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Generalized von Neumann-Jordan constant and its relationship to the fixed point property

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Abstract

We introduce a new geometric constant $C_{NJ}^{(p)}(X)$ for a Banach space X, called a generalized von Neumann-Jordan constant. Next, it is shown that $1 \le C_{NJ}^{(p)}(X) \le 2$ for any Banach space X and that the right hand side inequality is sharp if and only if X is uniformly non-square. Moreover, a relationship between the James constant $J(X)$ and $C_{NJ}^{(p)}(X)$ is presented. Finally, the generalized von Neumann-Jordan constant of the Lebesgue space $L_r([0,1])$ is calculated and a relationship between $C_{NJ}^{(p)}(X)$ and the fixed point property is found.

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Keywords: the generalized von Neumann-Jordan constant; James constant; uniform non-squareness; Lebesgue space; normal structure; weak fixed point property; fixed point property

1 Introduction

Recently many geometric constants for a Banach space X have been investigated. In particular, the von Neumann-Jordan constant $C_{NJ}(X)$ and the James constant $J(X)$ are widely treated. We introduce a new geometric constant, called the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$, which is related to the von Neumann-Jordan constant of a Banach space X and can be used for much better characterization of a Banach space X.

In connection with the famous work [1] (see also [2]) of Jordan and von Neumann concerning inner products, the von Neumann-Jordan constant $C_{NJ}(X)$ for a Banach space X was introduced by Clarkson [3] as the smallest constant C, for which the estimates

$$\frac{1}{C} \le \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \le C$$

hold for all $x, y \in X$ with $(x,y) \ne (0,0)$. Equivalently,

$$C_{NJ}(X) := \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ with } (x,y) \ne (0,0) \right\}.$$

The classical von Neumann-Jordan constant $C_{NJ}(X)$ was investigated in many papers (see for instance [4–7]).

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A Banach space X is said to be uniformly non-square in the sense of James if there exists a positive number $\delta < 2$ such that for any $x, y \in S_X := \{x \in X : \|x\| = 1\}$, we have

$$\min\bigl(\|x+y\|, \|x-y\|\bigr) \leq \delta.$$

The James constant $J(X)$ of a Banach space X is defined by

$$J(X) := \sup\bigl\{\min\bigl(\|x+y\|, \|x-y\|\bigr) : x, y \in S_X\bigr\}.$$

It is obvious that X is uniformly non-square if and only if $J(X) < 2$.

In this paper we introduce a new constant $C_{NJ}^{(p)}(X)$, generalizing the von Neumann-Jordan constant $C_{NJ}(X)$. By the definition of $C_{NJ}^{(p)}(X)$, we will get a relationship between $C_{NJ}^{(p)}(X)$ and $J(X)$, as well as we will estimate the value of $C_{NJ}^{(p)}(X)$. Furthermore, the constant $C_{NJ}^{(p)}(X)$ enable us to establish some new equivalent conditions for the uniform non-squareness of a Banach space X. Since any uniformly non-square Banach space X has the fixed point property (see [8]), our constant $C_{NJ}^{(p)}(X)$ is related to the fixed point theory. Moreover, the value of the generalized von Neumann-Jordan constant for the space $L_r[0,1]$ will be calculated. Finally, we will find a relationship between the constant $C_{NJ}^{(p)}(X)$ and normal structure of X, and in such a way we have again its relationship to the fixed point theory.

2 Preliminaries

Let $X = (X, \|\cdot\|)$ be a real Banach space. Geometrical properties of a Banach space X are determined by its unit sphere S_X or its unit ball $B(X)$.

Definition 1 The generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ is defined by

$$C_{NJ}^{(p)}(X) := \sup\left\{\frac{\|x+y\|^p + \|x-y\|^p}{2^{p-1}\bigl(\|x\|^p + \|y\|^p\bigr)} : x, y \in X, (x, y) \neq (0, 0)\right\},$$

where $1 \leq p < \infty$.

We will also use the following parametrized formula for the constant $C_{NJ}^{(p)}(X)$ (see [9] and [7] in the case of the classical von Neumann-Jordan constant):

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} : x, y \in S_X, 0 \leq t \leq 1\right\},$$

where $1 \leq p < \infty$. By taking $t = 1$ and $x = y$, we obtain the estimate

$$C_{NJ}^{(p)}(X) \geq \frac{\|2x\|^p}{2^{p-1}(1+1)} = \frac{2^p}{2^{p-1} \cdot 2} = 1.$$

Definition 2 (see [10]) The modulus of uniform smoothness of X is defined as

$$\rho_X(t) := \sup\left\{\frac{\|x+ty\| + \|x-ty\|}{2} - 1 : x, y \in S_X, t > 0\right\}.$$

It is clear that $\rho_X(t)$ is a convex function on the interval $[0,\infty)$ satisfying $\rho_X(0) = 0$, whence it follows that ρ_X is nondecreasing on $[0,\infty)$. It is also easy to show that $\max\{0, t-1\} \leq \rho_X(t) \leq t$.

Definition 3 (see [11]) A Banach space X is said to be uniformly smooth if $(\rho_X)'_+(0) := \lim_{t\to 0^+} \frac{\rho_X(t)}{t} = 0$.

Definition 4 (see [12] or [13]) A Banach space X is said to be q-uniformly smooth ($1 < q \leq 2$) if there exists a constant $K > 0$ such that $\rho_X(t) \leq Kt^q$ for all $t > 0$.

Definition 5 (see [13]) Given any Banach space X and a number $p \in [1,\infty)$, another function $J_{X,p}(t)$ is defined by

$$J_{X,p}(t) := \sup\left\{\left(\frac{\|x+ty\|^p + \|x-ty\|^p}{2}\right)^{\frac{1}{p}} : x, y \in S_X\right\}$$

on the interval $[0,\infty)$.

By the inequality

$$\frac{\|x+ty\|^p + \|x-ty\|^p}{2} \geq \left(\frac{\|x+ty\| + \|x-ty\|}{2}\right)^p,$$

which follows by convexity of the function $f(u) = u^p$ on $[0,\infty)$, we get $J_{X,p}(t) \geq \rho_X(t) + 1$ when $1 \leq p < \infty$. For $p = 1$ and $p = 2$, we have the equalities $J_{X,1}(t) = \rho_X(t) + 1$ and $2J^2_{X,2}(t) = E(t,X)$, respectively, where the constant $E(t,X)$ was introduced by Gao [14] in 2005, and it is defined by the formula

$$E(t,X) = \sup\left\{\|x+ty\|^2 + \|x-ty\|^2 : x, y \in S_X\right\}.$$

Definition 6 (see [15]) For any Banach space X, we define

$$\mu(X) := \inf\Bigl\{r > 0 : \limsup_{n\to\infty} \|x + x_n\| \leq r \limsup_{n\to\infty} \|x - x_n\|, \text{for any } (x_n) \subset X \\ \text{with } x_n \xrightarrow{w} 0 \text{ and any } x \in X\Bigr\}.$$

Definition 7 A Banach space X is said to have normal (resp. weak normal) structure if X contains no bounded and closed (resp. weakly compact) convex subset C with more than one point which is diametral in the sense that, for all $x \in C$,

$$\sup\left\{\|y-x\| : y \in C\right\} = \mathrm{diamC} := \sup\left\{\|y-z\| : y, z \in C\right\}.$$

Recall that the weak normal structure (so the normal structure as well) of a Banach space X implies the weak fixed point property for X (see [16, 17]).

Remark 2.1 (see [18]) A sufficient condition for normal structure of a Banach space X is the following: there exists $\varepsilon \in (0,2)$ such that

$$\frac{1}{\mu(X)} > \max\left\{\frac{\varepsilon}{2}, 1 - \delta_x(\varepsilon)\right\},$$

where $\delta_x : [0,2] \to [0,1]$ is the classical modulus of convexity of X defined as

$$\delta_x(\varepsilon) = \inf\left\{1 - \frac{1}{2}\|x+y\| : x, y \in B_X, \|x-y\| \geq \varepsilon\right\}.$$

Lemma 2.2 (see [13]) *For any Banach space X and any $1 \leq p < \infty$ the following statements are true*:

(1) $J_{X,p}(\cdot)$ *is nondecreasing on* $(0,\infty)$.
(2) $J_{X,p}(\cdot)$ *is convex on* $(0,\infty)$.
(3) $J_{X,p}(\cdot)$ *is continuous on* $(0,\infty)$.
(4) $\frac{J_{X,p}(\cdot)-1}{t}$ *is nondecreasing on* $(0,\infty)$.

The proof of this lemma can be found in [13].

Lemma 2.3 *For any $1 \leq p < \infty$ a Banach space X is uniformly smooth if and only if* $\lim_{t\to 0^+} \frac{J_{X,p}(t)-1}{t} = 0$.

Proof Since $J_{X,p}(t) \geq \rho_X(t) + 1$ for any $t > 0$ and $1 \leq p < \infty$, the sufficiency is obvious. Now we will prove the necessity. Assume, to derive a contradiction, that $\lim_{t\to 0^+} \frac{J_{X,p}(t)-1}{t} > 0$. By Lemma 2.2(4), there exists $0 < c < 1$ such that $\lim_{t\to 0^+} \frac{J_{X,p}(t)-1}{t} \geq c$. In particular, we can choose $0 < t < 1$ and x, y in X with $\|x\| = 1$, $\|y\| = t$ satisfying

$$\|x+y\|^p + \|x-y\|^p \geq 2(1+ct)^p. \tag{2.1}$$

We can assume without loss of generality that $\min\{\|x+y\|, \|x-y\|\} = \|x-y\|$. Then, denoting $\|x-y\| = h$, we have $h \in [1-t, 1+t]$, which follows from the inequalities $|\|x\| - \|y\|| \leq \|x-y\| \leq \|x\| + \|y\|$. By inequality (2.1), we obtain

$$\|x+y\| + \|x-y\| \geq h + \left(2(1+ct)^p - h^p\right)^{\frac{1}{p}} =: f(h).$$

Since

$$f'(h) = 1 - \frac{h^{p-1}}{(2(1+ct)^p - h^p)^{\frac{p-1}{p}}},$$

it is easy to see that f is an increasing function with respect to h on the interval $[1-t, 1+ct]$ and decreasing on the interval $[1+ct, 1+t]$. Hence the minimum value of the function $f(h)$ can be attained either at $h = 1-t$ or at $h = 1+t$. In the case when the minimum value is attained at the point $1-t$, we have by the definition of the modulus of uniform smoothness that

$$\frac{\rho_X(t)}{t} \geq \frac{f(1-t)-2}{2t} = \frac{1-t+(2(1+ct)^p - (1-t)^p)^{\frac{1}{p}} - 2}{2t}.$$

In the second case, we have

$$\frac{\rho_X(t)}{t} \geq \frac{f(1+t)-2}{2t} = \frac{1+t+(2(1+ct)^p - (1+t)^p)^{\frac{1}{p}} - 2}{2t}.$$

In both cases, letting $t \to 0^+$ and using the L'Hôpital rule, we easily obtain $\lim_{t\to 0^+} \frac{\rho_X(t)}{t} \geq c > 0$. Obviously, this contradicts the definition of uniform smoothness of X, and thus we completed the proof. □

Lemma 2.4 (see [12]) *Let $1 \leq p < \infty$ and $1 < q \leq 2$. A Banach space X is q-uniformly smooth if and only if there exists a constant $K \geq 1$ such that*

$$\frac{\|x+y\|^p + \|x-y\|^p}{2} \leq \|x\|^q + \|Ky\|^q, \quad \forall x, y \in X.$$

Therefore, according to Lemma 2.4 and the definition of $J_{X,p}(\cdot)$, the following lemma holds.

Lemma 2.5 *Let $1 \leq p < \infty$ and $1 < q \leq 2$. The following statements are equivalent*:

(1) *X is q-uniformly smooth.*

(2) *There exists a constant $K \geq 1$ such that the inequality $J_{X,p}(t) \leq (1 + Kt^q)^{\frac{1}{q}}$ is satisfied for any $t > 0$.*

3 Main results

Theorem 3.1 *For any Banach space X and any $1 \leq p < \infty$ the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$ satisfies the inequality $C_{NJ}^{(p)}(X) \leq 2$.*

Proof We will use in the proof the following parametrized formula for the generalized von Neumann-Jordan constant $C_{NJ}^{(p)}(X)$, where $1 \leq p < \infty$:

$$C_{NJ}^{(p)}(X) = \sup\left\{ \frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} : x, y \in S_X, 0 \leq t \leq 1 \right\}.$$

Since

$$\begin{aligned} \|x+ty\|^p + \|x-ty\|^p &\leq \big(\|x\| + t\|y\|\big)^p + \big(\|x\| + t\|y\|\big)^p \\ &= 2\big(\|x\| + t\|y\|\big)^p \\ &= 2(1+t)^p, \end{aligned}$$

so

$$\frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} \leq \frac{2(1+t)^p}{2^{p-1}(1+t^p)}. \tag{3.1}$$

Applying convexity of the function $\varphi(u) = |u|^p$, we get

$$(1+t)^p = \left(2 \cdot \frac{1+t}{2}\right)^p = 2^p \left(\frac{1+t}{2}\right)^p \leq 2^p \cdot \frac{1+t^p}{2} = 2^{p-1}\big(1+t^p\big).$$

Combining this estimate with inequality (3.1), we get

$$\frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} \leq \frac{2(1+t)^p}{2^{p-1}(1+t^p)} \leq \frac{1}{2^{p-2}} \cdot 2^{p-1} = 2.$$

Hence

$$C_{NJ}^{(p)}(X) = \sup\left\{\frac{\|x+ty\|^p + \|x-ty\|^p}{2^{p-1}(1+t^p)} : x,y \in S_X, 0 \le t \le 1\right\} \le 2,$$

and the proof is completed. □

Lemma 3.2 (see [6]) *Let* $1 < p < \infty$. *A Banach space* X *is uniformly non-square if and only if there exists* $\delta \in (0,1)$ *such that for any* $x, y \in X$, *we have*

$$\left\|\frac{x+y}{2}\right\|^p + \left\|\frac{x-y}{2}\right\|^p \le (2-\delta)\frac{\|x\|^p + \|y\|^p}{2}.$$

According to Lemma 3.2, we directly obtain the following theorem.

Theorem 3.3 *Let* $1 \le p < \infty$. *A Banach space* X *is uniformly non-square if and only if* $C_{NJ}^{(p)}(X) < 2$.

Now let us present the following theorem indicating the relationship between constants $J(X)$ and $C_{NJ}^{(p)}(X)$.

Theorem 3.4 *For any* $1 < p < \infty$ *and any Banach space* X, *the following inequality holds*:

$$J(X) \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(X)}.$$

Proof Indeed, if $1 < p < \infty$, then for any $x, y \in S_X$, we have

$$\begin{aligned} 2\bigl(\min\bigl\{\|x+y\|, \|x-y\|\bigr\}\bigr)^p &\le \|x+y\|^p + \|x-y\|^p \\ &\le 2^{p-1}\bigl(\|x\|^p + \|y\|^p\bigr)C_{NJ}^{(p)}(X) \\ &= 2^{p-1} \cdot 2C_{NJ}^{(p)}(X), \end{aligned}$$

so

$$\min\bigl\{\|x+y\|, \|x-y\|\bigr\} \le 2^{\frac{p-1}{p}} \sqrt[p]{C_{NJ}^{(p)}(X)},$$

and the proof is completed. □

By Theorem 3.4, we obtain the following corollary.

Corollary 3.5 *For any Banach space* X *and any* $1 \le p < \infty$ *the inequalities* $C_{NJ}^{(p)}(X) < 2$ *and* $J(X) < 2$ *are equivalent. Moreover, if* X *is a Banach space with* $C_{NJ}^{(p)}(X) < 2$, *then* X *has the fixed point property.*

Proof It is well known that $J(X) < 2$ if and only if a Banach space X is uniformly non-square. However, by Theorem 3.3, we know that a Banach space X is uniformly non-square if and only if $C_{NJ}^{(p)}(X) < 2$. Hence, $J(X) < 2$ if and only if $C_{NJ}^{(p)}(X) < 2$. Moreover, every uniformly non-square Banach space have the fixed point property (see [8]), so if X is a Banach space with $C_{NJ}^{(p)}(X) < 2$, then X has the fixed point property. □

Now we will calculate the generalized von Neumann-Jordan constant for the space $L_r[0,1]$.

Theorem 3.6 *Let X be the Banach space $L_r[0,1]$. Let $1 < r \leq 2$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Then*

(1) *if $1 < p \leq r$ then $C_{NJ}^{(p)}(L_r[0,1]) = 2^{2-p}$ and if $r < p \leq r'$ then $C_{NJ}^{(p)}(L_r[0,1]) = 2^{\frac{p}{r}-p+1}$;*

(2) *if $r' < p < \infty$ then $C_{NJ}^{(p)}(L_r[0,1]) = 1$.*

Proof Let us note that $r \leq 2 \leq r'$ and

(1) for any $x, y \in S_X$ and any $0 \leq t \leq 1$, if $1 < p \leq r'$, then in virtue of Remark 2.3 from [19], we have

$$\left(\|x+ty\|_r^p + \|x-ty\|_r^p\right)^{\frac{1}{p}} \leq 2^{\frac{1}{p}}\left(\|x\|_r^r + \|ty\|_r^r\right)^{\frac{1}{r}} = 2^{\frac{1}{p}}\left(1+t^r\right)^{\frac{1}{r}},$$

which is equivalent to

$$\|x+ty\|_r^p + \|x-ty\|_r^p \leq 2\left(1+t^r\right)^{\frac{p}{r}}.$$

Consequently,

$$\frac{\|x+ty\|_r^p + \|x-ty\|_r^p}{2^{p-1}(1+t^p)} \leq \frac{2(1+t^r)^{\frac{p}{r}}}{2^{p-1}(1+t^p)},$$

whence

$$\sup\left\{\frac{\|x+ty\|_r^p + \|x-ty\|_r^p}{2^{p-1}(1+t^p)} : x, y \in S_X\right\} \leq \frac{2(1+t^r)^{\frac{p}{r}}}{2^{p-1}(1+t^p)},$$

and from the definition of $C_{NJ}^{(p)}(L_r[0,1])$, we have

$$C_{NJ}^{(p)}\left(L_r[0,1]\right) \leq \sup\left\{\frac{2(1+t^r)^{\frac{p}{r}}}{2^{p-1}(1+t^p)} : 0 \leq t \leq 1\right\}.$$

Defining $f(t) = \frac{(1+t^r)^{\frac{p}{r}}}{1+t^p}$, we get $(f(t))^r = \frac{(1+t^r)^p}{(1+t^p)^r} =: G(t)$. Obviously, both functions $f(t)$ and $G(t)$ are continuous and

$$G'(t) = \frac{p(1+t^r)^{p-1}rt^{r-1}(1+t^p)^r - r(1+t^p)^{r-1}pt^{p-1}(1+t^r)^p}{(1+t^p)^{2r}},$$

whence it follows that $G'(t) = 0$ if and only if

$$p\left(1+t^r\right)^{p-1}rt^{r-1}\left(1+t^p\right)^r - r\left(1+t^p\right)^{r-1}pt^{p-1}\left(1+t^r\right)^p = 0,$$

i.e. $t^r(1+t^p) - t^p(1+t^r) = 0$, which means that $t^r = t^p$. Let us observe that if $p = r$, then $G(t) = 1$ for any $t \in [0,1]$, so $G'(t) = 0$ on the whole interval $[0,1]$.

Notice also that if $1 < p \neq r$, then there is no interior point of the interval $[0,1]$ at which the derivative $G'(t)$ vanishes. Therefore, the function $f(t)$ can reach its biggest value on the interval $[0,1]$ either at the point 0 ($f(0) = 1$) or at the point 1 ($f(1) = 2^{\frac{p}{r}-1}$), depending on the relationship between p and r. Namely:

- if $1 < p \le r$, then $2^{\frac{p}{r}-1} \le 1$, so $C_{NJ}^{(p)}(L_r[0,1]) \le \frac{2}{2^{p-1}} \cdot 1 = 2^{2-p}$;
- if $r < p \le r'$, then $2^{\frac{p}{r}-1} > 1$, so $C_{NJ}^{(p)}(L_r[0,1]) \le \frac{2}{2^{p-1}} \cdot 2^{\frac{p}{r}-1} = 2^{\frac{p}{r}-p+1}$.

On the other hand, notice that the space $L_r[0,1]$ is r-uniformly smooth if $1 < r \le 2$, and the following Clarkson inequality is satisfied:

$$\left(\frac{\|x+ty\|^{r'} + \|x-ty\|^{r'}}{2}\right)^{\frac{1}{r'}} \le \left(\|x\|^r + \|y\|^r\right)^{\frac{1}{r}}.$$

If $1 < p \le r'$, the thesis in Lemma 2.4 holds with $K = 1$. Therefore, we have the inequality $J_{X,p}(t) \le (1+t^r)^{\frac{1}{r}}$ for any $t \ge 0$. Take x and y from the space $L_r[0,1]$, satisfying $\int_0^b |x(s)|^r \, ds = 1$ and $\int_b^1 |y(s)|^r \, ds = 1$ with some $b \in (0,1)$ and let

$$x_1(s) = \begin{cases} x(s), & 0 \le s < b, \\ 0, & b \le s \le 1, \end{cases} \qquad y_1(s) = \begin{cases} 0, & 0 \le s < b, \\ y(s), & b \le s \le 1. \end{cases}$$

Then $\|x_1(s)\|_r = \|y_1(s)\|_r = 1$, and if $1 < p < r'$, we have

$$\left(\frac{\|x_1(s) + ty_1(s)\|_r^p + \|x_1(s) - ty_1(s)\|_r^p}{2}\right)^{\frac{1}{p}} = \left(1 + t^r\right)^{\frac{1}{r}}.$$

Thus

$$\frac{\|x_1(s) + ty_1(s)\|_r^p + \|x_1(s) - ty_1(s)\|_r^p}{2^{p-1}(1+t^p)} = \frac{2(1+t^r)^{\frac{p}{r}}}{2^{p-1}(1+t^p)},$$

which means that if $1 < p \le r'$. Therefore

$$C_{NJ}^{(p)}\left(L_r[0,1]\right) \ge \frac{2(1+t^r)^{\frac{p}{r}}}{2^{p-1}(1+t^p)} \quad \left(\forall t \in [0,1]\right).$$

Taking $t = 1$, we get $C_{NJ}^{(p)}(L_r[0,1]) \ge 2^{\frac{p}{r}-p+1}$, while taking $t = 0$, we obtain $C_{NJ}^{(p)}(L_r[0,1]) \ge 2^{2-p}$. Therefore:

- if $1 < p \le r$ then $2^{2-p} \ge 2^{\frac{p}{r}-p+1}$ and $C_{NJ}^{(p)}(L_r[0,1]) \ge 2^{2-p}$;
- if $r < p \le r'$ then $2^{\frac{p}{r}-p+1} > 2^{2-p}$ and $C_{NJ}^{(p)}(L_r[0,1]) \ge 2^{\frac{p}{r}-p+1}$.

From what has been discussed above, the results from the thesis (1) of the theorem follow immediately.

(2) In the case when $r' < p < \infty$, in virtue of Remark 2.3 from [19] we know that for any $x, y \in S_X$ and any $0 \le t \le 1$, we have

$$\left(\|x+ty\|_r^p + \|x-ty\|_r^p\right)^{\frac{1}{p}} \le 2^{\frac{1}{r'}}\left(\|x\|_r^r + t\|y\|_r^r\right)^{\frac{1}{r}} = 2^{\frac{1}{r'}}\left(1+t^r\right)^{\frac{1}{r}},$$

which is equivalent to

$$\|x+ty\|_r^p + \|x-ty\|_r^p \le 2^{\frac{p}{r'}}\left(1+t^r\right)^{\frac{p}{r}}.$$

Consequently,

$$\frac{\|x+ty\|_r^p + \|x-ty\|_r^p}{2^{p-1}(1+t^p)} \le \frac{2^{\frac{p}{r'}}(1+t^r)^{\frac{p}{r}}}{2^{p-1}(1+t^p)} = 2^{\frac{p}{r'}-p+1} \cdot \frac{(1+t^r)^{\frac{p}{r}}}{1+t^p}.$$

By the proof of thesis (1), if $r < p$ then the supremum of the function f is equal to $2^{\frac{p}{r}-1}$, so we have

$$C_{NJ}^{(p)}\bigl(L_r[0,1]\bigr) \leq 2^{\frac{p}{r}-p+1} \cdot 2^{\frac{p}{r}-1} = 1.$$

By the observation just after Definition 1 of $C_{NJ}^{(p)}(X)$, we have $C_{NJ}^{(p)}(X) \geq 1$, so thesis (2) is proved and the proof of the theorem is completed. □

The following theorem gives a relationship between the constant $C_{NJ}^{(p)}(X)$ and the normal structure of X. It is a generalization of a similar result from [20] concerning only the case $p = 2$.

Theorem 3.7 *If $1 \leq p < \infty$ and X is a Banach space with $C_{NJ}^{(p)}(X) < \frac{1}{2^{p-1}}(1 + \frac{1}{\mu(X)})^p$, then X has normal structure.*

Proof Let us observe that by the inequality $\mu(X) \geq 1$, we have $C_{NJ}^{(p)}(X) < 2$. We know that if $J(X) < 2$, then X is reflexive (see [21]). Therefore, by Corollary 3.5, $C_{NJ}^{(p)}(X) < 2$, and so X is reflexive and it has normal structure if and only if it has weak normal structure.

Looking for a contradiction, suppose that X fails to have weak normal structure. Then it is well known (see [17]) that there exists a bounded sequence (x_n) in X satisfying the following statements:

(i) (x_n) is weakly convergent to 0 in X,
(ii) $\operatorname{diam}(\{x_n : n = 1, 2, \ldots\}) = 1$,
(iii) for all $x \in \overline{\operatorname{conv}}(\{x_n : n = 1, 2, \ldots\})$, we have

$$\lim_{n\to\infty} \|x - x_n\| = \operatorname{diam}\bigl(\{x_n : n = 1, 2, \ldots\}\bigr) = 1.$$

Let us fix $\varepsilon > 0$ as small as needed. Then, using the above properties of (x_n) and the definition of $\mu := \mu(X)$, we can find two positive integers n, m, with $m > n$, such that

(1) $\|x_n\| \geq 1 - \varepsilon$,
(2) $\|x_m - x_n\| \leq 1$,
(3) $\|x_m + x_n\| \leq \mu + \varepsilon$,
(4) $\|(1 + \frac{1}{\mu+\varepsilon})x_m - (1 - \frac{1}{\mu+\varepsilon})x_n\| \geq (1 + \frac{1}{\mu+\varepsilon})(1 - \varepsilon)$,
(5) $\|(1 - \frac{1}{\mu+\varepsilon})x_m - (1 + \frac{1}{\mu+\varepsilon})x_n\| \geq (1 + \frac{1}{\mu+\varepsilon})\|x_n\| - \varepsilon$.

Since

$$\limsup_{n\to\infty} \|x_m + x_n\| \leq \mu \limsup_{n\to\infty} \|x_m - x_n\|,$$

by condition (2), when m is big enough, we get

$$\|x_m + x_n\| \leq \mu + \varepsilon,$$

and condition (3) is proved. We just need to prove conditions (4) and (5).

Let us fix $n \in \mathbb{N}$ and define again $\mu := \mu(X)$. Notice that we can easily get from the Mazur theorem

$$\left[\left(1 - \frac{1}{\mu + \varepsilon}\right) \Big/ \left(1 + \frac{1}{\mu + \varepsilon}\right)\right] x_n \in \overline{\operatorname{conv}}\bigl(\{x_k : k \in \mathbb{N}\}\bigr) \tag{3.2}$$

for any $n \in \mathbb{N}$. Indeed, since $x_n \to 0$ weakly as $n \to \infty$, then by the Mazur theorem $0 \in \overline{\mathrm{conv}}(\{x_k : k \in \mathbb{N}\})$, whence (3.2) follows immediately. Since (3.2) holds, so by the assumption that X fails to have weak normal structure, for some $m > n$, we have

$$\left\| x_m - \frac{1 - \frac{1}{\mu+\varepsilon}}{1 + \frac{1}{\mu+\varepsilon}} x_n \right\| \geq 1 - \varepsilon,$$

and condition (4) follows. In the same way, we can get condition (5).

Next, put $x = x_m - x_n$, $y = (\mu + \varepsilon)^{-1}(x_m + x_n)$ and use the previous estimates to obtain $\|x\| \leq 1$, $\|y\| \leq 1$, and

$$\begin{aligned}
\|x + y\| &= \left\| \left(1 + \frac{1}{\mu + \varepsilon}\right) x_m - \left(1 - \frac{1}{\mu + \varepsilon}\right) x_n \right\| \\
&\geq \left(1 + \frac{1}{\mu + \varepsilon}\right)(1 - \varepsilon), \\
\|x - y\| &= \left\| \left(1 - \frac{1}{\mu + \varepsilon}\right) x_m - \left(1 + \frac{1}{\mu + \varepsilon}\right) x_n \right\| \\
&\geq \left(1 + \frac{1}{\mu + \varepsilon}\right) \|x_n\| - \varepsilon \\
&\geq \left(1 + \frac{1}{\mu + \varepsilon}\right)(1 - \varepsilon) - \varepsilon.
\end{aligned}$$

By the definition of $C_{NJ}^{(p)}(X)$, we get the estimate

$$\begin{aligned}
C_{NJ}^{(p)}(X) &\geq \frac{\|x + y\|^p + \|x - y\|^p}{2^{p-1}(\|x\|^p + \|y\|^p)} \\
&\geq \frac{(1 + \frac{1}{\mu+\varepsilon})^p (1 - \varepsilon)^p + [(1 + \frac{1}{\mu+\varepsilon})(1 - \varepsilon) - \varepsilon]^p}{2^{p-1}(1 + 1)}.
\end{aligned}$$

Finally, letting $\varepsilon \to 0^+$, we obtain

$$C_{NJ}^{(p)}(X) \geq \frac{1}{2^{p-1}} \left(1 + \frac{1}{\mu}\right)^p,$$

which contradicts the hypothesis. This contradiction finishes the proof of the theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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References

1. Jordan, P, von Neumann, J: On inner products in linear metric spaces. Ann. Math. **36**, 719-723 (1935)
2. Zuo, Z, Cui, Y: Some sufficient conditions for fixed points of multivalued nonexpansive mappings. Fixed Point Theory Appl. **2009**, Article ID 319804 (2009). doi:10.1155/2009/319804
3. Clarkson, JA: The von Neumann-Jordan constant for the Lebesgue space. Ann. Math. **38**, 114-115 (1937)
4. Kato, M, Maligranda, L, Takahashi, Y: On James and Jordan-von Neumann constants and the normal structure coefficient of Banach spaces. Stud. Math. **144**(3), 275-295 (2001)
5. Kato, M, Maligranda, L: On James and Jordan-von Neumann constants of Lorentz sequence spaces. J. Math. Anal. Appl. **258**(2), 457-465 (2001)
6. Takahashi, Y, Kato, M: Von Neumann-Jordan constant and uniformly non-square Banach spaces. Nihonkai Math. J. **9**, 155-169 (1998)
7. Yang, C: Jordan-von Neumann constant for Banaś-Frączek space. Banach J. Math. Anal. **8**(2), 185-192 (2014)
8. García-Falset, J, Llorens-Fuster, E, Mazcuńan-Navarro, EM: Uniformly nonsquare Banach spaces have the fixed point property for nonexpansive mappings. J. Funct. Anal. **233**, 494-514 (2006)
9. Yang, C, Wang, F: On a new geometric constant related to the von Neumann-Jordan constant. J. Math. Anal. Appl. **324**(1), 555-565 (2006)
10. Lindenstrauss, J: On the modulus of smoothness and divergent series in Banach spaces. Mich. Math. J. **10**, 241-252 (1963)
11. Day, MM: Uniform convexity in factor and conjugate spaces. Ann. Math. **45**, 375-385 (1944)
12. Takahashi, Y, Hashimoto, K, Kato, M: On sharp uniform convexity, smoothness and strong type, cotype inequalities. J. Nonlinear Convex Anal. **3**(2), 267-281 (2002)
13. Zuo, Z, Cui, Y: A coefficient related to some geometric properties of a Banach space. J. Inequal. Appl. **2009**, Article ID 934321 (2009). doi:10.1155/2009/934321
14. Gao, J: Normal structure and Pythagorean approach in Banach space. Period. Math. Hung. **51**(2), 19-30 (2005)
15. Jiménez-Melado, A, Llorens-Fuster, E: The fixed point property for some uniformly nonsquare Banach spaces. Boll. Unione Mat. Ital., A (7) **10**, 587-595 (1996)
16. Kirk, WA: A fixed point theorem for mappings which do not increase distances. Am. Math. Mon. **72**, 1004-1006 (1965)
17. Goebel, K, Kirk, WA: Topics in Metric Fixed Point Theory. Cambridge University Press, Cambridge (1990)
18. Kirk, WA, Sims, B: Handbook of Metric Fixed Point Theory. Kluwer Academic, Dordrecht (2001)
19. Kato, M, Takahashi, Y: Type, cotype constants and Clarkson's inequalities for Banach spaces. Math. Nachr. **186**, 187-195 (1997)
20. Jiménez-Melado, A, Llorens-Fuster, E, Saejung, S: The von Neumann-Jordan constant, weak orthogonality and normal structure in Banach spaces. Proc. Am. Math. Soc. **134**(2), 355-364 (2005)
21. James, RC: Uniformly non-square Banach spaces. Ann. Math. **80**, 542-550 (1964)

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