : X \times X \to \mathbb{R}$ satisfy the following properties:

(i) $D(x,y) = 0$ if and only if $x = y$;

(ii) $D(x,y) = D(y,x)$ for all $x, y \in X$;

(iii) $D(x,z) \leq K(D(x,y) + D(y,z))$ for all $x, y \in X$.

Then (X, D, K) is called a metric-type space.

Obviously, for $K = 1$, a metric-type space is simply a metric space.

Afterward, other authors proved fixed point theorems in metric-type space [18–20].

Given two non-empty subsets A and B of a metric-type space (X, D, K), the following notions and notations are used in the sequel.

$$D(A,B) = \inf\{d(x,y) : x \in A, y \in B\};$$

$$A_0 = \{x \in A : D(x,y) = D(A,B) \text{ for some } y \in B\};$$

$$B_0 = \{y \in B : D(x,y) = D(A,B) \text{ for some } x \in A\}.$$

This study focuses upon resolving a more general problem as regards the existence of common best proximity points for pairs of non-self-mappings in metric-type space. As a result, the finding of this study verifies a common global minimal solution to the problem of minimizing the real valued multi-objective functions $x \to d(x, Sx)$ and $x \to d(x, Tx)$, which in turn gives rise to a common optimal approximate solution of the fixed point equations $Sx = x$ and $Tx = x$, where D is a metric-type space and the non-self-mappings $S : A \to B$ and $T : A \to B$ satisfy a contraction-like condition. Our best proximity point theorem generalizes a result due to Sadiq Basha [21]. Further, a common fixed point theorem for commuting self-mappings is a special case of our common best proximity point theorem. Now, we review some definitions used throughout the paper.

Definition 1.2 An element $x \in A$ is said to be a common best proximity point of the non-self-mappings $f_1, f_2, \ldots, f_n : A \to B$ if it satisfies the condition that

$$D(x, f_1 x) = D(x, f_2 x) = \cdots = D(x, f_n x) = D(A,B).$$

Definition 1.3 The mappings $S : A \to B$ and $T : A \to B$ are said to be commute proximally if they satisfy the condition that

$$\bigl[D(u, Sx) = D(v, Tx) = D(A,B)\bigr] \quad \Rightarrow \quad Sv = Tu.$$

Definition 1.4 If $A_0 \neq \varnothing$ then the pair (A,B) is said to have P-property if and only if for any $x_1, x_2 \in A_0$ and $y_2, y_2 \in B_0$

$$\begin{cases} D(x_1, y_1) = D(A,B), \\ D(x_2, y_2) = D(A,B) \end{cases} \implies \quad D(x_1, x_2) = D(y_1, y_2).$$

2 Main result

We begin our study with the following definition.

Definition 2.1 Let A and B be two non-empty subsets of a metric-type space (X, D, K). Non-self-mappings $f, g, S, T : A \to B$ are said to satisfy a K-contractive condition if there exists a non-negative number $\alpha < \frac{1}{K}$ such that for each $x, y \in A$

$$D(fx, gy) \leq \alpha \max\left\{D(Sx, Ty), D(fx, Sx), D(Ty, gy), \frac{1}{2K}\left[D(Sx, gy) + D(fx, Ty)\right]\right\}.$$

Theorem 2.2 *Let A and B be non-empty subsets of a complete metric-type space (X, D, K). Moreover, assume that A_0 and B_0 are non-empty and A_0 is closed. Let the non-self-mappings $f, g, S, T : A \to B$ satisfy the following conditions*:

(i) $\{f, S\}$ *and* $\{g, T\}$ *commute proximally*;

(ii) *the pair* (A, B) *has the P-property*;

(iii) f, g, S *and* T *are continuous*;

(iv) f, g, S, *and* T *satisfy the* K*-contractive condition*;

(v) $f(A_0) \subseteq T(A_0)$, $g(A_0) \subseteq S(A_0)$ *and* $g(A_0) \subseteq B_0$, $f(A_0) \subseteq B_0$.

Then f, g, S, *and* T *have a unique common best proximity point.*

Proof Fix x_0 in A_0, since $f(A_0) \subseteq T(A_0)$, then there exists an element x_1 in A_0 such that $f(x_0) = T(x_1)$. Similarly, a point $x_2 \in A_0$ can be chosen such that $g(x_1) = S(x_2)$. Continuing this process, we obtain a sequence $\{x_n\} \in A_0$ such that $f(x_{2n}) = T(x_{2n+1})$ and $g(x_{2n+1}) = S(x_{2n+2})$.

Since $f(A_0) \subseteq B_0$ and $g(A_0) \subseteq B_0$, there exists $\{u_n\} \in A_0$ such that

$$D\bigl(u_{2n}, f(x_{2n})\bigr) = D(A, B) \quad \text{and} \quad D\bigl(u_{2n+1}, g(x_{2n+1})\bigr) = D(A, B). \tag{1}$$

Since the pair (A, B) has the P-property, by (1) we have

$$\begin{aligned} D(u_{2n}, u_{2n+1}) &= D(fx_{2n}, gx_{2n+1}) \\ &\leq \alpha \max\Bigl\{D(Sx_{2n}, Tx_{2n+1}), D(fx_{2n}, Sx_{2n}), D(Tx_{2n+1}, gx_{2n+1}), \\ &\quad \frac{1}{2K}\left[D(Sx_{2n}, gx_{2n+1}) + D(fx_{2n}, Tx_{2n+1})\right]\Bigr\} \\ &\leq \alpha \max\Bigl\{D(u_{2n-1}, u_{2n}), D(u_{2n}, u_{2n-1}), D(u_{2n}, u_{2n+1}), \\ &\quad \frac{1}{2K}\left[D(u_{2n-1}, u_{2n+1}) + D(u_{2n}, u_{2n})\right]\Bigr\}, \end{aligned}$$

thus (note that $\frac{1}{2K} D(u_{2n-1}, u_{2n+1}) \leq \frac{1}{2}[D(u_{2n-1}, u_{2n}) + D(u_{2n}, u_{2n+1})]$ and $\alpha < 1$)

$$D(u_{2n}, u_{2n+1}) \leq \alpha D(u_{2n-1}, u_{2n}). \tag{2}$$

Similarly

$$\begin{aligned} D(u_{2n+1}, u_{2n+2}) &= D(fx_{2n+2}, gx_{2n+1}) \\ &\leq \alpha \max\Bigl\{D(Sx_{2n+2}, Tx_{2n+1}), D(fx_{2n+2}, Sx_{2n+2}), D(Tx_{2n+1}, gx_{2n+1}), \end{aligned}$$

$$\left.\frac{1}{2K}\big[D(Sx_{2n+2},gx_{2n+1})+D(fx_{2n+2},Tx_{2n+1})\big]\right\}$$
$$\le \alpha\max\left\{D(u_{2n+1},u_{2n}),D(u_{2n+2},u_{2n+1}),D(u_{2n},u_{2n+1}),\right.$$
$$\left.\frac{1}{2K}\big[D(u_{2n+1},u_{2n+1})+D(u_{2n+2},u_{2n})\big]\right\},$$

thus (note that $\frac{1}{2K}D(u_{2n+2},u_{2n})\le\frac{1}{2}[D(u_{2n+2},u_{2n+1})+D(u_{2n+1},u_{2n})]$ and $\alpha<1$)

$$D(u_{2n+1},u_{2n+2})\le\alpha D(u_{2n},u_{2n+1}). \tag{3}$$

Therefore, by (2) and (3) we have

$$D(u_n,u_{n+1})\le\alpha D(u_{n-1},u_n),$$

and then

$$D(u_n,u_{n+1})\le\alpha^n D(u_0,u_1). \tag{4}$$

Let $m,n\in\mathbb{N}$ and $m<n$; we have

$$\begin{aligned}
D(u_m,u_n)&\le K\big[D(u_m,u_{m+1})+D(u_{m+1},u_n)\big]\\
&\le KD(u_m,u_{m+1})+K^2\big[D(u_{m+1},u_{m+2})+D(u_{m+2},u_n)\big]\\
&\le\cdots\\
&\le KD(u_m,u_{m+1})+K^2D(u_{m+1},u_{m+2})+\cdots\\
&\quad+K^{n-m-1}\big[D(u_{n-2},u_{n-1})+D(u_{n-1},u_n)\big]\\
&\le KD(u_m,u_{m+1})+K^2D(u_{m+1},u_{m+2})+\cdots\\
&\quad+K^{n-m-1}D(u_{n-2},u_{n-1})+K^{n-m}D(u_{n-1},u_n).
\end{aligned}$$

Now (4) and $K\alpha<1$ imply that

$$\begin{aligned}
D(u_m,u_n)&\le\big(K\alpha^m+K^2\alpha^{m+1}+\cdots+K^{n-m}\alpha^{n-1}\big)D(u_0,u_1)\\
&\le K\alpha^m\big(1+K\alpha+\cdots+(K\alpha)^{n-m-1}\big)D(u_0,u_1)\\
&\le\frac{K\alpha^m}{1-K\alpha}D(u_0,u_1)\to 0\quad\text{when } m\to\infty;
\end{aligned}$$

then $\{u_n\}$ is a Cauchy sequence.

Since $\{u_n\}\subset A_0$ and A_0 is a closed subset of the complete metric-type space (X,D,K), we can find $u\in A_0$ such that $\lim_{n\to\infty}u_n=u$.

By (1) and because of the fact $\{f,S\}$ and $\{g,T\}$ commute proximally, $fu_{2n-1}=Su_{2n}$ and $gu_{2n}=Tu_{2n+1}$. Therefore, the continuity of f, g, S, and T and $n\to\infty$ ascertain that $fu=gu=Tu=Su$.

Since $f(A_0)\subseteq B_0$, there exists $x\in A_0$ such that

$$D(A,B)=D(x,fu)=D(x,gu)=D(x,Su)=D(x,Tu).$$

As $\{f,S\}$ and $\{g,T\}$ commute proximally, $fx = gx = Sx = Tx$. Since $f(A_0) \subseteq B_0$, there exists $z \in A_0$ such that

$$D(A,B) = D(z,fx) = D(z,gx) = D(z,Sx) = D(z,Tx).$$

Because the pair (A,B) has the P-property

$$\begin{aligned} D(x,z) &= D(fu,gx) \\ &\le \alpha \max\left\{D(Su,Tx), D(fu,Su), D(Tx,gx), \frac{1}{2K}\left[D(Su,gx) + D(fu,Tx)\right]\right\} \\ &\le \alpha \max\left\{D(x,z), D(x,x), D(z,z), \frac{1}{2K}\left[D(x,z) + d(x,z)\right]\right\} \\ &\le \alpha D(x,z), \end{aligned}$$

which implies that $x = z$. Thus, it follows that

$$D(A,B) = D(x,fx) = (x,gx) = (x,Tx) = (x,Sx), \tag{5}$$

then x is a common best proximity point of the mappings f, g, S, and T.

Suppose that y is another common best proximity point of the mappings f, g, S, and T, so that

$$D(A,B) = D(y,fy) = (y,gy) = (y,Ty) = (y,Sy). \tag{6}$$

As the pair (A,B) has the P-property, from (5) and (6), we have

$$D(x,y) \le \alpha D(x,y),$$

which implies that $x = y$. □

Now we illustrate our common best proximity point theorem by the following example.

Example 2.3 *Let $X = [0,1] \times [0,1]$. Suppose that $D(x,y) = d^2(x,y)$ for all $x,y \in X$, where d is the Euclidean metric. Then (X,D,K) is a complete metric-type space with $K = 2$. Let*

$$A := \{(0,x) : 0 \le x \le 1\}, \qquad B := \{(1,y) : 0 \le y \le 1\}.$$

Then $D(A,B) = 1$, $A_0 = A$, and $B_0 = B$. Let f, g, S, and T be defined as $f(0,y) = (1, \frac{y}{8})$, $g(0,y) = (1, \frac{y}{32})$, $S(0,y) = (1,y)$, and $T(0,y) = (1, \frac{y}{4})$. Then for all x and $y \in X$ we have

$$D(fx,gy) = \left(\frac{x}{8} - \frac{y}{32}\right)^2 = \frac{1}{64} D(Sx,Ty).$$

Now, all the required hypotheses of Theorem 2.2 are satisfied. Clearly $(0,0)$ is unique common best proximity point of f, g, S, and T.

By Theorem 2.2 we also obtain the following common fixed point theorem in metric-type space.

Theorem 2.4 *Let (X, D, K) be a complete metric-type space. Let $f, g, S, T : X \to X$ be given continuous mappings satisfying the K-contractive condition such that S and T commute with f and g, respectively. Further let $f(X) \subseteq T(X)$, $g(X) \subseteq S(x)$. Then f, g, S, and T have a unique common fixed point.*

Proof We take the same sequence $\{u_n\}$ and u as in the proof of Theorem 2.2. Due to the fact that S and T commute with f and g, respectively, we have

$$fu_{2n-1} = Su_{2n}, \qquad gu_{2n} = Tu_{2n+1}.$$

By continuity of f, g, S, T, and $n \to \infty$ we have

$$fu = Su, \qquad gu = Tu. \tag{7}$$

Since $f, g, S, T : X \to X$ satisfy the K-contractive condition, and by (7),

$$\begin{aligned} D(fu, gu) &\leq \alpha \max\left\{D(Su, Tu), D(fu, Su), D(Tu, gu), \frac{1}{2K}\big[D(Su, gu) + D(fu, Tu)\big]\right\} \\ &\leq \alpha \max\left\{D(fu, gu), D(fu, fu), D(gu, gu), \frac{1}{2K}\big[(fu, gu) + (fu, gu)\big]\right\}, \end{aligned}$$

we have $D(fu, gu) \leq \alpha D(fu, gu)$. Therefore $fu = gu$, and by (7), $fu = gu = Su = Tu$.

We set $w = fu = gu = Su = Tu$. Because of the fact that T commutes with g we obtain

$$gw = gTu = Tgu = Tw,$$

and

$$\begin{aligned} D(w, gw) &= D(fu, gw) \\ &\leq \alpha \max\left\{D(Su, Tw), D(fu, Su), D(Tw, gw), \frac{1}{2K}\big[D(Su, gw) + D(fu, Tw)\big]\right\} \\ &\leq \alpha \max\left\{D(w, gw), D(w, w), D(gw, gw), \frac{1}{2K}\big[(w, gw) + (w, gw)\big]\right\}. \end{aligned}$$

Therefore, $D(w, gw) \leq \alpha D(w, gw)$ and consequently

$$w = gw = Tw. \tag{8}$$

Similarly, we can show that

$$w = fw = Sw. \tag{9}$$

Hence, by (8) and (9) we deduce that $w = fw = gw = Sw = Tw$. Therefore, w is a common fixed point of f, g, S, and T.

Assume to the contrary that $p = fp = gp = Sp = Tp$ and $q = fq = gq = Sq = Tq$ but $p \neq q$.

We have

$$\begin{aligned} D(p,q) &= D(fp,gq) \\ &\le \alpha \max\left\{D(Sp,Tq), D(fp,Sp), D(Tq,gq), \frac{1}{2K}\big[D(Sp,gq)+D(fp,Tq)\big]\right\} \\ &\le \alpha \max\left\{D(p,q), D(p,p), D(q,q), \frac{1}{2K}\big[(p,q)+(p,q)\big]\right\}. \end{aligned}$$

Consequently $D(p,q) \le \alpha D(p,q)$ and $\alpha < 1$; then $D(p,q) = 0$, a contradiction. Therefore, f, g, S, and T have a unique fixed point. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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