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Fixed point theorems for Kannan-type maps

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Abstract

We introduce the new classes of Kannan-type maps with respect to u-distance and prove some fixed point theorems for these mappings. Then we present several examples to illustrate the main theorems.

1 Introduction

A mapping T on a metric space (X, d) is called Kannan if there exists $\alpha \in [0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le \alpha d(x, Tx) + \alpha d(y, Ty) \tag{1.1}$$

for all $x, y \in X$. Kannan [1] proved that if X is complete, then a Kannan mapping has a fixed point. It is interesting that Kannan's theorem is independent of the Banach contraction principle [2]. Also, Kannan's fixed point theorem is very important because Subrahmanyam [3] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space X is complete if and only if every Kannan mapping on X has a fixed point.

Using the concept of Hausdorff metric, Nadler [4] proved the fixed point theorem for multi-valued contraction maps, which is a generalization of the Banach contraction principle [2]. Since then various fixed point results concerning multi-valued contractions have appeared; for example, see [5–7] and the references cited there.

Without using the concept of Hausdorff metric, most recently Dehaish and Latif [8] generalized fixed point theorems of Latif and Abdou [9], Suzuki [10], Suzuki and Takahashi [11].

In 1996, Kada *et al.* [12] introduced the notion of w-distance and improved several classical results including Caristi's fixed point theorem. Suzuki and Takahashi [11] introduced single-valued and multi-valued weakly contractive maps with respect to w-distance and proved fixed point results for such maps. Generalizing the concept of w-distance, in 2001, Suzuki [10] introduced the notion of τ-distance on a metric space and improved several classical results including the corresponding results of Suzuki and Takahashi [11]. In 2010, Ume [13] introduced the new concept of a distance called u-distance, which generalizes w-distance, Tataru's distance and τ-distance. Then he proved a new minimization theorem and a new fixed point theorem by using u-distance on a complete metric space.

Distances in uniform spaces were given by Vályi [14]. More general concepts of distances were given by Wlodarczyk and Plebaniak [15–18] and Wlodarczyk [19].

In this paper, we introduce the new classes of Kannan-type multi-valued maps without using the concept of Hausdorff metric and Kannan-type single-valued maps with respect to u-distance and prove some fixed point theorems for these mappings. Then we present several examples to illustrate the main theorems.

2 Preliminaries

Throughout this paper we denote by N the set of all positive integers, by R the set of all real numbers and by R_+ the set of all nonnegative real numbers.

Ume [13] introduced u-distance as follows: Let X be metric space with metric d. Then a function $p : X \times X \to R_+$ is called u-distance on X if there exists a function $\theta : X \times X \times R_+ \times R_+ \to R_+$ such that the following hold for $x, y, z \in X$:

(u_1) $p(x,z) \le p(x,y) + p(y,z)$.

(u_2) $\theta(x,y,0,0) = 0$ and $\theta(x,y,s,t) \ge \min\{s,t\}$ for all $x, y \in X$ and $s, t \in R_+$, for any $x \in X$ and for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|s - s_0| < \delta$, $|t - t_0| < \delta$, $s, s_0, t, t_0 \in R_+$ and $y \in X$ imply $|\theta(x,y,s,t) - \theta(x,y,s_0,t_0)| < \varepsilon$.

(u_3) $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} \sup\{\theta(w_n, z_n, p(w_n, x_m), p(z_n, x_m)) : m \ge n\} = 0$ imply $p(y,x) \le \lim_{n\to\infty} \inf p(y, x_n)$ for all $y \in X$.

(u_4) $\lim_{n\to\infty} \sup\{p(x_n, w_m) : m \ge n\} = 0$, $\lim_{n\to\infty} \sup\{p(y_n, z_m) : m \ge n\} = 0$, $\lim_{n\to\infty} \theta(x_n, w_n, s_n, t_n) = 0$ and $\lim_{n\to\infty} \theta(y_n, z_n, s_n, t_n) = 0$ imply $\lim_{n\to\infty} \theta(w_n, z_n, s_n, t_n) = 0$ or $\lim_{n\to\infty} \sup\{p(w_m, x_n) : m \ge n\} = 0$, $\lim_{n\to\infty} \sup\{p(z_m, y_n) : m \ge n\} = 0$, $\lim_{n\to\infty} \theta(x_n, w_n, s_n, t_n) = 0$ and $\lim_{n\to\infty} \theta(y_n, z_n, s_n, t_n) = 0$ imply $\lim_{n\to\infty} \theta(w_n, z_n, s_n, t_n) = 0$.

(u_5) $\lim_{n\to\infty} \theta(w_n, z_n, p(w_n, x_n), p(z_n, x_n)) = 0$ and $\lim_{n\to\infty} \theta(w_n, z_n, p(w_n, y_n), p(z_n, y_n)) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$ or $\lim_{n\to\infty} \theta(a_n, b_n, p(x_n, a_n), p(x_n, b_n)) = 0$ and $\lim_{n\to\infty} \theta(a_n, b_n, p(y_n, a_n), p(y_n, b_n)) = 0$ imply $\lim_{n\to\infty} d(x_n, y_n) = 0$.

We recall remark, examples, definition and lemmas which will be useful in what follows.

Remark 2.1 ([13]) (a) Suppose that θ from $X \times X \times R_+ \times R_+$ into R_+ is a mapping satisfying (u_2)$\sim$(u_5). Then there exists a mapping η from $X \times X \times R_+ \times R_+$ into R_+ such that η is nondecreasing in its third and fourth variable, satisfying $(u_2)_\eta \sim (u_5)_\eta$, where $(u_2)_\eta \sim (u_5)_\eta$ stand for substituting η for θ in (u_2)$\sim$(u_5), respectively.

(b) On account of (a), we may assume that θ is nondecreasing in its third and fourth variables, respectively, for a function θ from $X \times X \times R_+ \times R_+$ into R_+ satisfying (u_2)$\sim$(u_5).

(c) Each τ-distance p on a metric space (X,d) is also a u-distance on X.

We present some examples of u-distance which are not τ-distance (for details, see [13]).

Example 2.2 Let $X = R_+$ with the usual metric. Define $p : X \times X \to R_+$ by $p(x,y) = (\frac{1}{4})x^2$. Then p is a u-distance on X but not a τ-distance on X.

Example 2.3 Let X be a normed space with $\|\cdot\|$, then a function $p : X \times X \to R_+$ defined by $p(x,y) = \|x\|$ for every $x, y \in X$ is a u-distance on X but not a τ-distance.

It follows from the above examples and (c) of Remark 2.1 that u-distance is a proper extension of τ-distance.

Definition 2.4 ([13]) Let X be a metric space with a metric d and let p be a u-distance on X. Then a sequence $\{x_n\}$ in X is called p-Cauchy if there exists a function θ from $X \times$

$X \times R_+ \times R_+$ into R_+ satisfying $(u_2)\sim(u_5)$ and a sequence $\{z_n\}$ of X such that

$$\lim_{n\to\infty} \sup\{\theta(z_n, z_n, p(z_n, x_m), p(z_n, x_m)) : m \geq n\} = 0 \quad \text{or}$$
$$\lim_{n\to\infty} \sup\{\theta(z_n, z_n, p(x_m, z_n), p(x_m, z_n)) : m \geq n\} = 0.$$

Lemma 2.5 ([13]) *Let X be a metric space with a metric d and let p be a u-distance on X. If $\{x_n\}$ is a p-Cauchy sequence, then $\{x_n\}$ is a Cauchy sequence.*

Lemma 2.6 ([13]) *Let X be a metric space with a metric d and let p be a u-distance on X.*

(1) *If sequences $\{x_n\}$ and $\{y_n\}$ of X satisfy $\lim_{n\to\infty} p(z, x_n) = 0$ and $\lim_{n\to\infty} p(z, y_n) = 0$ for some $z \in X$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*
(2) *If $p(z, x) = 0$ and $p(z, y) = 0$, then $x = y$.*
(3) *Suppose that sequences $\{x_n\}$ and $\{y_n\}$ of X satisfy $\lim_{n\to\infty} p(x_n, z) = 0$ and $\lim_{n\to\infty} p(y_n, z) = 0$ for some $z \in X$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*
(4) *If $p(x, z) = 0$ and $p(y, z) = 0$, then $x = y$.*

Lemma 2.7 ([13]) *Let X be a metric space with a metric d and let p be a u-distance on X. Suppose that a sequence $\{x_n\}$ of X satisfies*

$$\lim_{n\to\infty} \sup\{p(x_n, x_m) : m > n\} = 0 \quad \textit{or}$$
$$\lim_{n\to\infty} \sup\{p(x_m, x_n) : m > n\} = 0.$$

Then $\{x_n\}$ is a p-Cauchy sequence.

3 Main result

The following lemma plays an important role in proving our theorems.

Lemma 3.1 *Let (X, d) be a metric space with a u-distance p on X and $\{a_n\}$ and $\{b_n\}$ be sequences of X such that*

$$\lim_{n\to\infty} \sup\{p(a_n, a_m) : m > n\} = 0 \quad \textit{and}$$
$$\lim_{n\to\infty} \sup\{p(a_n, b_m) : m > n\} = 0.$$

Then there exist a subsequence $\{a_{k_n}\}$ of $\{a_n\}$ and a subsequence $\{b_{k_n}\}$ of $\{b_n\}$ such that $\lim_{n\to\infty} d(a_{k_n}, b_{k_n}) = 0$.

Proof Since p is a u-distance on X,

$$\begin{aligned} &\text{there exists a mapping } \theta : X \times X \times R_+ \times R_+ \to R_+ \\ &\text{such that } \theta \text{ is nondecreasing in its third and} \\ &\text{fourth variable respectively, satisfying } (u_2)\sim(u_5). \end{aligned} \tag{3.1}$$

For each $n \in N$, let

$$\alpha_n = \sup\{p(a_n, a_m) : m > n\} \quad \text{and} \quad \beta_n = \sup\{p(a_n, b_m) : m > n\}. \tag{3.2}$$

By hypotheses and (3.2), we have

$$\lim_{n\to\infty}(\alpha_n + \beta_n) = 0. \tag{3.3}$$

Let $k_1 \in N$ be an arbitrary and fixed element. Then, by (u_2), for this $a_{k_1} \in X$ and $\varepsilon = 1$, there exists $\delta_1 > 0$ such that

$$|s| = s < \delta_1, \qquad |t| = t < \delta_1, \quad y \in X \quad \text{imply} \quad \theta(a_{k_1}, y, s, t) < 1. \tag{3.4}$$

By virtue of (3.3) and (3.4), for this $\delta_1 > 0$, there exists $M_1 \in N$ such that

$$n \geq M_1 \quad \text{implies} \quad \alpha_n + \beta_n < \delta_1. \tag{3.5}$$

Let $k_2 \in N$ be such that

$$k_2 \geq \max\{1 + k_1, M_1\}. \tag{3.6}$$

Due to (3.6), we have

$$k_1 < k_2 \quad \text{and} \quad k_2 \geq M_1. \tag{3.7}$$

From (3.4), (3.5), (3.6) and (3.7) we get

$$\theta(a_{k_1}, a_{k_2}, \alpha_{k_2} + \beta_{k_2}, \alpha_{k_2} + \beta_{k_2}) < 1. \tag{3.8}$$

In terms of (u_2) and (3.6), for this $a_{k_2} \in X$ and $\varepsilon = \frac{1}{2}$, there exists $\delta_2 > 0$ such that $|s| = s < \delta_2$, $|t| = t < \delta_2$, $y \in X$ imply

$$\theta(a_{k_2}, y, s, t) < \frac{1}{2}. \tag{3.9}$$

In view of (3.3) and (3.9), for this $\delta_2 > 0$, there exists $M_2 \in N$ such that

$$n \geq M_2 \quad \text{implies} \quad \alpha_n + \beta_n < \delta_2. \tag{3.10}$$

Let $k_3 \in N$ be such that

$$k_3 \geq \max\{1 + k_2, M_2\}. \tag{3.11}$$

On account of (3.9), (3.10), (3.11), we obtain

$$k_2 < k_3 \quad \text{and} \quad \theta(a_{k_2}, a_{k_3}, \alpha_{k_3} + \beta_{k_3}, \alpha_{k_3} + \beta_{k_3}) < \frac{1}{2}. \tag{3.12}$$

Continuing this process, there exist a subsequence $\{a_{k_n}\}$ of $\{a_n\}$ and a subsequence $\{b_{k_n}\}$ of $\{b_n\}$ such that for all $n \in N$,

$$\theta(a_{k_n}, a_{k_{n+1}}, \alpha_{k_{n+1}} + \beta_{k_{n+1}}, \alpha_{k_{n+1}} + \beta_{k_{n+1}}) < \frac{1}{n}. \tag{3.13}$$

Using (3.2), (3.3) and (3.13), we know that

$$\begin{aligned} &\lim_{n\to\infty}\bigl\{\sup\bigl[p(a_{k_n},a_{k_{m+1}}):m\geq n\bigr]\bigr\} \\ &\quad\leq \lim_{n\to\infty}\bigl\{\sup\bigl[p(a_{k_n},a_l):l>k_n\bigr]\bigr\} \\ &\quad= \lim_{n\to\infty}\alpha_{k_n}=0 \quad\text{and} \\ &\lim_{n\to\infty}\theta(a_{k_n},a_{k_{n+1}},\alpha_{k_{n+1}}+\beta_{k_{n+1}},\alpha_{k_{n+1}}+\beta_{k_{n+1}})=0. \end{aligned} \tag{3.14}$$

Using (3.1), (3.2), (3.14) and putting $x_n = y_n = a_{k_n}$, $w_m = z_m = a_{k_{m+1}}$ and $s_n = t_n = \alpha_{k_{n+1}} + \beta_{k_{n+1}}$ in (u_4), we deduce

$$\begin{aligned} &\lim_{n\to\infty}\theta\bigl(a_{k_{n+1}},a_{k_{n+1}},p(a_{k_{n+1}},a_{k_{n+2}}),p(a_{k_{n+1}},a_{k_{n+2}})\bigr)=0 \quad\text{and} \\ &\lim_{n\to\infty}\theta\bigl(a_{k_{n+1}},a_{k_{n+1}},p(a_{k_{n+1}},b_{k_{n+2}}),p(a_{k_{n+1}},b_{k_{n+2}})\bigr)=0. \end{aligned} \tag{3.15}$$

Using (3.15) and putting $w_n = z_n = a_{k_{n+1}}$, $x_n = a_{k_{n+2}}$ and $y_n = b_{k_{n+2}}$ in (u_5), we have

$$\lim_{n\to\infty} d(a_{k_{n+2}},b_{k_{n+2}})=0. \tag{3.16}$$

Due to (3.13) and (3.16), there exist a subsequence $\{a_{k_n}\}$ of $\{a_n\}$ and a subsequence $\{b_{k_n}\}$ of $\{b_n\}$ such that

$$\lim_{n\to\infty} d(a_{k_n},b_{k_n})=0. \tag{3.17}$$

□

Definition 3.2 Let (X,d) be a metric space, 2^X be a set of all nonempty subsets of X and $\mathrm{Cl}(X)$ be a set of all nonempty closed subsets of X. Let $T : X \to 2^X$. Then an element $z \in X$ is a fixed point of T if $z \in Tz$.

A mapping $T : X \to 2^X$ is called Kannan-type multi-valued p-contractive mapping if there exist a u-distance p on X and $r \in [0, \frac{1}{2})$ such that

(i) $p(y_1,y_2) \leq r[p(x_1,y_1) + p(x_2,y_2)]$ for any $x_1,x_2 \in X$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$,

(ii) $Ty \subseteq Tx$ for all $x,y \in X$ with $y \in Tx$.

In the next example we shall show that if (X,d) is a complete metric space with a u-distance p and a mapping $T : X \to \mathrm{Cl}(X)$ is not Kannan-type multi-valued p-contractive, in general, T may have no fixed point in X.

Example 3.3 Let $X = [0,1]$ be a closed interval with the usual metric and $p : X \times X \to R_+$ and $T : X \to \mathrm{Cl}(X)$ be mappings defined as follows:

$$p(x,y) = \begin{cases} 2, & x = 0, \\ x, & x \neq 0, \end{cases} \tag{3.18}$$

$$Tx = \begin{cases} \{\frac{1}{4}\}, & x = 0, \\ [\frac{x}{8(1+x)}, \frac{x}{4(1+x)}], & x \neq 0. \end{cases} \tag{3.19}$$

Define $\theta : X \times X \times R_+ \times R_+ \to R_+$ by

$$\theta(x, y, s, t) = s \tag{3.20}$$

for all $x, y \in X$ and $s, t \in R_+$.

From (3.18) and (3.20) easily we can obtain that p is a u-distance on X.

In terms of (3.18) and (3.19), we have

$$p(y_1, y_2) \leq \frac{1}{4}\left[p(x_1, y_1) + p(x_2, y_2)\right] \tag{3.21}$$

for all $x_1, x_2 \in X$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$.

To show that (3.21) is satisfied, we need to consider several possible cases.

$$\begin{aligned}
&\text{Case 1. Let } x_1 = x_2 = 0. \text{ Then } y_1 \in Tx_1 = \left\{\frac{1}{4}\right\}, y_2 \in Tx_2 = \left\{\frac{1}{4}\right\},\\
&p(y_1, y_2) = y_1 = \frac{1}{4}, p(x_1, y_1) = 2 \text{ and } p(x_2, y_2) = 2 \text{ and}\\
&\frac{1}{4}\left[p(x_1, y_1) + p(x_2, y_2)\right] = \frac{1}{4}[2 + 2] = 1 \geq \frac{1}{4} = p(y_1, y_2). \text{ Thus}\\
&p(y_1, y_2) \leq \frac{1}{4}\left[p(x_1, y_1) + p(x_2, y_2)\right].
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&\text{Case 2. Let } x_1 = 0 \text{ and } x_2 \neq 0. \text{ Then } y_1 \in Tx_1 = \left\{\frac{1}{4}\right\},\\
&y_2 \in Tx_2 = \left[\frac{x_2}{8(1 + x_2)}, \frac{x_2}{4(1 + x_2)}\right], p(y_1, y_2) = y_1 = \frac{1}{4},\\
&p(x_1, y_1) = 2 \text{ and } p(x_2, y_2) = x_2. \text{ Thus}\\
&\frac{1}{4}\left[p(x_1, y_1) + p(x_2, y_2)\right] = \frac{1}{4}[2 + x_2] \geq \frac{1}{4} = p(y_1, y_2).
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
&\text{Case 3. Let } x_1 \neq 0 \text{ and } x_2 = 0. \text{ Then } y_1 \in Tx_1 = \left[\frac{x_1}{8(1 + x_1)}, \frac{x_1}{4(1 + x_1)}\right],\\
&y_2 \in Tx_2 = \left\{\frac{1}{4}\right\}, p(y_1, y_2) = y_1 \leq \frac{x_1}{4(1 + x_1)}, p(x_1, y_1) = x_1\\
&\text{and } p(x_2, y_2) = 2. \text{ Thus}\\
&\frac{1}{4}\left[p(x_1, y_1) + p(x_2, y_2)\right] = \frac{1}{4}[x_1 + 2] \geq \frac{x_1}{4(1 + x_1)} \geq p(y_1, y_2).
\end{aligned} \tag{3.24}$$

$$\begin{aligned}
&\text{Case 4. Let } x_1 \neq 0 \text{ and } x_2 \neq 0. \text{ Then } y_1 \in Tx_1 = \left[\frac{x_1}{8(1 + x_1)}, \frac{x_1}{4(1 + x_1)}\right],\\
&y_2 \in Tx_2 = \left[\frac{x_2}{8(1 + x_2)} \frac{x_2}{4(1 + x_2)}\right], p(y_1, y_2) = y_1 \leq \frac{x_1}{4(1 + x_1)},\\
&p(x_1, y_1) = x_1 \text{ and } p(x_2, y_2) = x_2. \text{ Thus}\\
&\frac{1}{4}\left[p(x_1, y_1) + p(x_2, y_2)\right] = \frac{1}{4}[x_1 + x_2] \geq \max\left\{\frac{x_1}{4(1 + x_1)}, \frac{x_2}{4(1 + x_2)}\right\} \geq p(y_1, y_2).
\end{aligned} \tag{3.25}$$

From (3.18)∼(3.25), we have

$$p(y_1, y_2) \leq \frac{1}{4}\big[p(x_1, y_1) + p(x_2, y_2)\big] \tag{3.26}$$

for all $x_1, x_2 \in X$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$.

But there exist $x = 0 \in X$ and $y = \frac{1}{4} \in X$ with $y \in Tx$ such that $Ty = T\frac{1}{4} = [\frac{1}{40}, \frac{1}{20}] \nsubseteq \{\frac{1}{4}\} = T0$. Therefore T is not Kannan-type multi-valued p-contractive and T does not have a fixed point.

Using Lemma 3.1, we have the following main theorem.

Theorem 3.4 *Let (X, d) be a complete metric space and let $T : X \to \mathrm{Cl}(X)$ be a Kannan-type multi-valued p-contractive mapping. Then T has a unique fixed point in X.*

Proof Let $a_1 \in X$ be arbitrary, $a_2 \in Ta_1$ and $a_3 \in Ta_2$ be chosen. Since T is Kannan-type p-contractive,

$$p(a_2, a_3) \leq r\big[p(a_1, a_2) + p(a_2, a_3)\big], \tag{3.27}$$

where $r \in [0, \frac{1}{2})$.

From (3.27), we get

$$p(a_2, a_3) \leq kp(a_1, a_2), \tag{3.28}$$

where $k = \frac{r}{1-r} \in [0, 1)$.

By (3.27) and (3.28), we obtain a sequence $\{a_n\}$ in X such that

$$a_{n+1} \in Ta_n \quad \text{and} \quad p(a_{n+1}, a_{n+2}) \leq kp(a_n, a_{n+1}) \tag{3.29}$$

for all $n \in N$.

By repeated application of (3.29), we have

$$p(a_n, a_{n+1}) \leq k^{n-1} p(a_1, a_2) \tag{3.30}$$

for all $n \in N$.

Now we shall know that $\{a_n\}$ is a Cauchy sequence.

Let $n, m \in N$ be such that $n < m$. Then, by virtue of (3.30), we deduce

$$\begin{aligned} p(a_n, a_m) &\leq p(a_n, a_{n+1}) + p(a_{n+1}, a_{n+2}) + \cdots + p(a_{m-1}, a_m) \\ &= \sum_{i=n}^{m-1} p(a_i, a_{i+1}) \leq \sum_{i=n}^{m-1} k^{i-1} p(a_1, a_2) \\ &\leq \left(\frac{k^{n-1}}{1-k}\right) p(a_1, a_2). \end{aligned} \tag{3.31}$$

In view of (3.31), we get

$$\lim_{n\to\infty} \sup\big\{p(a_n, a_m) : m > n\big\} = 0. \tag{3.32}$$

On account of Lemma 2.5, Lemma 2.7 and (3.32), $\{a_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $b_1 \in X$ such that

$$\lim_{n\to\infty} a_n = b_1. \tag{3.33}$$

By the same method as that in (3.27)∼(3.33), there exists a sequence $\{b_n\}$ in X such that

$$b_{n+1} \in Tb_n \quad \text{and} \quad p(b_n, b_{n+1}) \le k^{n-1}p(b_1, b_2) \tag{3.34}$$

for all $n \in N$.

Combining the hypothesis, (3.27), (3.28), (3.30), (3.31) and (3.34), we have

$$\begin{aligned} p(a_n, b_m) &\le p(a_n, a_m) + p(a_m, b_m) \\ &\le p(a_n, a_m) + r\bigl[p(a_{m-1}, a_m) + p(b_{m-1}, b_m)\bigr] \\ &\le \left(\frac{k^{n-1}}{1-k}\right)p(a_1, a_2) + r\bigl[k^{m-2}p(a_1, a_2) + k^{m-2}p(b_1, b_2)\bigr] \\ &\le \left(\frac{k^{n-1}}{1-k}\right)p(a_1, a_2) + k^{n-1}p(a_1, a_2) + k^{n-1}p(b_1, b_2) \\ &= k^{n-1}\left\{\left(\frac{1}{1-k}\right)p(a_1, a_2) + p(a_1, a_2) + p(b_1, b_2)\right\} \end{aligned} \tag{3.35}$$

for all $n, m \in N$ with $m > n$.

By (3.35), we have

$$\lim_{n\to\infty} \sup\bigl\{p(a_n, b_m) : m > n\bigr\} = 0. \tag{3.36}$$

Due to Lemma 3.1, (3.32) and (3.36), there exist a subsequence $\{a_{k_n}\}$ of $\{a_n\}$ and a subsequence $\{b_{k_n}\}$ of $\{b_n\}$ such that

$$\lim_{n\to\infty} d(a_{k_n}, b_{k_n}) = 0. \tag{3.37}$$

On account of the hypothesis and (3.34), we obtain

$$b_{n+1} \in Tb_1 \tag{3.38}$$

for all $n \in N$.

By virtue of the hypothesis, (3.33), (3.37) and (3.38), we have

$$b_1 \in Tb_1. \tag{3.39}$$

Due to (3.39), b_1 is a fixed point of T. To prove the unique fixed point of T, let c_1 be another fixed point of T. Then

$$c_1 \in Tc_1. \tag{3.40}$$

Since T is Kannan-type multi-valued p-contractive, by (3.39) and (3.40), we have

$$p(b_1,b_1) \le r\bigl[p(b_1,b_1)+p(b_1,b_1)\bigr], \tag{3.41}$$

$$p(c_1,c_1) \le r\bigl[p(c_1,c_1)+p(c_1,c_1)\bigr], \tag{3.42}$$

$$p(b_1,c_1) \le r\bigl[p(b_1,b_1)+p\bigl(c_1,c_1'\bigr)\bigr]. \tag{3.43}$$

Since $r \in [0,\frac{1}{2})$, from (3.41), (3.42) and (3.43), we get

$$p(b_1,b_1) = p(c_1,c_1) = p(b_1,c_1) = 0. \tag{3.44}$$

By virtue of Lemma 2.6 and (3.44), we have

$$b_1 = c_1. \tag{3.45}$$

On account of (3.39), (3.40) and (3.45), T has a unique fixed point. □

Now we give an example to support Theorem 3.4.

Example 3.5 Let $X = [0,1]$ be a closed interval with the usual metric, and $p : X \times X \to R_+$ and $T : X \to \mathrm{Cl}(X)$ be mappings defined as follows:

$$\begin{aligned} &p(x,y) = x, \\ &Tx = \left[0, \frac{1}{4}x\right]. \end{aligned} \tag{3.46}$$

Let $\theta : X \times X \times R_+ \times R_+ \to R_+$ be as in (3.20). Then, due to (3.46), we easily can obtain that p is a u-distance on X.

From (3.46), we have

$$p(y_1,y_2) \le \frac{1}{4}\bigl[p(x_1,y_1)+p(x_2,y_2)\bigr] \tag{3.47}$$

for all $x_1,x_2 \in X$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$. To show that (3.47) is satisfied, let $x_1,x_2 \in X$, $y_1 \in Tx_1$ and $y_2 \in Tx_2$. Then $p(y_1,y_2) = y_1 \le \frac{1}{4}x_1$ and $\frac{1}{4}[p(x_1,y_1)+p(x_2,y_2)] = \frac{1}{4}(x_1+x_2) \ge \frac{1}{4}x_1$. Thus (3.47) is satisfied. Let $x,y \in X$ be such that $y \in Tx$. Then $0 \le y \le \frac{1}{4}x$ and $Ty \subseteq [0,\frac{1}{16}x] \subseteq [0,\frac{1}{4}x] = Tx$. Thus $Ty \subseteq Tx$ for all $x,y \in X$ with $y \in Tx$. Therefore all the conditions of Theorem 3.4 are satisfied and T has a unique fixed point 0 in X.

Definition 3.6 Let (X,d) be a metric space. A mapping $T : X \to X$ is called Kannan-type single-valued p-contractive mapping if there exist a u-distance p on X and $r \in [0,\frac{1}{2})$ such that

(iii) $p(Tx_1,Tx_2) \le r[p(x_1,Tx_1)+p(x_2,Tx_2)]$ for any $x_1,x_2 \in X$,

(iv) if $\{x_n\}$ is a sequence in X such that $x_{n+1} = Tx_n$ for each $n \in N$ and $\lim_{n\to\infty} x_n = c \in X$, then $p(Tc,c) \le r[p(Tc,Tc)+p(c,Tc)]$ and $p(c,Tc) \le r[p(Tc,Tc)+p(Tc,c)]$.

In the following example we show that if (X,d) is a complete metric space and a mapping $T : X \to X$ is not Kannan-type single-valued p-contractive, in general, T may have no fixed point in X.

Example 3.7 Let $X = [0,1]$ be a closed interval with the usual metric, and $p : X \times X \to R_+$ and $T : X \to X$ be mappings defined as follows:

$$p(x,y) = \begin{cases} 2, & x = 0, \\ x, & x \neq 0, \end{cases} \tag{3.48}$$

$$Tx = \begin{cases} \frac{1}{4}, & x = 0, \\ \frac{x}{4(1+x)}, & x \neq 0. \end{cases} \tag{3.49}$$

Define $\theta : X \times X \times R_+ \times R_+ \to R_+$ by

$$\theta(x,y,s,t) = s. \tag{3.50}$$

By the same methods as in Example 3.3, we know that p is a u-distance on X and T is not Kannan-type single-valued p-contractive and T has no fixed point in X.

Theorem 3.8 *Let (X,d) be a metric space with a u-distance p on X.*

Let $T : X \to X$ be a Kannan-type single-valued p-contractive mapping such that there exist a sequence $\{x_n\}$ of X and $c \in X$ satisfying $x_{n+1} = Tx_n$ for each $n \in N$ and $\lim_{n\to\infty} x_n = c \in X$. Then c is a fixed point of T, i.e., $Tc = c$.

Proof By hypotheses, we obtain

$$p(Tc,Tc) \leq r\bigl[p(c,Tc) + p(c,Tc)\bigr] = 2rp(c,Tc), \tag{3.51}$$

$$\begin{aligned} p(Tc,c) &\leq r\bigl[p(Tc,Tc) + p(c,Tc)\bigr] \\ &\leq r\bigl[2rp(c,Tc) + p(c,Tc)\bigr] \\ &= r(2r+1)p(c,Tc), \end{aligned} \tag{3.52}$$

$$\begin{aligned} p(c,Tc) &\leq r\bigl[p(Tc,Tc) + p(Tc,c)\bigr] \\ &\leq r\bigl\{2rp(c,Tc) + r(2r+1)p(c,Tc)\bigr\} \\ &= \bigl(2r^3 + 3r^2\bigr)p(c,Tc). \end{aligned} \tag{3.53}$$

Since $r \in [0, \frac{1}{2})$, $2r, r(2r+1), (2r^3 + 3r^2) \in [0,1)$ and thus, by (3.51), (3.52) and (3.53), we have

$$p(Tc,Tc) = p(Tc,c) = p(c,Tc). \tag{3.54}$$

In view of Lemma 2.6 and (3.54),

$$Tc = c. \tag{3.55}$$

This means that c is a fixed point of T. □

Theorem 3.9 *Let (X,d) be a complete metric space and let $T : X \to X$ be a Kannan-type single-valued p-contractive mapping.*

Then T has a unique fixed point in X.

Proof Since T is Kannan type single-valued p-contractive, there exists a sequence $\{x_n\}$ of X such that

$$x_{n+1} = Tx_n \quad \text{and} \quad p(x_{n+1}, x_{n+2}) \le kp(x_n, x_{n+1}) \tag{3.56}$$

for all $n \in N$, where $k \in [0,1)$.

By repeated application of (3.56), we have

$$p(x_n, x_{n+1}) \le k^{n-1}p(x_1, x_2) \tag{3.57}$$

for all $n \in N$.

On account of (3.57), we get

$$\begin{aligned} p(x_n, x_m) &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{m-1}, x_m) \\ &\le \sum_{i=n}^{m-1} p(x_i, x_{i+1}) \le \sum_{i=n}^{m-1} k^{i-1}p(x_1, x_2) \\ &\le \left(\frac{k^{n-1}}{1-k}\right)p(x_1, x_2) \end{aligned} \tag{3.58}$$

for all $n, m \in N$ with $n < m$.

In view of (3.58), we deduce that

$$\lim_{n\to\infty} \sup\left\{p(x_n, x_m) : m > n\right\} = 0. \tag{3.59}$$

By virtue of Lemma 2.5, Lemma 2.7 and (3.59), we know that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $c \in X$ such that

$$\lim_{n\to\infty} x_n = c. \tag{3.60}$$

On account of the hypothesis, Theorem 3.8, (3.56) and (3.60), we know that c is a fixed point, *i.e.*,

$$Tc = c. \tag{3.61}$$

By the same method as that in (3.40)∼(3.45), we can prove that T has a unique fixed point X. □

From Theorem 3.9, we have the following corollary.

Corollary 3.10 ([1]) *Let (X, d) be a complete metric space and let $T : X \to X$ be a mapping such that*

$$d(Tx, Ty) \le r\left[d(x, Tx) + d(y, Ty)\right]$$

for all $x, y \in X$ and some $r \in [0, \frac{1}{2})$.

Then T has a unique fixed point in X.

Proof By the same methods as in (3.56)∼(3.59), we deduce that

$$\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0, \tag{3.62}$$

where $x_{n+1} = Tx_n$ for all $n \in N$.

Since X is complete, there exists $c \in X$ such that

$$\lim_{n\to\infty} x_n = c. \tag{3.63}$$

Due to the hypothesis and (3.62), we get

$$\begin{aligned} d(x_{n+1}, Tc) &= d(Tx_n, Tc) \\ &\le r\big[d(x_n, Tx_n) + d(c, Tc)\big] \\ &= r\big[d(x_n, x_{n+1}) + d(c, Tc)\big] \end{aligned} \tag{3.64}$$

for all $n \in N$ and some $r \in [0, \frac{1}{2})$.

Taking the limit as $n \to \infty$ in (3.64), we obtain

$$d(c, Tc) \le rd(c, Tc) \tag{3.65}$$

for some $r \in [0, \frac{1}{2})$.

From (3.65), we have

$$d(c, Tc) = 0. \tag{3.66}$$

Since metric d is a u-distance, by view of (3.62), (3.63), (3.66) and the hypothesis, conditions of Corollary 3.10 satisfy all conditions of Theorem 3.9.

Therefore T has a unique fixed point. □

Finally we shall present an example to show that all conditions of Theorem 3.9 are satisfied, but all conditions of Corollary 3.10 are not satisfied.

Example 3.11 Let $X = [0, 1]$ be a closed interval with the usual metric, and $p : X \times X \to R_+$ and $T : X \times X$ be mappings defined as follows:

$$\begin{aligned} &p(x, y) = x, \\ &Tx = \frac{1}{4}x. \end{aligned} \tag{3.67}$$

Then, due to (3.67), we easily can obtain that p is a u-distance on X, but not metric and

$$p(Tx, Ty) \le \frac{1}{4}\big[p(x, Tx) + p(y, Ty)\big] \tag{3.68}$$

for all $x, y \in X$.

Suppose that $\{x_n\}$ is a sequence of X such that $x_{n+1} = Tx_n$ for all $x \in N$.

Then, by (3.67), we have

$$x_{n+1} = Tx_n = \frac{1}{4}x_n \tag{3.69}$$

for all $n \in N$.

By virtue of (3.69),

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(\frac{1}{4}\right)^{n-1} x_1 = 0. \tag{3.70}$$

On account of (3.67)∼(3.70), all conditions of Theorem 3.9 are satisfied, but all conditions of Corollary 3.10 are not satisfied since p is not metric.

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author completed the paper himself. The author read and approved the final manuscript.

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