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Convergence theorems for split equality mixed equilibrium problems with applications

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Abstract
In this paper, we introduce a new algorithm for solving split equality mixed equilibrium problems in the framework of infinite-dimensional real Hilbert spaces. The strong and weak convergence theorems are obtained. As application, we shall utilize our results to study the split equality mixed variational inequality problem and the split equality convex minimization problem. Our results presented in this paper improve and extend some recent corresponding results.

MSC: 47H09; 47J25

Keywords: split equality mixed equilibrium problems; split equality mixed variational inequality problem; split equality convex minimization problem

1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm$\|\cdot\|$. Let D be a nonempty closed convex subset of H. Let $T:D\to D$ be a nonlinear mapping. The fixed point set of T is denoted by $F(T)$, that is, $F(T)=\{x\in D: Tx=x\}$. A mapping T is said to be nonexpansive if

$$\|Tx-Ty\|\le\|x-y\|,\quad \forall x,y\in D. \tag{1.1}$$

If D is a bounded nonempty closed convex subset of H and T is a nonexpansive mapping of D into itself, then $F(T)$ is nonempty [1].

A mapping T is said to be quasi-nonexpansive if $F(T)\neq\emptyset$, and

$$\|Tx-p\|\le\|x-p\|\quad \text{for each } x\in D \text{ and } p\in F(T). \tag{1.2}$$

For modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2], in 1994 Censor and Elfving [3] firstly introduced the following split feasibility problem (SFP) in finite-dimensional Hilbert spaces:

Let C and Q be nonempty closed convex subsets of the Hilbert spaces H_1 and H_2, respectively, let $A:H_1\to H_2$ be a bounded linear operator. The split feasibility problem (SFP) is formulated as finding a point x^* with the property

$$x^*\in C\quad\text{and}\quad Ax^*\in Q. \tag{1.3}$$

It has been found that the *SFP* can be used in many areas such as image restoration, computer tomograph, and radiation therapy treatment planing [4–6]. Some methods have been proposed to solve split feasibility problems; see, for instance, [2, 7–14].

Assuming that the *SFP* is consistent (*i.e.*, (1.3) has a solution), it is not hard to see that

$$x^* = P_C\bigl(I + \gamma A^*(P_Q - I)\bigr)Ax^*, \quad \forall x \in C, \tag{1.4}$$

where P_C and P_Q are the (orthogonal) projections onto C and Q, respectively, $\gamma > 0$, and A^* denotes the adjoint of A. That is, x^* solves *SFP* (1.3) if and only if x^* solves fixed point equation (1.4) (see [15]). This implies that *SFP* can be solved by using fixed point algorithms.

Recently, Moudafi [16] introduced the following new split feasibility problem, which is also called general split equality problem:

Let H_1, H_2, H_3 be real Hilbert spaces, $C \subset H_1$, $Q \subset H_2$ be two nonempty closed convex sets, $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. The new split feasibility problem is to

$$\text{find } x^* \in C, y^* \in Q \text{ such that } Ax^* = By^*. \tag{1.5}$$

This allows asymmetric and partial relations between the variables x and y.

It is easy to see that problem (1.5) reduces to problem (1.3) as $H_2 = H_3$ and $B = I$ (I stands for the identity mapping from H_2 to H_2) in (1.5). Therefore the new split feasibility problem (1.5) proposed by Moudafi is a generalization of split feasibility problem (1.3). The interest of this problem is to cover many situations, for instance, in decomposition methods for *PDE*'s, applications in game theory and in intensity-modulated radiation therapy.

Many authors have proposed some useful methods to solve some kinds of general split feasibility problems and general split equality problems in real Hilbert spaces, and under suitable conditions some strong convergence theorems have been proved; see, for instance, [17–19] and the references therein.

The equilibrium problem (for short, *EP*) is to find $x^* \in C$ such that

$$F\bigl(x^*, y\bigr) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The set of solutions of *EP* is denoted by $EP(F)$. Given a mapping $T : C \to C$, let $F(x,y) = \langle Tx, y-x\rangle$ for all $x, y \in C$. Then $x^* \in EP(F)$ if and only if $x^* \in C$ is a solution of the variational inequality $\langle Tx, y - x\rangle \geq 0$ for all $y \in C$, *i.e.*, x^* is a solution of the variational inequality.

Let $\phi : C \to R \cup \{+\infty\}$ be a function. The mixed equilibrium problem (for short, *MEP*) is to find $x^* \in C$ such that

$$F\bigl(x^*, y\bigr) + \phi(y) - \phi\bigl(x^*\bigr) \geq 0, \quad \forall y \in C. \tag{1.7}$$

The set of solutions of *MEP* is denoted by $MEP(F, \phi)$.

If $\phi = 0$, then the mixed equilibrium problem (1.7) reduces to (1.6).

If $F = 0$, then the mixed equilibrium problem (1.7) reduces to the following convex minimization problem:

$$\text{find } x^* \in C \text{ such that } \phi(y) \geq \phi(x^*), \quad \forall y \in C. \tag{1.8}$$

The set of solutions of (1.8) is denoted by $CMP(\phi)$.

The mixed equilibrium problem (MEP) includes serval important problems arising in physics, engineering, science optimization, economics, transportation, network and structural analysis, Nash equilibrium problems in noncooperative games and others. It has been shown that variational inequalities and mathematical programming problems can be viewed as a special realization of the abstract equilibrium problems (*e.g.*, [20–23]).

Recently, Bnouhachem [24] introduced the following split equilibrium problems:

Let $F : C \times C \to R$ and $G : Q \times Q \to R$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.9}$$

and such that

$$y^* = Ax^* \in Q \text{ solves } G(y^*, y) \geq 0, \quad \forall y \in Q. \tag{1.10}$$

In this paper, we consider the following pair of equilibrium problems called split equality equilibrium problems (SEEP).

Definition 1.1 Let $F : C \times C \to R$ and $G : Q \times Q \to R$ be nonlinear bifunctions, let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators, then the split equality equilibrium problem (SEEP) is to find $x^* \in C$ and $y^* \in Q$ such that

$$F(x^*, x) \geq 0, \quad \forall x \in C, \qquad G(y^*, y) \geq 0, \quad \forall y \in Q \quad \text{and} \quad Ax^* = By^*. \tag{1.11}$$

The set of solutions of (1.11) is denoted by $SEEP(F, G)$.

The split equality mixed equilibrium problem (SEMEP) is defined as follows.

Definition 1.2 Let $F : C \times C \to R$ and $G : Q \times Q \to R$ be nonlinear bifunctions, let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap \operatorname{dom} \phi \neq \emptyset$ and $Q \cap \operatorname{dom} \varphi \neq \emptyset$, and let $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operators, then the split equality mixed equilibrium problem (SEMEP) is to find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{aligned} &F(x^*, x) + \phi(x) - \phi(x^*) \geq 0, \quad \forall x \in C, \qquad G(y^*, y) + \varphi(y) - \varphi(y^*) \geq 0, \quad \forall y \in Q \\ &\quad \text{and} \quad Ax^* = By^*. \end{aligned} \tag{1.12}$$

The set of solutions of (1.12) is denoted by $SEMEP(F, G, \phi, \varphi)$.

Remark 1.3 (1) In (1.12), if $\phi = 0$, then the split equality mixed equilibrium problem (1.12) reduces to (1.11).

(2) If $F = 0$ and $G = 0$, then the split equality mixed equilibrium problem (1.12) reduces to the following split equality convex minimization problem: find $x^* \in C$ and $y^* \in Q$ such that

$$\phi(x) \geq \phi(x^*), \quad \forall x \in C, \qquad \varphi(y) \geq \varphi(y^*), \quad \forall y \in Q \quad \text{and} \quad Ax^* = By^*. \tag{1.13}$$

The set of solutions of (1.13) is denoted by $SECMP(\phi, \varphi)$.

(3) If $F = 0$, $G = 0$, $B = I$ and $y^* = Ax^*$, then the split equality mixed equilibrium problem (1.12) reduces to the following split convex minimization problem: find $x^* \in C$ such that

$$\phi(x) \geq \phi(x^*), \quad \forall x \in C, \quad \text{and} \quad y^* = Ax^* \in Q, \quad \varphi(y) \geq \varphi(y^*), \quad \forall y \in Q. \tag{1.14}$$

The set of solutions of (1.14) is denoted by $SCMP(\phi, \varphi)$.

In order to solve the split equality problem (1.5), Moudafi and Al-Shemas [25] presented the following simultaneous iterative method and obtained weak convergence theorem:

$$(SIM - FPP) \begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)); \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \tag{1.15}$$

where H_1, H_2, H_3 are real Hilbert spaces, $U : H_1 \to H_1$, $T : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are two bounded linear operators, A^* and B^* are the adjoint of A and B, respectively. Under some suitable conditions, they obtained some weak convergence theorems.

In this paper, motivated by the above works and related literature, we introduce a new algorithm for solving split equality mixed equilibrium problems in the framework of infinite-dimensional real Hilbert spaces. Under suitable conditions some strong and weak convergence theorems are obtained. As application, we shall utilize our results to study the split equality mixed variational inequality problem and the split equality convex minimization problem. Our results presented in this paper improve and extend some recent corresponding results.

2 Preliminaries

Throughout this paper, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point $x \in X$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection from H onto C. It is well known that P_C is a firmly nonexpansive mapping from H to C, *i.e.*,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle, \quad \forall x, y \in H.$$

Further, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y \rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

For solving mixed equilibrium problems, we assume that the bifunction $F : C \times C \to R$ satisfies the following conditions:

(A1) $F(x,x) = 0$, $\forall x \in C$;
(A2) $F(x,y) + F(y,x) \leq 0$, $\forall x,y \in C$;
(A3) For all $x,y,z \in C$, $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) For each $x \in C$, the function $y \longmapsto F(x,y)$ is convex and lower semi-continuous;
(A5) For fixed $r > 0$ and $z \in C$, there exists a bounded subset K of H_1 and $x \in C \cap K$ such that

$$F(z,x) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \quad \forall y \in C\backslash K.$$

Lemma 2.1 ([26]) *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)-(A5), *and let $\phi : C \to R \cup \{+\infty\}$ be a proper lower semi-continuous and convex function such that $C \cap \operatorname{dom} \phi \neq \emptyset$. For $r > 0$ and $x \in H_1$, define a mapping $T_r^F : H_1 \to C$ as follows*:

$$T_r^F(x) = \left\{ z \in C : F(z,y) + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}. \tag{2.2}$$

Then

(1) *For each $x \in H$, $T_r^F(x) \neq \emptyset$*;
(2) *T_r^F is single-valued*;
(3) *T_r^F is firmly nonexpansive, that is*, $\forall x,y \in H_1$,

$$\left\| T_r^F x - T_r^F y \right\|^2 \leq \left\langle T_r^F x - T_r^F y, x - y \right\rangle;$$

(4) $F(T_r^F) = MEP(F,\phi)$;
(5) *$MEP(F,\phi)$ is closed and convex.*

Assume that $G : Q \times Q$ to R satisfying (A1)-(A5), and let $\varphi : Q \to R \cup \{+\infty\}$ be a proper lower semi-continuous and convex function such that $Q \cap \operatorname{dom} \varphi \neq \emptyset$, and for $s > 0$ and $\forall u \in H_2$, define a mapping $T_s^G : H_2 \to Q$ as follows:

$$T_s^G(u) = \left\{ v \in Q : G(v,w) + \varphi(w) - \varphi(v) + \frac{1}{s}\langle w - v, v - u\rangle \geq 0, \forall w \in Q \right\}. \tag{2.3}$$

Then it follows from Lemma 2.1 that T_s^G satisfies (1)-(5) of Lemma 2.1, and $F(T_s^G) = MEP(G,\varphi)$.

Definition 2.2 Let H be a Hilbert space.

(1) A single-value mapping $T : H \to H$ is said to be demiclosed at origin if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$ and $\|x_n - Tx_n\| \to 0$, we have $x^* = Tx^*$.
(2) A single-value mapping $T : H \to H$ is said to be semi-compact if, for any bounded sequence $\{x_n\} \subset H$ with $\|x_n - Tx_n\| \to 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to a point $x^* \in H$.

Lemma 2.3 ([27]) *Let C be a nonempty closed convex subset of a Hilbert space and T be a nonexpansive mapping from C into itself. If T has a fixed point, then $I - T$ is demiclosed* *at origin, where I is the identity mapping of H, that is, whenever $\{x_n\}$ is a sequence in C weakly converging to some $x \in C$ and the sequence $\{(I - T)x_n\}$ converges strongly to some 0, it follows that $Tx = x$.*

Lemma 2.4 ([25]) *Let H be a Hilbert space and $\{\mu_n\}$ be a sequence in H such that there exists a nonempty set $W \subset H$ satisfying*:

(i) *For every $\mu^* \in W$, $\lim_{n\to\infty} \|\mu_n - \mu^*\|$ exists.*

(ii) *Any weak-cluster point of the sequence $\{\mu_n\}$ belongs to W.*

Then there exists $\mu^* \in W$ such that $\{\mu_n\}$ weakly converges to μ^*.

Lemma 2.5 ([28]) *Let H be a real Hilbert space, then for all $x, y \in H$, we have*

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle. \tag{2.4}$$

3 Main results

Theorem 3.1 *Let H_1, H_2, H_3 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Assume that $F : C \times C \to R$ and $G : Q \times Q \to R$ are bifunctions satisfying* (A1)-(A5), *and let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap \operatorname{dom}\phi \neq \emptyset$ and $Q \cap \operatorname{dom}\varphi \neq \emptyset$. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonexpansive mappings, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Let $(x_1, y_1) \in C \times Q$ and the iteration scheme $\{(x_n, y_n)\}$ be defined as follows*:

$$\begin{cases} F(u_n, u) + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, & \forall u \in C; \\ G(v_n, v) + \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, & \forall v \in Q; \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)T(u_n - \rho_n A^*(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n v_n + (1 - \alpha_n)S(v_n + \rho_n B^*(Au_n - Bv_n)), & \forall n \geq 1; \end{cases} \tag{3.1}$$

where λ_A and λ_B stand for the spectral radii of A^*A and B^*B respectively, $\{\rho_n\}$ is a positive real sequence such that $\rho_n \in (\varepsilon, \frac{2}{\lambda_A + \lambda_B} - \varepsilon)$ (for ε small enough), $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies the following conditions:

(1) $0 < \alpha \leq \alpha_n \leq \beta < 1$ *(for some $\alpha, \beta \in (0, 1)$)*;

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

If $\Gamma := F(T) \cap F(S) \cap SEMEP(F, G, \phi, \varphi) \neq \emptyset$, then

(I) *The sequence $\{(x_n, y_n)\}$ converges weakly to a solution of problem* (1.12).

(II) *In addition, if S, T are also semi-compact, then $\{(x_n, y_n)\}$ converges strongly to a solution of problem* (1.12).

Proof Now we prove conclusion (I).

Taking $(x, y) \in \Gamma$, it follows from Lemma 2.1 that $x = T^F_{r_n} x$ and $y = T^G_{r_n} y$, we have

$$\|u_n - x\| = \left\|T^F_{r_n} x_n - T^F_{r_n} x\right\| \leq \|x_n - x\|, \tag{3.2}$$

$$\|v_n - y\| = \left\|T^G_{r_n} y_n - T^G_{r_n} y\right\| \leq \|y_n - y\|. \tag{3.3}$$

Let $(x,y) \in \Gamma$. Since $\|\cdot\|^2$ is convex and S, T are nonexpansive mappings, we have

$$\begin{aligned}
\|x_{n+1}-x\|^2 &= \bigl\|\alpha_n u_n + (1-\alpha_n)T\bigl(u_n - \rho_n A^*(Au_n - Bv_n)\bigr) - x\bigr\|^2 \\
&= \alpha_n^2\|u_n - x\|^2 + (1-\alpha_n)^2\bigl\|T\bigl(u_n - \rho_n A^*(Au_n - Bv_n)\bigr) - x\bigr\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\bigl\langle u_n - x, T\bigl(u_n - \rho_n A^*(Au_n - Bv_n)\bigr) - x\bigr\rangle \\
&\le \alpha_n^2\|u_n - x\|^2 + (1-\alpha_n)^2\bigl\|u_n - \rho_n A^*(Au_n - Bv_n) - x\bigr\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\bigl\langle u_n - x, T\bigl(u_n - \rho_n A^*(Au_n - Bv_n)\bigr) - x\bigr\rangle \\
&\le \alpha_n^2\|u_n - x\|^2 + (1-\alpha_n)^2\bigl\|u_n - \rho_n A^*(Au_n - Bv_n) - x\bigr\|^2 \\
&\quad + \alpha_n(1-\alpha_n)\bigl(\|u_n - x\|^2 + \bigl\|u_n - \rho_n A^*(Au_n - Bv_n) - x\bigr\|^2\bigr) \\
&= \alpha_n\|u_n - x\|^2 + (1-\alpha_n)\bigl\|u_n - \rho_n A^*(Au_n - Bv_n) - x\bigr\|^2 \\
&\le \alpha_n\|u_n - x\|^2 + (1-\alpha_n)\bigl(\|u_n - x\|^2 + \bigl\|\rho_n A^*(Au_n - Bv_n)\bigr\|^2 \\
&\quad - 2\rho_n\langle Au_n - Ax, Au_n - Bv_n\rangle\bigr) \\
&\le \|x_n - x\|^2 + (1-\alpha_n)\bigl\|\rho_n A^*(Au_n - Bv_n)\bigr\|^2 \\
&\quad - 2(1-\alpha_n)\rho_n\langle Au_n - Ax, Au_n - Bv_n\rangle.
\end{aligned} \tag{3.4}$$

Since

$$\begin{aligned}
\bigl\|\rho_n A^*(Au_n - Bv_n)\bigr\|^2 &= \rho_n^2\bigl\langle A^*(Au_n - Bv_n), A^*(Au_n - Bv_n)\bigr\rangle \\
&= \rho_n^2\bigl\langle Au_n - Bv_n, AA^*(Au_n - Bv_n)\bigr\rangle \\
&\le \lambda_A\rho_n^2\langle Au_n - Bv_n, Au_n - Bv_n\rangle \\
&= \lambda_A\rho_n^2\|Au_n - Bv_n\|^2.
\end{aligned} \tag{3.5}$$

Combine (3.4) and (3.5), then we have

$$\begin{aligned}
\|x_{n+1}-x\|^2 &\le \|x_n - x\|^2 + (1-\alpha_n)\lambda_A\rho_n^2\|Au_n - Bv_n\|^2 \\
&\quad - 2(1-\alpha_n)\rho_n\langle Au_n - Ax, Au_n - Bv_n\rangle.
\end{aligned} \tag{3.6}$$

Similarly, from the fourth equality in (3.1), we can get

$$\begin{aligned}
\|y_{n+1}-y\|^2 &= \|y_n - x\|^2 + (1-\alpha_n)\lambda_B\rho_n^2\|Au_n - Bv_n\|^2 \\
&\quad + 2(1-\alpha_n)\rho_n\langle Bv_n - By, Au_n - Bv_n\rangle.
\end{aligned} \tag{3.7}$$

Since $(x,y) \in \Gamma$, so we know that $Ax = By$, and finally we have

$$\begin{aligned}
&\|x_{n+1}-x\|^2 + \|y_{n+1}-y\|^2 \\
&\quad \le \|x_n - x\|^2 + \|y_n - y\|^2 - \rho_n(1-\alpha_n)\bigl(2 - \rho_n(\lambda_A + \lambda_B)\bigr)\|Au_n - Bv_n\|^2.
\end{aligned} \tag{3.8}$$

Let $\Gamma_n(x,y) := \|x_n - x\|^2 + \|y_n - y\|^2$, then we have

$$\Gamma_{n+1}(x,y) \le \Gamma_n(x,y) - \rho_n(1-\alpha_n)\bigl(2 - \rho_n(\lambda_A + \lambda_B)\bigr)\|Au_n - Bv_n\|^2. \tag{3.9}$$

Obviously the sequence $\{\Gamma_n(x,y)\}$ is decreasing and is lower bounded by 0, so it converges to some finite limit, say $\omega(x,y)$. This means that the first condition of Lemma 2.4 (Opial's lemma) is satisfied with $\mu_n = (x_n, y_n)$, $\mu^* = (x,y)$ and $W = \Gamma$. And by passing to limit in (3.9), we obtain that

$$\lim_{n\to\infty} \|Au_n - Bv_n\| = 0. \tag{3.10}$$

Since $\|x_n - x\|^2 \leq \Gamma_n(x,y)$, $\|y_n - y\|^2 \leq \Gamma_n(x,y)$ and $\lim_{n\to\infty} \Gamma_n(x,y)$ exists, we know that $\{x_n\}$ and $\{y_n\}$ are bounded, and $\limsup_{n\to\infty} \|x_n - x\|$ and $\limsup_{n\to\infty} \|y_n - y\|$ exist. From (3.2) and (3.3), we have $\limsup_{n\to\infty} \|u_n - x\|$ and $\limsup_{n\to\infty} \|v_n - y\|$ also exist. Let x^* and y^* be respectively weak cluster points of the sequences $\{x_n\}$ and $\{y_n\}$. From Lemma 2.5, we have

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \|x_{n+1} - x - x_n + x\|^2 \\ &= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2\langle x_{n+1} - x_n, x_n - x\rangle \\ &= \|x_{n+1} - x\|^2 - \|x_n - x\|^2 - 2\big\langle x_{n+1} - x^*, x_n - x\big\rangle + 2\big\langle x_n - x^*, x_n - x\big\rangle.\end{aligned}$$

So

$$\limsup_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.11}$$

Similarly, we have

$$\limsup_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.12}$$

This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \tag{3.13}$$

and

$$\lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.14}$$

It follows from Lemma 2.1 that $u_n = T^F_{r_n} x_n$ and $u_{n+1} = T^F_{r_{n+1}} x_{n+1}$, we have

$$F(u_{n+1}, u) + \phi(u) - \phi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall u \in C,$$

and

$$F(u_n, u) + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, \quad \forall u \in C.$$

Particularly, we have

$$F(u_{n+1}, u_n) + \phi(u_n) - \phi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0 \tag{3.15}$$

and

$$F(u_n, u_{n+1}) + \phi(u_{n+1}) - \phi(u_n) + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0. \tag{3.16}$$

Summing up (3.15) and (3.16) and using (A2), we obtain

$$\frac{1}{r_{n+1}}\langle u_n - u_{n+1}, u_{n+1} - x_{n+1}\rangle + \frac{1}{r_n}\langle u_{n+1} - u_n, u_n - x_n\rangle \geq 0,$$

thus

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_n - x_{n+1}}{r_{n+1}}\right\rangle \geq 0,$$

which implies that

$$\begin{aligned} 0 &\leq \left\langle u_{n+1} - u_n, u_n - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle \\ &= \left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1})\right\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1})\right\rangle \\ &\leq \|u_{n+1} - u_n\| \cdot \left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \cdot \|u_{n+1} - x_{n+1}\|\right]. \end{aligned}$$

Thus, we have

$$\|u_{n+1} - u_n\| \leq \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \cdot \|u_{n+1} - x_{n+1}\|. \tag{3.17}$$

Since $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$, $\{u_n\}$ and $\{x_n\}$ are bounded, from (3.13) we have

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0. \tag{3.18}$$

Using the same argument as the proof of the above, we have

$$\lim_{n\to\infty} \|v_{n+1} - v_n\| = 0. \tag{3.19}$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} \|x_{n+1} - x\|^2 &\leq \|u_n - x\|^2 + (1 - \alpha_n)\lambda_A\rho_n^2\|Au_n - Bv_n\|^2 \\ &\quad - 2(1 - \alpha_n)\rho_n\langle Au_n - Ax, Au_n - Bv_n\rangle \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} \|y_{n+1} - y\|^2 &= \|v_n - x\|^2 + (1 - \alpha_n)\lambda_B\rho_n^2\|Au_n - Bv_n\|^2 \\ &\quad + 2(1 - \alpha_n)\rho_n\langle Bv_n - By, Au_n - Bv_n\rangle. \end{aligned} \tag{3.21}$$

By adding the last two inequalities and by taking into account the fact that $Ax = By$, we have

$$\begin{aligned}&\|x_{n+1}-x\|^2+\|y_{n+1}-y\|^2\\&\quad\le\|u_n-x\|^2+\|v_n-y\|^2-\rho_n(1-\alpha_n)\bigl(2-\rho_n(\lambda_A+\lambda_B)\bigr)\|Au_n-Bv_n\|^2,\end{aligned}\tag{3.22}$$

where

$$\begin{aligned}\|u_n-x\|^2&=\bigl\|T^F_{r_n}x_n-T^F_{r_n}x\bigr\|^2\\&\le\langle x_n-x,u_n-x\rangle\\&=\frac{1}{2}\bigl(\|x_n-x\|^2+\|u_n-x\|^2-\|x_n-u_n\|^2\bigr),\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}\|v_n-y\|^2&=\bigl\|T^G_{r_n}y_n-T^G_{r_n}y\bigr\|^2\\&\le\langle y_n-y,v_n-y\rangle\\&=\frac{1}{2}\bigl(\|y_n-y\|^2+\|v_n-x\|^2-\|y_n-v_n\|^2\bigr).\end{aligned}\tag{3.24}$$

It follows from (3.22), (3.23) and (3.24) that

$$\begin{aligned}&\|x_n-u_n\|^2+\|y_n-v_n\|^2\\&\quad\le\|x_n-x\|^2-\|x_{n+1}-x\|^2+\|y_n-y\|^2-\|y_{n+1}-y\|^2\\&\qquad-\rho_n(1-\alpha_n)\bigl(2-\rho_n(\lambda_A+\lambda_B)\bigr)\|Au_n-Bv_n\|^2.\end{aligned}\tag{3.25}$$

By (3.13) and (3.14), we obtain

$$\lim_{n\to\infty}\|x_n-u_n\|=0,\tag{3.26}$$

$$\lim_{n\to\infty}\|y_n-v_n\|=0.\tag{3.27}$$

It follows from (3.26) and (3.27) that $u_n \rightharpoonup x^*$ and $v_n \rightharpoonup y^*$, respectively.

Since T and S are nonexpansive mappings, so

$$\begin{aligned}\|u_n-Tu_n\|&=\|u_n-x_{n+1}+x_{n+1}-Tu_n\|\\&\le\|u_n-x_{n+1}\|+\|x_{n+1}-Tu_n\|\\&=\|u_n-u_{n+1}-u_{n+1}-x_{n+1}\|\\&\quad+\bigl\|\alpha_nu_n+(1-\alpha_n)T\bigl(u_n-\rho_nA^*(Au_n-Bv_n)\bigr)-Tu_n\bigr\|\\&\le\|u_n-u_{n+1}\|+\|u_{n+1}-x_{n+1}\|+\alpha_n\|u_n-Tu_n\|\\&\quad+(1-\alpha_n)\bigl\|T\bigl(u_n-\rho_nA^*(Au_n-Bv_n)\bigr)-Tu_n\bigr\|\\&\le\|u_n-u_{n+1}\|+\|u_{n+1}-x_{n+1}\|+\alpha_n\|u_n-Tu_n\|\\&\quad+(1-\alpha_n)\bigl\|-\rho_nA^*(Au_n-Bv_n)\bigr\|.\end{aligned}$$

That is,

$$(1-\alpha_n)\|u_n - Tu_n\| \le \|u_n - u_{n+1}\| + \|u_{n+1} - x_{n+1}\| + (1-\alpha_n)\big\|-\rho_n A^*(Au_n - Bv_n)\big\|. \quad (3.28)$$

By (3.10), (3.18) and (3.26), we get

$$\lim_{n\to\infty} \|Tu_n - u_n\| = 0. \quad (3.29)$$

Similarly,

$$\lim_{n\to\infty} \|Sv_n - v_n\| = 0. \quad (3.30)$$

Since

$$\begin{aligned} \|x_n - Tx_n\| &\le \|x_n - u_n + u_n - Tu_n + Tu_n - Tx_n\| \\ &\le \|x_n - u_n\| + \|u_n - Tu_n\| + \|Tu_n - Tx_n\| \\ &\le 2\|x_n - u_n\| + \|u_n - Tu_n\|. \end{aligned} \quad (3.31)$$

It follows from (3.26) and (3.29) that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \quad (3.32)$$

In addition, since

$$\|y_n - Sy_n\| \le \|y_n - v_n\| + \|v_n - Sv_n\| + \|Sv_n - Sy_n\| \le 2\|y_n - v_n\| + \|v_n - Sv_n\|, \quad (3.33)$$

then, from (3.27) and (3.30), we have

$$\lim_{n\to\infty} \|y_n - Sy_n\| = 0. \quad (3.34)$$

Since $\{x_n\}$ and $\{y_n\}$ converge weakly to x^* and y^*, respectively, then it follows from (3.32), (3.34) and Lemma 2.3 that $x^* \in F(T)$ and $y^* \in F(S)$. Since every Hilbert space satisfies Opial's condition, Opial's condition guarantees that the weakly subsequential limit of $\{(x_n, y_n)\}$ is unique.

We now prove $x^* \in MEP(F, \phi)$ and $y^* \in MEP(G, \varphi)$.

Since $u_n = T^F_{r_n} x_n$, we have

$$F(u_n, u) + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge 0, \quad \forall u \in C. \quad (3.35)$$

From (A2) we obtain

$$\phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \ge -F(u_n, u) \ge F(u, u_n), \quad \forall u \in C. \quad (3.36)$$

And hence

$$\phi(u) - \phi(u_{n_j}) + \frac{1}{r_{n_j}}\langle u - u_{n_j}, u_{n_j} - x_{n_j}\rangle \ge F(u, u_{n_j}), \quad \forall u \in C. \quad (3.37)$$

From (3.26) we obtain $u_{n_j} \rightharpoonup x^*$. It follows from (A4) that $\lim_{j\to\infty} \frac{\|u_{n_j}-x_{n_j}\|}{r_{n_j}} = 0$, and from the proper lower semicontinuity of ϕ that

$$F(u,x^*) + \phi(x^*) - \phi(u) \le 0, \quad \forall u \in C. \tag{3.38}$$

Put $z_t = tu + (1-t)x^*$ for all $t \in (0,1]$ and $u \in C$. Consequently, we get $z_t \in C$ and hence $F(z_t,x^*) + \phi(x^*) - \phi(z_t) \le 0$. So from (A1) and (A4) we have

$$\begin{aligned} 0 &= F(z_t,z_t) - \phi(z_t) + \phi(z_t) \\ &\le tF(z_t,u) + (1-t)G(z_t,x^*) + t\phi(u) + (1-t)\phi(x^*) - \phi(z_t) \\ &\le t\bigl[F(z_t,x) + \phi(u) - \phi(z_t)\bigr]. \end{aligned} \tag{3.39}$$

Hence, we have

$$F(z_t,u) + \phi(u) - \phi(z_t) \ge 0, \quad \forall u \in C. \tag{3.40}$$

Letting $t \to 0$, from (A4) and the proper lower semicontinuity of ϕ, we have

$$F(x^*,u) + \phi(u) - \phi(x^*) \ge 0, \quad \forall u \in C. \tag{3.41}$$

This implies that $x^* \in MEP(F,\phi)$.

Following a similar argument as the proof of the above, we have $y^* \in MEP(G,\varphi)$.

On the other hand, since the squared norm is weakly lower semicontinuous, we have

$$\|Ax^* - By^*\|^2 \le \liminf_{n\to\infty} \|Au_n - Bv_n\|^2 = 0,$$

therefore $Ax^* = By^*$. This implies that $(x^*,y^*) \in SEMEP(F,G,\phi,\varphi)$. Therefore, $(x^*,y^*) \in \Gamma$. Thus from Lemma 2.4 we know that $\{(x_n,y_n)\}$ converges weakly to (x^*,y^*). The proof of conclusion (I) is completed.

Next, we prove conclusion (II).

Since T and S are semi-compact, $\{x_n\}$ and $\{y_n\}$ are bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, $\lim_{n\to\infty} \|y_n - Sy_n\| = 0$, then there exist subsequences $\{x_{n_j}\}$ and $\{y_{n_j}\}$ of $\{x_n\}$ and $\{y_n\}$ such that $\{x_{n_j}\}$ and $\{y_{n_j}\}$ converge strongly to u^* and v^* (some point in H_1 and H_2, respectively), respectively. Since $\{x_{n_j}\}$ and $\{y_{n_j}\}$ converge weakly to x^* and y^*, respectively, this implies that $x^* = u^*$ and $y^* = v^*$. From Lemma 2.3, we have $x^* \in F(T)$ and $y^* \in F(S)$. Using the same argument as in the proof in conclusion (I), we have $x^* \in MEP(F,\phi)$ and $y^* \in MEP(G,\varphi)$. Further, since the norm is weakly lower semicontinuous and $Au_{n_j} - Bv_{n_j} \to Ax^* - By^*$, we have

$$\|Ax^* - By^*\|^2 \le \liminf_{j\to\infty} \|Au_{n_j} - Bv_{n_j}\|^2 = 0,$$

so $Ax^* = By^*$. This implies that $(x^*,y^*) \in \Gamma$.

On the other hand, since $\Gamma_n(x,y) = \|x_n - x\|^2 + \|y_n - y\|^2$ for any $(x,y) \in \Gamma$, we know that $\lim_{j\to\infty} \Gamma_{n_j}(x^*,y^*) = 0$. From conclusion (I), we have $\lim_{n\to\infty} \Gamma_n(x^*,y^*)$ exists, therefore

$\lim_{n\to\infty} \Gamma_n(x^*, y^*) = 0$. Further, we can obtain that $\lim_{n\to\infty} \|x_n - x^*\| = 0$ and $\lim_{n\to\infty} \|y_n - y^*\| = 0$. This completes the proof of conclusion (II). □

Taking $\phi = 0$ and $\varphi = 0$ in Theorem 3.1, we also have the following result.

Corollary 3.2 *Let H_1, H_2, H_3 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Assume that $F : C \times C \to R$ and $G : Q \times Q \to R$ are bifunctions satisfying* (A1)-(A4). *Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonexpansive mappings, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Let $(x_1, y_1) \in C \times Q$ and the iteration scheme $\{(x_n, y_n)\}$ be defined as follows*:

$$\begin{cases} F(u_n, u) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, & \forall u \in C; \\ G(v_n, v) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, & \forall v \in Q; \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)T(u_n - \rho_n A^*(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n v_n + (1 - \alpha_n)S(v_n + \rho_n B^*(Au_n - Bv_n)), & \forall n \geq 1; \end{cases}$$

where λ_A and λ_B stand for the spectral radii of A^*A and B^*B, respectively, $\{\rho_n\}$ is a positive real sequence such that $\rho_n \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$ (for ε small enough), $\{\alpha_n\}$ is a sequence in $(0,1)$ *and $\{r_n\} \subset (0,\infty)$ satisfies the following conditions*:

(1) $0 < \alpha \leq \alpha_n \leq \beta < 1$ (*for some* $\alpha, \beta \in (0,1)$);

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

If $\Gamma := F(T) \cap F(S) \cap SEEP(F, G, \phi, \varphi) \neq \emptyset$, then

(I) *The sequence $\{(x_n, y_n)\}$ converges weakly to a solution of problem* (1.11).

(II) *In addition, if S, T are also semi-compact, then $\{(x_n, y_n)\}$converges strongly to a solution of problem* (1.11).

In Theorem 3.1 taking $B = I$ and $H_2 = H_3$, from Theorem 3.1 we can obtain the following convergence theorem for general split equilibrium problem (1.10)

Corollary 3.3 *Let H_1 and H_2 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Assume that $F : C \times C \to R$ and $G : Q \times Q \to R$ are bifunctions satisfying* (A1)-(A4), *and let $\phi : C \to R \cup \{+\infty\}$, $\varphi : Q \to R \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap \operatorname{dom}\phi \neq \emptyset$ and $Q \cap \operatorname{dom}\varphi \neq \emptyset$. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonexpansive mappings, and $A : H_1 \to H_2$ be a bounded linear operator. Let $(x_1, y_1) \in C \times Q$ and the iteration scheme $\{(x_n, y_n)\}$ be defined as follows*:

$$\begin{cases} F(u_n, u) + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, & \forall u \in C; \\ G(v_n, v) + \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, & \forall v \in Q; \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)T(u_n - \rho_n A^*(Au_n - v_n)); \\ y_{n+1} = \alpha_n v_n + (1 - \alpha_n)S(v_n + \rho_n(Au_n - v_n)), & \forall n \geq 1; \end{cases}$$

where λ_A stands for the spectral radius of A^*A, $\{\rho_n\}$ is a positive real sequence such that $\rho_n \in (\varepsilon, \frac{1}{\lambda_A} - \varepsilon)$ (for ε small enough), $\{\alpha_n\}$ is a sequence in $(0,1)$ *and $\{r_n\} \subset (0,\infty)$ satisfies the following conditions*:

(1) $0 < \alpha \leq \alpha_n \leq \beta < 1$ (*for some* $\alpha, \beta \in (0,1)$);

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

If $\Gamma := F(T) \cap F(S) \cap SMEP(F, G, \phi, \varphi) \neq \emptyset$, *then*

(I) *The sequence* $\{(x_n, y_n)\}$ *converges weakly to a solution of problem* (1.10).

(II) *In addition, if* S, T *are also semi-compact, then* $\{(x_n, y_n)\}$*converges strongly to a solution of problem* (1.10).

4 Applications

4.1 Application to the split equality mixed variational inequality problem

The variational inequality problem (VIP) is formulated as the problem of finding a point x^* with property $x^* \in C$, $\langle Ax^*, z - x^*\rangle \geq 0$, $\forall z \in C$. We will denote the solution set of VIP by $VI(A, C)$.

In [29], the mixed variational inequality of Browder type (VI) is shown to be equivalent to finding a point $u \in C$ such that

$$\langle Au, y - u\rangle + \varphi(y) - \varphi(u) \geq 0, \quad \forall y \in C.$$

We will denote the solution set of a mixed variational inequality of Browder type by $VI(A, C, \varphi)$.

A mapping $A : C \to H$ is said to be an α-inverse-strongly monotone mapping if there exists a constant $\alpha > 0$ such that $\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2$ for any $x, y \in C$. Setting $F(x, y) = \langle Ax, y - x\rangle$, it is easy to show that F satisfies conditions (A1)-(A4) as A is an α-inverse-strongly monotone mapping.

In 2012, Censor *et al.* [30] introduced the split variational inequality problem (SVIP) which is formulated as follows:

$$\begin{aligned} &\text{find a point } x^* \in C \text{ such that } \langle f(x^*), x - x^*\rangle \geq 0 \quad \text{for all } x \in C, \\ &\text{and such that} \\ &y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^*\rangle \geq 0 \quad \text{for all } y \in Q. \end{aligned} \tag{4.1}$$

The so-called split equality mixed variational inequality problem is shown to be equivalent to finding $x^* \in C$, $y^* \in Q$ such that

$$\begin{aligned} &\langle B_1(x^*), x - x^*\rangle + \phi(x) - \phi(x^*) \geq 0 \quad \text{for all } x \in C, \quad \text{and} \\ &\langle B_2(y^*), y - y^*\rangle + \varphi(y) - \varphi(y^*) \geq 0 \quad \text{for all } y \in Q, \\ &\text{and such that} \\ &Ax^* = By^*. \end{aligned} \tag{4.2}$$

We will denote the solution set of a split equality mixed variational inequality problem by $SEMVIP(\phi, \varphi)$.

The so-called split mixed variational inequality problem is shown to be equivalent to

$$\begin{aligned} &\text{finding a point } x^* \in C \text{ such that } \langle B_1(x^*), x - x^*\rangle + \phi(x) - \phi(x^*) \geq 0 \quad \text{for all } x \in C, \\ &\text{and such that} \\ &y^* = Ax^* \in Q \text{ solves } \langle B_2(y^*), y - y^*\rangle + \varphi(y) - \varphi(y^*) \geq 0 \quad \text{for all } y \in Q. \end{aligned} \tag{4.3}$$

The set of solutions of a split mixed variational inequality problem is denoted by $SMVIP(\phi,\varphi)$.

Setting $F(x,y) = \langle B_1 x, y-x\rangle$ and $G(x,y) = \langle B_2 x, y-x\rangle$, it is easy to show that F and G satisfy conditions (A1)-(A5) as B_i $(i=1,2)$ is an η_i-inverse-strongly monotone mapping. Then it follows from Theorem 3.1 that the following result holds.

Theorem 4.1 *Let H_1, H_2, H_3 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let B_i $(i=1,2)$ be η_i-inverse strongly monotone mappings, and let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap \operatorname{dom}\phi \neq \emptyset$ and $Q \cap \operatorname{dom}\varphi \neq \emptyset$. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonexpansive mappings, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Assume that $(x_1,y_1) \in C \times Q$ and the iteration scheme $\{(x_n,y_n)\}$ is defined as follows*:

$$\begin{cases} \langle B_1 u_n, u-u_n\rangle + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \geq 0, & \forall u \in C; \\ \langle B_2(v_n), v-v_n\rangle + \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v-v_n, v_n-y_n\rangle \geq 0, & \forall v \in Q; \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)T(u_n - \rho_n A^*(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n v_n + (1-\alpha_n)S(v_n + \rho_n B^*(Au_n - Bv_n)), & \forall n \geq 1; \end{cases}$$

where λ_A and λ_B stand for the spectral radii of A^*A and B^*B, respectively, $\{\rho_n\}$ is a positive real sequence such that $\rho_n \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$ (for ε small enough), $\eta_i > 0$ $(i=1,2)$, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies the following conditions:

(1) $0 < \alpha \leq \alpha_n \leq \beta < 1$ (*for some* $\alpha, \beta \in (0,1)$);

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

If $\Gamma := F(T) \cap F(S) \cap SEMVIP(\phi,\varphi) \neq \emptyset$, then

(I) *The sequence $\{(x_n,y_n)\}$ converges weakly to a solution of the split equality mixed variational inequality problem* (4.2).

(II) *In addition, if S, T are also semi-compact, then $\{(x_n,y_n)\}$converges strongly to a solution of the split equality mixed variational inequality problem* (4.2).

In Theorem 4.1 taking $B = I$ and $H_2 = H_3$, from Theorem 4.1 we can obtain the following convergence theorem for split mixed variational inequality problem $SMVIP(\phi,\varphi)$.

Corollary 4.2 *Let H_1 and H_2 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let B_i $(i=1,2)$ be η_i-inverse strongly monotone mappings, and let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper lower semi-continuous and convex functions such that $C \cap \operatorname{dom}\phi \neq \emptyset$ and $Q \cap \operatorname{dom}\varphi \neq \emptyset$. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonexpansive mappings, and $A : H_1 \to H_2$ be a bounded linear operator. Assume that $(x_1,y_1) \in C \times Q$ and the iteration scheme $\{(x_n,y_n)\}$ is defined as follows*:

$$\begin{cases} \langle B_1 u_n, u-u_n\rangle + \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u-u_n, u_n-x_n\rangle \geq 0, & \forall u \in C; \\ \langle B_2(v_n), v-v_n\rangle + \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v-v_n, v_n-y_n\rangle \geq 0, & \forall v \in Q; \\ x_{n+1} = \alpha_n u_n + (1-\alpha_n)T(u_n - \rho_n A^*(Au_n - v_n)); \\ y_{n+1} = \alpha_n v_n + (1-\alpha_n)S(v_n + \rho_n(Au_n - v_n)), & \forall n \geq 1; \end{cases}$$

where λ_A stands for the spectral radius of A^*A, $\{\rho_n\}$ is a positive real sequence such that $\rho_n \in (\varepsilon, \frac{2}{\lambda_A} - \varepsilon)$ (for ε small enough), $\eta_i > 0$ $(i = 1,2)$, $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies the following conditions:

(1) $0 < \alpha \leq \alpha_n \leq \beta < 1$ (*for some* $\alpha, \beta \in (0,1)$);

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

If $\Gamma := F(T) \cap F(S) \cap SMVIP(\phi,\varphi) \neq \emptyset$, *then*

(I) *The sequence* $\{(x_n, y_n)\}$ *converges weakly to a solution of the split mixed variational inequality problem* (4.3).

(II) *In addition, if* S, T *are also semi-compact, then* $\{(x_n, y_n)\}$*converges strongly to a solution of the split mixed variational inequality problem* (4.3).

4.2 Application to the split equality convex minimization problem

It is easy to see that the split equality mixed equilibrium problem (1.12) reduces to the split equality convex minimization problem (1.13) as $F = 0$ and $G = 0$. Therefore, Theorem 3.1 can be used to solve split equality convex minimization problem (1.13), and the following result can be directly deduced from Theorem 3.1.

Theorem 4.3 *Let H_1, H_2, H_3 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper lower semi-continuous and convex functions. Let $T : H_1 \to H_1$, $S : H_2 \to H_2$ be two nonexpansive mappings, and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be two bounded linear operators. Assume that $(x_1, y_1) \in C \times Q$ and the iteration scheme $\{(x_n, y_n)\}$ is defined as follows*:

$$\begin{cases} \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, & \forall u \in C; \\ \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, & \forall v \in Q; \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)T(u_n - \rho_n A^*(Au_n - Bv_n)); \\ y_{n+1} = \alpha_n v_n + (1 - \alpha_n)S(v_n + \rho_n B^*(Au_n - Bv_n)), & \forall n \geq 1; \end{cases}$$

where λ_A and λ_B stand for the spectral radii of A^*A and B^*B, respectively, $\{\rho_n\}$ is a positive real sequence such that $\rho_n \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$ (for ε small enough), $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies the following conditions:

(1) $0 < \alpha \leq \alpha_n \leq \beta < 1$ (*for some* $\alpha, \beta \in (0,1)$);

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

If $\Gamma := F(T) \cap F(S) \cap SECMP(\phi,\varphi) \neq \emptyset$, *then*

(I) *The sequence* $\{(x_n, y_n)\}$ *converges weakly to a solution of problem* (1.13).

(II) *In addition, if* S, T *are also semi-compact, then* $\{(x_n, y_n)\}$*converges strongly to a solution of problem* (1.13).

In Theorem 4.3 taking $B = I$ and $H_2 = H_3$, from Theorem 4.3 we can obtain the following convergence theorem for split convex minimization problem (1.14) $SCMP(\phi,\varphi)$.

Corollary 4.4 *Let H_1 and H_2 be real Hilbert spaces, $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $\phi : C \to R \cup \{+\infty\}$ and $\varphi : Q \to R \cup \{+\infty\}$ be proper lower semi-continuous and convex functions. Let $T : H_1 \to H_1$,*

$S : H_2 \to H_2$ be two nonexpansive mappings, and $A : H_1 \to H_2$ be a bounded linear operator. Assume that $(x_1, y_1) \in C \times Q$ and the iteration scheme $\{(x_n, y_n)\}$ is defined as follows:

$$\begin{cases} \phi(u) - \phi(u_n) + \frac{1}{r_n}\langle u - u_n, u_n - x_n\rangle \geq 0, & \forall u \in C; \\ \varphi(v) - \varphi(v_n) + \frac{1}{r_n}\langle v - v_n, v_n - y_n\rangle \geq 0, & \forall v \in Q; \\ x_{n+1} = \alpha_n u_n + (1 - \alpha_n)T(u_n - \rho_n A^*(Au_n - v_n)); \\ y_{n+1} = \alpha_n v_n + (1 - \alpha_n)S(v_n + \rho_n(Au_n - v_n)), & \forall n \geq 1; \end{cases}$$

where λ_A stands for the spectral radius of A^*A, $\{\rho_n\}$ is a positive real sequence such that $\rho_n \in (\varepsilon, \frac{2}{\lambda_A} - \varepsilon)$ (for ε small enough), $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{r_n\} \subset (0,\infty)$ satisfies the following conditions:

(1) $0 < \alpha \leq \alpha_n \leq \beta < 1$ (*for some* $\alpha, \beta \in (0,1)$);

(2) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$.

If $\Gamma := F(T) \cap F(S) \cap SCMP(\phi, \varphi) \neq \emptyset$, *then*

(I) *The sequence* $\{(x_n, y_n)\}$ *converges weakly to a solution of problem* (1.14).

(II) *In addition, if* S, T *are also semi-compact, then* $\{(x_n, y_n)\}$*converges strongly to a solution of problem* (1.14).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors contributed equally to this work. The authors read and approved the final manuscript.

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