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Existence of solutions for generalized vector quasi-equilibrium problems in abstract convex spaces with applications

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Abstract

In this paper, several kinds of generalized vector quasi-equilibrium problems are introduced and studied in abstract convex spaces. Using the properties of Γ-convex and $\mathfrak{KC}$-maps, some sufficient conditions are given to guarantee the existence of solutions in connection with these generalized vector quasi-equilibrium problems. As applications, some existence theorems of solutions for the generalized semi-infinite programs with vector quasi-equilibrium constraints are also given.

Keywords: abstract convex space; generalized vector quasi-equilibrium problem; generalized semi-infinite program; set-valued mapping; KKM mapping

1 Introduction

It is well known that the vector quasi-equilibrium problem is an important generalization of the vector equilibrium problem which provides a unified model for vector quasi-variational inequalities, vector quasi-complementarity problems, vector optimization problems and vector saddle point problems. In 2000, Fu [1] established the existence theorems for the generalized vector quasi-equilibrium problems and the set-valued vector equilibrium problems. In 2003, Ansari and Fabián [2] considered a generalized vector quasi-equilibrium problem with or without involving Φ-condensing mappings and proved the existence of its solution in real topological vector spaces. In 2005, Li *et al.* [3] studied the existence of solutions for two classes of generalized vector quasi-equilibrium problems. Recently, Lin *et al.* [4] introduced and studied a class of generalized vector quasi-equilibrium problems involving pseudomonotonicity hemicontinuity mappings under different conditions in topological vector spaces. Lin *et al.* [5] proved the existence of equilibria for generalized abstract economy with a lower semicontinuous constraint correspondence and a fuzzy constraint correspondence defined on a noncompact/nonparacompact strategy set. They also considered a systems of generalized vector quasi-equilibrium problems in topological vector spaces. Very recently, Yang and Pu [6] studied the existence and essential components in connection with the set of solutions for the system of strong vector quasi-equilibrium problems. Fu and Wang [7] considered the generalized strong vector quasi-equilibrium problems with domination structure. On the other hand, Ding [8] studied the existence of solutions for generalized vector quasi-equilibrium problems in locally

G-convex spaces. Balaj and Lin [9] investigated existence of solutions for the generalized equilibrium problems in G-convex spaces.

The abstract convex space, introduced by Park [10] in 2006, includes the convex subset of a topological vector space, the convex space, the H-space, and the G-convex space as special cases. Moreover, Park [11] investigated the property of the abstract convex spaces and showed some applications. Recently, several authors have focused on the studies concerned with the set-valued maps and optimization problems in abstract convex spaces with applications. For instance, Cho *et al.* [12] studied some coincidence theorems and minimax inequalities in abstract convex spaces. Yang *et al.* [13] proved some maximal element theorems for set-valued maps in abstract convex spaces with applications. Yang and Huang [14] gave some coincidence theorems for compact and noncompact $\mathfrak{KC}$-maps in abstract convex spaces with applications. Lu and Hu [15] established a new collectively fixed point theorem in noncompact abstract convex spaces with applications to equilibria for generalized abstract economies. Park [16] gave some comments on fixed points, maximal elements, and equilibria of economies in abstract convex spaces. Yang and Huang [17] studied the existence of solutions for the generalized vector equilibrium problems in abstract convex spaces. At the end of the paper [17], Yang and Huang pointed out that it is an interesting and important work to study some types of generalized vector quasi-equilibrium problems with moving cones in topological spaces. To the best of our knowledge, it seems that there is no work concerned with the study of the generalized vector quasi-equilibrium problems in abstract convex spaces. Therefore, it is natural and interesting to study some generalized vector quasi-equilibrium problems in abstract convex spaces under some suitable conditions.

On the other hand, we know that semi-infinite programs are constrained optimization problems in which the number of decision variables is finite, but the number of constraints is infinite. Since John [18] initiated semi-infinite programming precisely to deduce important results about two such geometric problems: the problems of covering a compact body in finite dimensional spaces by the minimum-volume disk and the minimum-volume ellipsoid, many researchers have been investigated the theory, applications and methods for the semi-infinite programming (see, for example, [19–22]). As a generalization of semi-infinite programming, the generalized semi-infinite programming has been become a vivid field of active research in mathematical programming in recent years due to its important applications to numerous real-life problems such as Chebyshev approximation, design centering, robust optimization, optimal layout of an assembly line, time minimal control, and disjunctive optimization (see [23] and the references therein). Therefore, it is important and interesting to study the existence of solutions concerned with some generalized semi-infinite programs with vector quasi-equilibrium constraints in abstract convex spaces.

The main purpose of this paper is to study several classes of generalized vector quasi-equilibrium problems in abstract convex spaces with applications to generalized semi-infinite programs. We give some sufficient conditions to guarantee the existence of solutions for these generalized vector quasi-equilibrium problems in abstract convex spaces. As applications, we give some existence theorems of solutions for the generalized semi-infinite programs under suitable conditions.

2 Preliminaries

Let X, Y be two nonempty sets. A set-valued mapping $T : X \rightrightarrows Y$ is a mapping from X into the power set 2^Y. The inverse T^{-1} of T is the set-valued mapping from Y to X defined by

$$T^{-1}(y) = \{x \in X : y \in T(x)\}.$$

An abstract convex space (X, D, Γ) consists of a nonempty set X, a nonempty set D, and a set-valued mapping $\Gamma : \langle D \rangle \rightrightarrows X$ with nonempty values, where $\langle D \rangle$ denotes the set of all nonempty finite subset of a set D. If for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exists a continuous function $\phi_A : \triangle_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\triangle_J) \subseteq \Gamma(J)$, where $\triangle_n$ is the standard n-simplex and $\triangle_J$ the face of $\triangle_n$ corresponding to $J \in \langle A \rangle$, then the abstract convex space reduces to the G-convex space. Let $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$. When $D \subset X$, the space is defined by $(X \supseteq D, \Gamma)$. In this case, a subset M of X is said to be Γ-convex if, for any $A \in \langle M \cap D \rangle$, we have $\Gamma_A \subseteq M$. In the case $X = D$, let $(X, \Gamma) := (X, X, \Gamma)$.

It is easy to see that any vector space Y is an abstract convex space with $\Gamma :=$ co, where co denotes the convex hull in the vector space Y. Next we give more examples as follows.

Example 2.1 ([10]) Let E be a topological vector space with a neighborhood system $\mathcal{V}$ of its origin. A subset X of E is said to be almost convex (see [24] for more details) if for any $V \in \mathcal{V}$ and for any finite subset $A = \{x_1, x_2, \ldots, x_n\}$ of X, there exists a subset $B = \{y_1, y_2, \ldots, y_n\}$ of X such that $y_i - x_i \in V$ for all $i = 1, 2, \ldots, n$ and $\operatorname{co} B \subset X$. Let $\Gamma_A = \operatorname{co} B$ for any $A \in \langle X \rangle$. Then (X, Γ) is a G-convex space and hence an abstract convex space.

Example 2.2 ([10]) Usually, a convex space $(E, \mathcal{C})$ in the classical sense consists of a nonempty set E and a family $\mathcal{C}$ of subsets of E such that E itself is an element of $\mathcal{C}$ and $\mathcal{C}$ is closed under arbitrary intersection. For any given subset $X \subset E$, the $\mathcal{C}$-convex hull of X is defined as by

$$\operatorname{Co}_{\mathcal{C}} X = \bigcap \{Y \in \mathcal{C} : X \subset Y\}.$$

We say that X is $\mathcal{C}$-convex if $X = \operatorname{Co}_{\mathcal{C}} X$. Consider the mapping $\Gamma : \langle E \rangle \rightrightarrows E$ defined by $\Gamma_A = \operatorname{Co}_{\mathcal{C}} A$. Then (E, Γ) is an abstract convex space.

Example 2.3 Let (M, d) be a pseudo-metric space, that is, $d : M \times M \to [0, +\infty)$ such that, for every $x, y, z \in M$,

(i) $d(x, x) = 0$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$.

For any $A \in \langle M \rangle$, define a set-valued mapping $\Gamma : \langle M \rangle \rightrightarrows M$ by

$$\Gamma_A = \Gamma(A) = \bigcap \{B : B \text{ is a closed ball containing } A\}.$$

Then it is easy to see that (M, Γ) is an abstract convex space.

As pointed out by Park [25], the abstract convex space includes many generalized convex spaces as special cases such as L-spaces, spaces having property (H), pseudo H-spaces, M-spaces, G-H-spaces, another L-spaces, FC-spaces and others. Some more examples of the abstract convex space and comments on it can be found in the literature [10, 25, 26] and the references therein.

Let (X, Γ) be an abstract convex space and V be a real topological vector space. Let E be a nonempty subset of X. Assume that $S : E \rightrightarrows E$ and $B : E \rightrightarrows E$ are two set-valued mappings. Suppose that $F : X \times X \times X \rightrightarrows V$ and $C : X \rightrightarrows V$ are two set-valued mappings such that for each $x \in X$, $C(x)$ is a closed convex cone with $\operatorname{int} C(x) \neq \emptyset$, here $\operatorname{int} C(x)$ denotes the interior of $C(x)$. In this paper, we will consider the following generalized vector quasi-equilibrium problems in abstract convex spaces.

- (GVQEP1) Find $\tilde{x} \in E$ such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, z) \subseteq C(\tilde{x}), \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

We would like to mention that (GVQEP1) was considered by Lin *et al.* [4] in topological vector spaces. When $S(x) = B(x) = E$ for all $x \in E$, (GVQEP1) was considered by Yang and Huang [17] in abstract convex spaces and by Balaj and Lin [9] in G-convex spaces, respectively.

- (GVQEP2) Find $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, \tilde{z}) \subseteq C(\tilde{x}), \quad \forall y \in S(\tilde{x}).$$

When $C(x)$ was replaced by $-C(x)$, (GVQEP2) was considered by Li and Li [27] in topological vector spaces. If $S(x) = E$ for all $x \in E$, then (GVQEP2) was investigated by Fu and Wang [7] in topological vector spaces.

- (GVQEP3) Find $\tilde{x} \in E$ such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, z) \cap -\operatorname{int} C(\tilde{x}) = \emptyset, \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

We note that (GVQEP3) was considered by Lin *et al.* [4] in topological vector spaces. When $S(x) = B(x) = E$ for all $x \in E$, (GVQEP3) was studied by Yang and Huang [17] in abstract convex spaces and by Balaj and Lin [9] in G-convex spaces, respectively.

- (GVQEP4) Find $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, \tilde{z}) \cap -\operatorname{int} C(\tilde{x}) = \emptyset, \quad \forall y \in S(\tilde{x}).$$

We note that (GVQEP4) was considered by Li and Li [27] in topological vector spaces.

- (GVQEP5) Find $\tilde{x} \in E$ such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, z) \nsubseteq -\operatorname{int} C(\tilde{x}), \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

When $S(x) = B(x) = E$ for all $x \in E$, (GVQEP5) was investigated by Yang and Huang [17] in abstract convex spaces and by Balaj and Lin [9] in G-convex spaces, respectively.

- (GVQEP6) Find $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, \tilde{z}) \nsubseteq -\operatorname{int} C(\tilde{x}), \quad \forall y \in S(\tilde{x}).$$

It is worth mentioning that (GVQEP6) was considered by Lin *et al.* [4], and Li and Li [27] in topological vector spaces, respectively. Moreover, some special cases of (GVQEP6) were considered by Ansari and Fabián [2] in topological vector spaces.

- (GVQEP7) Find $\tilde{x} \in E$ such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, z) \cap C(\tilde{x}) \neq \emptyset, \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

When $S(x) = B(x) = E$ for all $x \in E$, (GVQEP7) was studied by Yang and Huang [17] in abstract convex spaces and by Balaj and Lin [9] in G-convex spaces, respectively.

- (GVQEP8) Find $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad F(\tilde{x}, y, \tilde{z}) \cap C(\tilde{x}) \neq \emptyset, \quad \forall y \in S(\tilde{x}).$$

When $S(x) = B(x) = E$ for all $x \in E$, (GVQEP8) was considered by Lin [28] in topological vector spaces.

We would like to point out that, for a suitable choice of the spaces E, X, V and the mappings S, B, F, C, one can obtain a number of well-known insights into the generalized vector quasi-equilibrium problem [2, 4, 5, 7, 8, 27], the generalized vector equilibrium problem [9, 17, 28], the vector equilibrium problem, and the vector variational inequality problem [29, 30] as special cases of the problems (GVQEP1)-(GVQEP8).

Furthermore, assume that $h : X \rightrightarrows L$ is a set-valued mapping, where L is a real topological vector space ordered by a closed convex pointed cone $H \subseteq L$ with $\operatorname{int} H \neq \emptyset$. It is clear that the existence of solutions for problems (GVQEP1)-(GVQEP8) is closely analogous to the existence of solutions in connection with the following generalized semi-infinite programs with generalized vector quasi-equilibrium constraints:

- (GSIP1) Generalized semi-infinite program with constraint (GVQEP1):

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \cap -\operatorname{int} C(x) = \emptyset, \forall y \in S(x), \forall z \in B(x)\}.$$

When $S(x) = B(x) = E$ for all $x \in E$, (GSIP1) was considered by Yang and Huang [17] in abstract convex spaces.

- (GSIP2) Generalized semi-infinite program with constraint (GVQEP2):

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \subseteq C(x), \forall y \in S(x)\}.$$

Some special cases of (GSIP2) were considered by Lin [28] in topological vector spaces.

- (GSIP3) Generalized semi-infinite program with constraint (GVQEP3):

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \cap -\operatorname{int} C(x) = \emptyset, \forall y \in S(x), \forall z \in B(x)\}.$$

When $S(x) = B(x) = E$ for all $x \in E$, (GSIP3) was studied by Yang and Huang [17] in abstract convex spaces.

- (GSIP4) Generalized semi-infinite program with constraint (GVQEP4):

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \bigl\{x \in E : x \in S(x), \exists z \in B(x), F(x,y,z) \cap -\operatorname{int} C(x) = \emptyset, \forall y \in S(x)\bigr\}.$$

We would like to mention that some special cases of (GSIP4) were studied by Lin [28] in topological vector spaces.

- (GSIP5) Generalized semi-infinite program with constraint (GVQEP5):

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \bigl\{x \in E : x \in S(x), F(x,y,z) \not\subseteq -\operatorname{int} C(x), \forall y \in S(x), \forall z \in B(x)\bigr\}.$$

When $S(x) = B(x) = E$ for all $x \in E$, (GSIP5) was investigated by Yang and Huang [17] in abstract convex spaces.

- (GSIP6) Generalized semi-infinite program with constraint (GVQEP6):

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \bigl\{x \in E : x \in S(x), \exists z \in B(x), F(x,y,z) \not\subseteq -\operatorname{int} C(x), \forall y \in S(x)\bigr\}.$$

We note that some special cases of (GSIP6) were considered by Lin [28] in topological vector spaces.

- (GSIP7) Generalized semi-infinite program with constraint (GVQEP7):

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \bigl\{x \in E : x \in S(x), F(x,y,z) \cap C(x) \neq \emptyset, \forall y \in S(x), \forall z \in B(x)\bigr\}.$$

When $S(x) = B(x) = E$ for all $x \in E$, (GSIP7) was studied by Yang and Huang [17] in abstract convex spaces.

- (GSIP8) Generalized semi-infinite program with constraint (GVQEP8):

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x,y,z) \cap C(x) \neq \emptyset, \forall y \in S(x)\}.$$

It is worth mentioning that (GSIP8) can be considered as a generalization of the generalized vector semi-infinite programming introduced and studied by Lin [28] in topological vector spaces.

In brief, for suitable choice of the spaces L, V, X, E and the mappings S, B, F, C, h, one can obtain a number of known the generalized semi-infinite program [17], the mathematical program with equilibrium constraint [19], the generalized semi-infinite program [23], the generalized vector semi-infinite programming [28], and the vector optimization problem [30–32] as special cases from the problems (GSIP1)-(GSIP8).

Now, we recall some useful definitions and lemmas as follows.

Definition 2.1 Let $K \subseteq V$ be a nonempty set and $C \subseteq V$ be the closed convex pointed cone with $\operatorname{int} C \neq \emptyset$. The set of all weak minimal points of K with respect to the ordering cone C is defined as

$$\mathrm{wMin}_C(K) = \{x \in K : (x - K) \cap \operatorname{int} C = \emptyset\}.$$

Definition 2.2 Let (X, D, Γ) be an abstract convex space and Z be a set. For a set-valued mapping $T : X \rightrightarrows Z$ with nonempty values, if a set-valued mapping $G : D \rightrightarrows Z$ satisfies

$$T(\Gamma_N) \subseteq G(N) := \bigcup_{y \in N} G(y) \quad \text{for all } N \in \langle D \rangle,$$

then G is called a KKM mapping with respect to T. A KKM mapping $G : D \rightrightarrows X$ is a KKM mapping with respect to the identity mapping I_X.

A set-valued mapping $F : X \rightrightarrows Z$ is called to be a $\mathfrak{KC}$-map if, for any closed-valued KKM mapping $G : D \rightrightarrows Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. We denote

$$\mathfrak{KC}(X, Z) := \{F : F \text{ is } \mathfrak{KC}\text{-map}\}.$$

Definition 2.3 ([33]) Let X and Y be two topological spaces. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(i) upper semicontinuous (u.s.c.) at x_0 if for any open set $V \supseteq F(x_0)$, there is an open neighborhood O_{x_0} of x_0 such that $F(x') \subseteq V$ for each $x' \in O_{x_0}$,
(ii) lower semicontinuous (l.s.c.) at x_0 if for any open set $V \cap F(x_0) \neq \emptyset$, there is an open neighborhood O_{x_0} of x_0 such that $F(x') \cap V \neq \emptyset$ for each $x' \in O_{x_0}$,
(iii) continuous at x_0 if it is both upper and lower semicontinuous at x_0,
(iv) upper semicontinuous (lower semicontinuous or continuous) on X if it is upper semicontinuous (lower semicontinuous or continuous) at every $x \in X$,
(v) closed if and only if its graph $\mathrm{Graph}(F) := \{(x,y) \in X \times Y : y \in F(x)\}$ is closed.

Lemma 2.1 ([34]) *Let X and Y be two topological spaces and $F : X \rightrightarrows Y$ a set-valued mapping.*

(i) *If Y is compact, then F is closed if and only if it is upper semicontinuous,*
(ii) *if X is a compact space and F is a u.s.c. mapping with compact values, then $F(X)$ is a compact subset of Y.*

Lemma 2.2 ([35]) *Let X and Y be two topological spaces and $F : X \rightrightarrows Y$ be upper semicontinuous and $F(x)$ is compact. Then for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x$ and $y_\alpha \in F(x_\alpha)$, there exists a subnet $\{y_\beta\} \subset y_\alpha$ such that $y_\beta \to y \in F(x)$.*

Lemma 2.3 ([36]) *Let X and Y be two topological spaces and $F : X \rightrightarrows Y$ be lower semicontinuous at $x \in X$ if and only if for any $y \in F(x)$ and any net $\{x_\alpha\}$ with $x_\alpha \to x$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ and $y_\alpha \to y$.*

Lemma 2.4 ([10]) *Let (X, D, Γ) be an abstract convex space, Z a set, and $T : X \rightrightarrows Z$ a set-valued mapping. Then $F \in \mathfrak{KC}(X, Z)$ if and only for any $G : D \rightrightarrows Z$ satisfying*
(i) *G is closed-values*;
(ii) *$F(\Gamma_N) \subseteq G(N)$ for any $N \in \langle D \rangle$,*
we have

$$F(E) \bigcap \{G(y) : y \in N\} \neq \emptyset$$

for each $N \in \langle D \rangle$.

Lemma 2.5 ([32]) *Assume that A is a nonempty compact subset of a real topological vector space V and D is a closed convex cone in V with $D \neq V$. Then one has* $\mathrm{wMin}_D A \neq \emptyset$.

An abstract convex space with any topology is called an abstract convex topological space. In the rest of this paper, let (X, Γ) be an abstract convex Hausdorff topological space and E be a nonempty compact subset of X. Let V be a topological vector spaces. Assume that $T : X \rightrightarrows X$, $B : E \rightrightarrows E$, $S : E \rightrightarrows E$, $F : E \times E \times E \rightrightarrows V$ and $Q : E \rightrightarrows V$ are five set-valued mappings. Let ρ be a binary relation on 2^V and ρ^c be the complementary relation of ρ. Let α be any of the quantifiers $\forall$, $\exists$, and $\bar{\alpha}$ be the other of the quantifiers $\forall$, $\exists$.

3 Main results

In order to show the existence of solutions for the vector quasi-equilibrium problems (GVQEP1)-(GVQEP8), we first give the following general result.

Theorem 3.1 *Suppose that the following conditions are satisfied*:
(i) *$T \in \mathfrak{KC}(X, X)$*;
(ii) *for each $y \in E$, the set $\{x \in E : (\bar{\alpha})z \in B(x), \rho^c(F(x, y, z), Q(x))\}$ is open in E*;
(iii) *$G_0 = \{x \in E : x \notin S(x)\}$ is open in E*;
(iv) *for each $x \in E$, $S(x)$ is nonempty Γ-convex, $S^{-1}(y)$ is open for all $y \in E$*;
(v) *for each $(x_0, y_0) \in E \times E$ with $x_0 \in T(y_0)$ such that $y_0 \notin S(x_0)$.*
Then there exists $\tilde{x} \in S(\tilde{x})$ such that $(\alpha)z \in B(\tilde{x})$, $\rho(F(\tilde{x}, y, z), Q(\tilde{x}))$ for any $y \in S(\tilde{x})$.

Proof For any $x \in E$, define $A : E \rightrightarrows E$ by

$$A(x) = \{y \in E : (\bar{\alpha})z \in B(x), \rho^c(F(x, y, z), Q(x))\}.$$

From the definition of $A(x)$, one has

$$A^{-1}(y) = \{x \in E : (\bar{\alpha})z \in B(x), \rho^c(F(x,y,z), Q(x))\}.$$

Define $P : E \rightrightarrows E$ by

$$P(x) = \begin{cases} S(x) \cap A(x), & x \in E \setminus G_0; \\ S(x), & x \in G_0. \end{cases} \tag{1}$$

Let $M(y) = E \setminus P^{-1}(y)$. We show that $M(y)$ is closed for all $y \in E$. In fact, it follows from (1) that

$$\begin{aligned} P^{-1}(y) &= \{x \in E \setminus G_0 : y \in S(x) \cap A(x)\} \cup \{x \in G_0 : y \in S(x)\} \\ &= \{x \in E \setminus G_0 : x \in S^{-1}(y) \cap A^{-1}(y)\} \cup \{x \in G_0 : x \in S^{-1}(y)\} \\ &= \{(E \setminus G_0) \cap S^{-1}(y) \cap A^{-1}(y)\} \cup \{G_0 \cap S^{-1}(y)\} \\ &= S^{-1}(y) \cap (G_0 \cup A^{-1}(y)). \end{aligned}$$

Since $S^{-1}(y)$, $A^{-1}(y)$, and G_0 are open, we know that $P^{-1}(y)$ is open and so $M(y)$ is closed.

We show that M is a KKM mapping with respect to T. Suppose that M is not a KKM mapping with respect to T. Then there exist a finite subset N and a point $x_0 \in E$ such that $x_0 \in T(\Gamma_N) \setminus M(N)$. This shows that there exists a point $y_0 \in \Gamma_N$ such that $x_0 \in T(y_0)$, $x_0 \in P^{-1}(y)$ for any $y \in N$, and so $N \subset P(x_0) \subset S(x_0)$. Since $S(x_0)$ is Γ-convex and $N \in \langle S(x_0)\rangle$, we know that $y_0 \in \Gamma_N \subset S(x_0)$, which is a contradiction. It follows that M is a KKM mapping with respect to T.

It follows from Lemma 2.4 that M has finite intersection property. From the facts that $M(y) \subset E$ is closed and E is compact, we know that $M(y)$ is compact for any $y \in E$ and so

$$\bigcap_{y \in E} M(y) \neq \emptyset.$$

Thus, there exists a point $\tilde{x} \in E$ such that

$$\tilde{x} \in \bigcap_{y \in E} M(y) = E \setminus \bigcup_{y \in E} P^{-1}(y).$$

This implies that $\tilde{x} \notin P^{-1}(y)$ for all $y \in E$ and so $P(\tilde{x}) = \emptyset$.

If $\tilde{x} \in G_0$, then it is easy to see that $S(\tilde{x}) = P(\tilde{x}) = \emptyset$, which is a contradiction. Therefore, we have

$$\tilde{x} \in E \setminus G_0 \quad \text{with } S(\tilde{x}) \cap A(\tilde{x}) = P(\tilde{x}) = \emptyset$$

and so

$$\tilde{x} \in S(\tilde{x}) \quad \text{and} \quad y \notin A(\tilde{x}), \forall y \in S(\tilde{x}),$$

that is, $\tilde{x} \in S(\tilde{x})$, $(\alpha)z \in B(\tilde{x})$, $\rho(F(\tilde{x},y,z), C(\tilde{x}))$ for all $y \in S(\tilde{x})$. This completes the proof. □

Remark 3.1 By Lemma 2.1, it is easy to see that the condition (iii) can be replaced by the following condition:

(iii)′ $S : E \rightrightarrows E$ is a u.s.c. set-valued mapping.

Next we give some existence theorems in connection with the solution of the vector quasi-equilibrium problems (GVQEP1)-(GVQEP8).

Theorem 3.2 *Assume that the conditions* (i), (iii), (iv), *and* (v) *in Theorem* 3.1 *are satisfied. Moreover, suppose that*

(a) *for each $y \in E$, $F(\cdot, y, \cdot)$ is l.s.c. and C is closed*;

(b) *B is l.s.c.*

Then there exists $\tilde{x} \in E$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y, z) \subset C(\tilde{x})$ for all $y \in S(\tilde{x})$ and $z \in B(\tilde{x})$.

Proof Let

$$A(x) = \{y \in E : \exists z \in B(x), F(x, y, z) \nsubseteq C(x)\}.$$

We show that

$$A^{-1}(y) = \{x \in E : \exists z \in B(x), F(x, y, z) \nsubseteq C(x)\}$$

is open. Let $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$ be a net with $x_\alpha \to x_0$. Then

$$F(x_\alpha, y, z') \subseteq C(x_\alpha), \quad \forall z' \in B(x_\alpha).$$

Since B and $F(\cdot, y, \cdot)$ are l.s.c., by Lemma 2.3, for any $z \in B(x_0)$ and $v \in F(x_0, y, z)$, there exist $z_\alpha \in B(x_\alpha)$ and $v_\alpha \in F(x_\alpha, y, z_\alpha)$ such that $z_\alpha \to z$ and $v_\alpha \to v$. Now the closedness of C with $v_\alpha \in C(x_\alpha)$ shows that $v \in C(x)$ and so $F(x_0, y, z) \subseteq C(x)$ for any $z \in B(x_0)$. This shows that $x_0 \in E \setminus A^{-1}(y)$ and so $E \setminus A^{-1}(y)$ is closed. Thus, $A^{-1}(y)$ is open. It follows from Theorem 3.1 that there exists $\tilde{x} \in E$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y, z) \subseteq C(\tilde{x})$ for all $y \in S(\tilde{x})$ and $z \in B(\tilde{x})$. This completes the proof. □

Remark 3.2 Theorem 3.2 can be considered as a generalization of Theorem 3.3 in [4] under different conditions from the topological vector space to the abstract convex space.

Corollary 3.1 *Assume that the conditions* (i), (iii), (iv), *and* (v) *in Theorem* 3.1 *are satisfied with $B = S$. Suppose that, for each $y \in E$, $F(\cdot, y)$ is l.s.c. and C is closed. Then there exists $\tilde{x} \in E$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y) \subseteq C(\tilde{x})$ for all $y \in S(\tilde{x})$.*

Proof The proof is similar to that of Theorem 3.2 and so we omit it here. □

Remark 3.3 When $S(x) = E$ for all $x \in E$, Corollary 3.1 was given by Theorem 1 of Yang and Huang [17] under quite different conditions.

Theorem 3.3 *Assume that the conditions* (i), (iii), (iv), *and* (v) *in Theorem* 3.1 *are satisfied. Moreover, suppose that*

(a) *for each $y \in E$, $F(\cdot, y, \cdot)$ is l.s.c. and C is closed*;

(b) *B is u.s.c. and $B(x)$ is compact for each $x \in E$.*
Then there exist $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y, \tilde{z}) \subseteq C(\tilde{x})$ for all $y \in S(\tilde{x})$.

Proof Let

$$A(x) = \{y \in E : \forall z \in B(x), F(x,y,z) \nsubseteq C(x)\}.$$

We first show that

$$A^{-1}(y) = \{x \in E : \forall z \in B(x), F(x,y,z) \nsubseteq C(x)\}$$

is open. Let $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$ be a net with $x_\alpha \to x_0$. Then there exists $z_\alpha \in B(x_\alpha)$ such that $F(x_\alpha, y, z_\alpha) \subseteq C(x_\alpha)$. Since B is u.s.c. with compact values, by Lemma 2.2, there exists a subset net of $\{z_\alpha\}$, denoted again by $\{z_\alpha\}$, such that $z_\alpha \to z_0 \in B(x_0)$. The fact that $F(\cdot, y, \cdot)$ is l.s.c. together with Lemma 2.3 shows that, for any $v \in F(x_0, y, z_0)$, there exists $v_\alpha \in F(x_\alpha, y, z_\alpha)$ such that $v_\alpha \to v$. Since $v_\alpha \in C(x_\alpha)$ and C is closed, we know that $v \in C(x_0)$ and so $F(x_0, y, z_0) \subseteq C(x_0)$ for some $z_0 \in B(x_0)$. This implies that $x_0 \in E \setminus A^{-1}(y)$ and so $E \setminus A^{-1}(y)$ is closed. Thus, $A^{-1}(y)$ is open. It follows from Theorem 3.1 that there exist $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y, \tilde{z}) \subseteq C(\tilde{x})$ for any $y \in S(\tilde{x})$. This completes the proof. □

Remark 3.4 When $S(x) = E$ for all $x \in E$, the existence of the solutions for generalized vector quasi-equilibrium was studied in Theorem 3.1 of [7] in real Hausdorff topological vector spaces.

Theorem 3.4 *Assume that the conditions* (i), (iii), (iv), *and* (v) *in Theorem* 3.1 *are satisfied. Moreover, suppose that*

(a) *for each $y \in E$, $F(\cdot, y, \cdot)$ is l.s.c., $C(x)$ is a set-valued mapping with a nonempty interior for each $x \in E$, the mapping $W : E \rightrightarrows V$, defined by $W(x) = V \setminus (-\operatorname{int} C(x))$, is closed;*
(b) *B is l.s.c.*

Then there exists $\tilde{x} \in E$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y, z) \cap (-\operatorname{int} C(\tilde{x})) = \emptyset$ for all $y \in S(\tilde{x})$ and $z \in B(\tilde{x})$.

Proof Let

$$A(x) = \{y \in E : \exists z \in B(x), F(x,y,z) \cap (-\operatorname{int} C(x)) \neq \emptyset\}.$$

We prove that

$$A^{-1}(y) = \{x \in E : \exists z \in B(x), F(x,y,z) \cap (-\operatorname{int} C(x)) \neq \emptyset\}$$

is open. Let $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$ be a net with $x_\alpha \to x_0$. Then

$$F(x_\alpha, y, z') \cap (-\operatorname{int} C(x_\alpha)) = \emptyset, \quad \forall z' \in B(x_\alpha)$$

and so

$$F(x_\alpha, y, z') \subseteq W(x_\alpha) = V \setminus (-\operatorname{int} C(x_\alpha)).$$

Similar to the proof of Theorem 3.2, we get

$$F(x_0, y, z) \subseteq W(x_0) = V \setminus \bigl(-\operatorname{int} C(x_0)\bigr).$$

This shows that $x_0 \in E \setminus A^{-1}(y)$ and so $E \setminus A^{-1}(y)$ is closed. Thus, $A^{-1}(y)$ is open. It follows from Theorem 3.1 that there exists $\tilde{x} \in E$ such that $\tilde{x} \in S(\tilde{x})$ and

$$F(\tilde{x}, y, z) \cap \bigl(-\operatorname{int} C(\tilde{x})\bigr) = \emptyset, \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

This completes the proof. □

Remark 3.5 Theorem 3.4 can be considered as a generalization of Theorem 3.2 in [4] under different conditions from the topological vector space to the abstract convex space.

Corollary 3.2 *Assume that the conditions* (i), (iii), (iv), *and* (v) *in Theorem* 3.1 *are satisfied with* $S = B$. *Moreover, suppose that*

(a) *for each* $y \in E$, $F(\cdot, y)$ *is l.s.c.,* $C(x)$ *has a nonempty interior for each* $x \in E$, *the map* $W : E \rightrightarrows V$, *defined by* $W(x) = V \setminus (-\operatorname{int} C(x))$, *is closed.*

Then there exists $\tilde{x} \in E$ *such that* $\tilde{x} \in S(\tilde{x})$ *and* $F(\tilde{x}, y) \cap (-\operatorname{int} C(\tilde{x})) = \emptyset$ *for all* $y \in S(\tilde{x})$.

Proof The proof is similar to that of Theorem 3.4 and so we omit it here. □

Remark 3.6 When $S(x) = E$ for all $x \in E$, Corollary 3.2 was given by Theorem 2 of Yang and Huang [17] under quite different conditions.

Theorem 3.5 *Suppose the conditions* (i), (iii), (iv), *and* (v) *in Theorem* 3.1 *are satisfied. Moreover, suppose that*

(a) *for each* $y \in E$, $F(\cdot, y, \cdot)$ *is l.s.c.,* $C(x)$ *has a nonempty interior for each* $x \in E$, *and the mapping* $W : E \rightrightarrows V$, *defined by* $W(x) = V \setminus (-\operatorname{int} C(x))$, *is closed;*

(b) B *is u.s.c. and* $B(x)$ *is compact for each* $x \in E$.

Then there exist $\tilde{x} \in E$ *and* $\tilde{z} \in B(\tilde{x})$ *such that* $\tilde{x} \in S(\tilde{x})$ *and* $F(\tilde{x}, y, \tilde{z}) \cap (-\operatorname{int} C(\tilde{x})) = \emptyset$ *for all* $y \in S(\tilde{x})$.

Proof Let

$$A(x) = \bigl\{y \in E : \forall z \in B(x), F(x, y, z) \cap \bigl(-\operatorname{int} C(x)\bigr) \neq \emptyset\bigr\}.$$

We show that

$$A^{-1}(y) = \bigl\{x \in E : \forall z \in B(x), F(x, y, z) \cap \bigl(-\operatorname{int} C(x)\bigr) \neq \emptyset\bigr\}$$

is open. Let $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$ be a net with $x_\alpha \to x_0$. Then

$$F(x_\alpha, y, z_\alpha) \cap \bigl(-\operatorname{int} C(x_\alpha)\bigr) = \emptyset$$

for some $z_\alpha \in B(x_\alpha)$, that is,

$$F(x_\alpha, y, z_\alpha) \subseteq V \setminus \bigl(-\operatorname{int} C(x_\alpha)\bigr).$$

Using similar arguments to the proof of Theorem 3.3, we have

$$F(x_0, y, z_0) \subseteq W(x_0) = V \setminus \bigl(-\operatorname{int} C(x_0)\bigr)$$

for some $z_0 \in B(x_0)$. This shows that $x_0 \in E \setminus A^{-1}(y)$ and so $E \setminus A^{-1}(y)$ is closed. Thus, $A^{-1}(y)$ is open. It follows from Theorem 3.1 that there exist $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that $\tilde{x} \in S(\tilde{x})$ and

$$F(\tilde{x}, y, \tilde{z}) \cap \bigl(-\operatorname{int} C(\tilde{x})\bigr) = \emptyset, \quad \forall y \in S(\tilde{x}).$$

This completes the proof. □

Remark 3.7 When E is a nonempty convex compact of a topological vector space, Li and Li [27] studied the existence of solutions for (GVQEP4).

Theorem 3.6 *Assume that the conditions* (i), (iii), (iv), *and* (v) *are satisfied in Theorem* 3.1. *Moreover, suppose that*

(a) *for each $y \in E$, $F(\cdot, y, \cdot)$ is u.s.c. with compact valued on $E \times E \times E$ and $C(x)$ has a nonempty interior for each $x \in E$, the mapping $W : E \rightrightarrows V$, defined by $W(x) = V \setminus (-\operatorname{int} C(x))$, is closed*;

(b) *B is l.s.c.*

Then there exists $\tilde{x} \in E$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y, z) \nsubseteq -\operatorname{int} C(\tilde{x})$ for all $y \in S(\tilde{x})$ and $z \in B(\tilde{x})$.

Proof Let

$$A(x) = \bigl\{y \in E : \exists z \in B(x), F(x, y, z) \subseteq -\operatorname{int} C(x)\bigr\}.$$

We prove that

$$A^{-1}(y) = \bigl\{x \in E : \exists z \in B(x), F(x, y, z) \subseteq -\operatorname{int} C(x)\bigr\}$$

is open. Let $x_\alpha \in E \setminus A^{-1}(y)$ be a net with $x_\alpha \to x_0$. Then

$$F\bigl(x_\alpha, y, z'\bigr) \nsubseteq -\operatorname{int} C(x_\alpha)$$

for any $z' \in B(x_\alpha)$ and so there exists $v_\alpha \in V$ such that

$$v_\alpha \in F\bigl(x_\alpha, y, z'\bigr) \setminus \bigl(-\operatorname{int} C(x_\alpha)\bigr).$$

Since B is l.s.c., by Lemma 2.3, for any $z \in B(x_0)$, there exists $z_\alpha \in B(x_\alpha)$ such that $z_\alpha \to z$. Since $F(\cdot, y, \cdot)$ is u.s.c. with compact valued, by Lemma 2.2, there exists a subset net of $\{v_\alpha\}$, denoted again by $\{v_\alpha\}$, such that $v_\alpha \to v_0 \in F(x_0, y, z)$. On the other hand, the fact that $v_\alpha \notin -\operatorname{int} C(x_\alpha)$ shows that $v_\alpha \in W(x_\alpha)$. Now the closedness of W shows that $v_0 \in W(x_0)$ and so $v_0 \notin -\operatorname{int} C(x_0)$. Thus $F(x_0, y, z) \nsubseteq -\operatorname{int} C(x_0)$ for any $z \in B(x_0)$. This implies that $x_0 \in E \setminus A^{-1}(y)$ and so $E \setminus A^{-1}(y)$ is closed. Thus, $A^{-1}(y)$ is open. It follows from Theorem 3.1 that there exists $\tilde{x} \in E$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y, z) \nsubseteq -\operatorname{int} C(\tilde{x})$ for all $y \in S(\tilde{x})$ and $z \in B(\tilde{x})$. This completes the proof. □

Corollary 3.3 *Assume that the conditions* (i), (iii), (iv), *and* (v) *in Theorem* 3.1 *are satisfied with $S = B$. Moreover, suppose that*

(a) *for each $y \in E$, $F(\cdot, y)$ is u.s.c. with compact valued on $E \times E$ and $C(x)$ has a nonempty interior for each $x \in E$, the mapping $W : E \rightrightarrows V$, defined by $W(x) = V \setminus (-\operatorname{int} C(x))$, is closed.*

Then there exists $\tilde{x} \in E$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y) \nsubseteq -\operatorname{int} C(\tilde{x})$ for all $y \in S(\tilde{x})$.

Proof The proof is similar to that of Theorem 3.6 and so we omit it here. □

Remark 3.8 When $S(x) = E$ for all $x \in E$, Corollary 3.3 was given by Theorem 4 of Yang and Huang [17] under quite different conditions.

Theorem 3.7 *Assume that the conditions* (i), (iii), (iv), *and* (v) *are satisfied in Theorem* 3.1. *Moreover, suppose that*

(a) *for each $y \in E$, $F(\cdot, y, \cdot)$ is u.s.c. with compact valued on $E \times E \times E$ and $C(x)$ has a nonempty interior for each $x \in E$, the mapping $W : E \rightrightarrows V$, defined by $W(x) = V \setminus (-\operatorname{int} C(x))$, is closed.*

(b) *B is u.s.c. and $B(x)$ is compact for each $x \in E$.*

Then there exist $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y, \tilde{z}) \nsubseteq -\operatorname{int} C(\tilde{x})$ for all $y \in S(\tilde{x})$.

Proof Let

$$A(x) = \{y \in E : \forall z \in B(x), F(x, y, z) \subseteq -\operatorname{int} C(x)\}.$$

We prove that

$$A^{-1}(y) = \{x \in E : \forall z \in B(x), F(x, y, z) \subseteq -\operatorname{int} C(x)\}$$

is open. Let $x_\alpha \in E \setminus A^{-1}(y)$ be a net with $x_\alpha \to x_0$. Then $F(x_\alpha, y, z_\alpha) \nsubseteq -\operatorname{int} C(x_\alpha)$ for some $z_\alpha \in B(x_\alpha)$ and so there exists $v_\alpha \in V$ such that

$$v_\alpha \in F(x_\alpha, y, z_\alpha) \setminus (-\operatorname{int} C(x_\alpha)).$$

Since B is u.s.c. with compact valued, by Lemma 2.2, there exists a subnet of $\{z_\alpha\}$, denoted again by $\{z_\alpha\}$, such that $z_\alpha \to z_0 \in B(x_0)$. The fact that $F(\cdot, y, \cdot)$ is u.s.c. with compact valued together with Lemma 2.2 shows that there exists a subset net of $\{v_\alpha\}$, denoted again by $\{v_\alpha\}$, such that $v_\alpha \to v_0 \in F(x_0, y, z_0)$. On the other hand, it is easy to see that $v_\alpha \in W(x_\alpha)$. Since W is closed, we know that $v_0 \in W(x_0)$ and so $v_0 \notin -\operatorname{int} C(x_0)$. Thus $F(x_0, y, z_0) \nsubseteq -\operatorname{int} C(x_0)$ for some $z_0 \in B(x_0)$ and so $x_0 \in E \setminus A^{-1}(y)$. This implies that $E \setminus A^{-1}(y)$ is closed and so $A^{-1}(y)$ is open. It follows from Theorem 3.1 that there exist $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that $\tilde{x} \in S(\tilde{x})$ and $F(\tilde{x}, y, \tilde{z}) \nsubseteq -\operatorname{int} C(\tilde{x})$ for all $y \in S(\tilde{x})$. This completes the proof. □

Remark 3.9 Theorem 3.7 can be considered as a generalization of Theorem 3.1 in [2, 4] under different conditions from the topological vector space to the abstract convex space.

Remark 3.10 When E is a nonempty convex compact of topological vector space, Li and Li [27] studied the existence of solutions for (GVQEP6).

Theorem 3.8 *Suppose the conditions* (i), (iii), (iv), *and* (v) *in Theorem* 3.1 *are satisfied. Moreover, assume that*

(a) *for each* $y \in E$, $F(\cdot, y, \cdot)$ *is u.s.c. with compact valued on* $E \times E \times E$ *and* C *is closed*;

(b) B *is l.s.c.*

There exists $\tilde{x} \in E$ *such that* $\tilde{x} \in S(\tilde{x})$ *and* $F(\tilde{x}, y, z) \cap C(\tilde{x}) \neq \emptyset$ *for all* $y \in S(\tilde{x})$ *and* $z \in B(x)$.

Proof Let

$$A(x) = \{y \in E : \exists z \in B(x), F(x, y, z) \cap C(x) = \emptyset\}.$$

We show that

$$A^{-1}(y) = \{x \in E : \exists z \in B(x), F(x, y, z) \cap C(x) = \emptyset\}$$

is open. Let $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$ be a net with $x_\alpha \to x_0$. Then

$$F(x_\alpha, y, z') \cap C(x_\alpha) \neq \emptyset, \quad \forall z' \in B(x_\alpha).$$

It follows that there exists $v_\alpha \in F(x_\alpha, y, z') \cap C(x_\alpha)$. Since B is l.s.c., by Lemma 2.3, there exists $z_\alpha \in B(x_\alpha)$ such that $z_\alpha \to z$ for any $z \in B(x_0)$. By the fact that $F(\cdot, y, \cdot)$ is u.s.c. with compact valued, there exists a subset of $\{v_\alpha\}$, denoted again by $\{v_\alpha\}$, such that $v_\alpha \to v_0 \in F(x_0, y, z)$. Since $v_\alpha \in C(x_\alpha)$ and C is closed, we know that $v_0 \in C(x_0)$ and so $v_0 \in F(x_0, y, z) \cap C(x_0)$. Thus,

$$F(x_0, y, z) \cap C(x_0) \neq \emptyset, \quad \forall z \in B(x_0).$$

This shows that $x_0 \in E \setminus A^{-1}(y)$ and so $E \setminus A^{-1}(y)$ is closed. Thus, $A^{-1}(y)$ is open. By Theorem 3.1, there exists $\tilde{x} \in E$ such that $\tilde{x} \in S(\tilde{x})$ and

$$F(\tilde{x}, y, z) \cap C(x) \neq \emptyset, \quad \forall y \in S(\tilde{x}), \forall z \in B(\tilde{x}).$$

This completes the proof. □

Corollary 3.4 *Assume that the conditions* (i), (iii), (iv), *and* (v) *in Theorem* 3.1 *are satisfied with* $S = B$. *Moreover, suppose that, for each* $y \in E$, $F(\cdot, y)$ *is u.s.c. with compact valued on* $E \times E$ *and* C *is closed. Then there exists* $\tilde{x} \in E$ *such that* $\tilde{x} \in S(\tilde{x})$ *and* $F(\tilde{x}, y) \cap C(\tilde{x}) \neq \emptyset$ *for all* $y \in S(\tilde{x})$.

Proof The proof is similar to that of Theorem 3.8 and so we omit it here. □

Remark 3.11 When $S(x) = E$ for all $x \in E$, Corollary 3.4 was given by Theorem 3 of Yang and Huang [17] under some different conditions.

Theorem 3.9 *Suppose the conditions* (i), (iii), (iv), *and* (v) *are satisfied in Theorem* 3.1. *Moreover, assume that*

(a) *for each* $y \in E$, $F(\cdot, y, \cdot)$ *is u.s.c. with compact valued on* $E \times E \times E$ *and* C *is closed*;

(b) B *is u.s.c. and* $B(x)$ *is compact for each* $x \in E$.

There exist $\tilde{x} \in E$ *and* $\tilde{z} \in B(\tilde{x})$ *such that* $\tilde{x} \in S(\tilde{x})$ *and* $F(\tilde{x}, y, \tilde{z}) \cap C(\tilde{x}) \neq \emptyset$ *for all* $y \in S(\tilde{x})$.

Proof Let

$$A(x) = \{y \in E : \forall z \in B(x), F(x,y,z) \cap C(x) = \emptyset\}.$$

We prove that

$$A^{-1}(y) = \{x \in E : \forall z \in B(x), F(x,y,z) \cap C(x) = \emptyset\}$$

is open. Let $\{x_\alpha\} \subseteq E \setminus A^{-1}(y)$ be a net with $x_\alpha \to x_0$. Then

$$F(x_\alpha, y, z_\alpha) \cap C(x_\alpha) \neq \emptyset$$

for some $z_\alpha \in B(x_\alpha)$, that is, there exists $v_\alpha \in F(x_\alpha, y, z_\alpha) \cap C(x_\alpha)$. Since B is u.s.c. and $B(x)$ is compact, it follows from Lemma 2.2 that there exists a subset of $\{z_\alpha\}$, denoted again by $\{z_\alpha\}$, such that $z_\alpha \to z_0 \in B(x_0)$. Similar to the proof of Theorem 3.8, we can prove that $F(x_0, y, z_0) \cap C(x_0) \neq \emptyset$ for some $z_0 \in B(x_0)$. This shows that $x_0 \in E \setminus A^{-1}(y)$ and so $E \setminus A^{-1}(y)$ is closed. Thus, $A^{-1}(y)$ is open. It follows from Theorem 3.1 that there exist $\tilde{x} \in E$ and $\tilde{z} \in B(\tilde{x})$ such that $\tilde{x} \in S(\tilde{x})$ and

$$F(\tilde{x}, y, \tilde{z}) \cap C(\tilde{x}) \neq \emptyset, \quad \forall y \in S(\tilde{x}).$$

This completes the proof. □

4 Applications to the generalized semi-infinite programs

In this section, by the results presented in Section 3, we give some existence theorems of solutions to the generalized semi-infinite programs. Let L be a real topological vector space ordered by a closed convex pointed cone $H \subseteq L$ with $\operatorname{int} H \neq \emptyset$ and $h : X \rightrightarrows L$ be a u.s.c. mapping with compact values.

Theorem 4.1 *Suppose that all the conditions of Theorem* 3.2 *are satisfied. Moreover, assume that* $F(\cdot,\cdot,\cdot)$ *and* S *are l.s.c. Then there is a solution to the problem*

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x,y,z) \subseteq C(x), \forall y \in S(x), \forall z \in B(x)\}.$$

Proof Theorem 3.2 shows that $K \neq \emptyset$. From Lemma 2.5, it is sufficient to show that $h(K)$ is compact. Since h is u.s.c. and $K \subseteq E$, by Lemma 2.1, we only need to prove that K is closed. Let $\{x_\alpha\} \subseteq K$ be a net with $x_\alpha \to x_0$. Then $x_\alpha \in S(x_\alpha)$ and

$$F(x_\alpha, y', z') \subseteq C(x_\alpha), \quad \forall y' \in S(x_\alpha), \forall z' \in B(x_\alpha).$$

Since S and B are l.s.c., for any $y \in S(x_0)$ and $z \in B(x_0)$, it follows from Lemma 2.3 that there exist $y_\alpha \in S(x_\alpha)$ and $z_\alpha \in B(x_\alpha)$ such that $y_\alpha \to y$ and $z_\alpha \to z$. By the lower semi-continuity of F and Lemma 2.3, for any $v \in F(x_0, y, z)$, there exists $v_\alpha \in F(x_\alpha, y_\alpha, z_\alpha)$ such that $v_\alpha \to v$. Now the closedness of C with $v_\alpha \in C(x_\alpha)$ shows that $v \in C(x_0)$ and so $F(x_0, y, z) \subseteq C(x_0)$ for all $y \in S(x_0)$ and $z \in B(x_0)$. Moreover, the closedness of $E \setminus G_0$ shows that $x_0 \in S(x_0)$. Thus, K is closed. This completes the proof. □

Corollary 4.1 *Suppose that all the conditions of Corollary* 3.1 *are satisfied. Moreover, assume that* $F(\cdot,\cdot)$ *and* S *are l.s.c. Then there is a solution to the problem*

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y) \subseteq C(x), \forall y \in S(x)\}.$$

Remark 4.1 When $S(x) = E$ for all $x \in E$, Corollary 4.1 was given by Theorem 5 of Yang and Huang [17] under some different conditions.

Theorem 4.2 *Suppose that all the conditions of Theorem* 3.3 *are satisfied. Moreover, assume that* $F(\cdot,\cdot,\cdot)$ *and* S *are l.s.c. Then there is a solution to the problem*

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \subseteq C(x), \forall y \in S(x)\}.$$

Proof Obviously, Theorem 3.3 shows that $K \neq \emptyset$. By Lemma 2.5, it is sufficient to prove that $h(K)$ is compact. Since h is u.s.c. and $K \subseteq E$, from Lemma 2.1, we only need to show that K is closed. Let $\{x_\alpha\} \subseteq K$ be a net with $x_\alpha \to x_0$. Then $x_\alpha \in S(x_\alpha)$ and there exists $z_\alpha \in B(x_\alpha)$ such that

$$F(x_\alpha, y', z_\alpha) \subseteq C(x_\alpha), \quad \forall y' \in S(x_\alpha).$$

Since B is a u.s.c. mapping with compact values, it follows from Lemma 2.2 that there exists a subnet of $\{z_\alpha\}$, denoted again by $\{z_\alpha\}$, such that $z_\alpha \to z_0 \in B(x_0)$. For any $y \in S(x_0)$, the lower semi-continuity of S together with Lemma 2.3 implies that there exists $y_\alpha \in S(x_\alpha)$ such that $y_\alpha \to y$. For $v \in F(x_0, y, z_0)$, by the fact that F is l.s.c., it follows from Lemma 2.3 that there exists $v_\alpha \in F(x_\alpha, y_\alpha, z_\alpha)$ such that $v_\alpha \to v$. Now the closedness of C with $v_\alpha \in C(x_\alpha)$ shows that $v \in C(x_0)$ and so there exists $z_0 \in B(x_0)$ such that $F(x_0, y, z) \subseteq C(x_0)$ for all $y \in S(x_0)$. Moreover, the closedness of $E \setminus G_0$ shows that $x_0 \in S(x_0)$. Thus, K is closed. This completes the proof. □

Theorem 4.3 *Suppose that all the conditions of Theorem* 3.4 *are satisfied. Moreover, assume that* $F(\cdot,\cdot,\cdot)$ *and* S *are l.s.c. Then there is a solution to the problem*

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x,y,z) \cap -\operatorname{int} C(x) = \emptyset, \forall y \in S(x), \forall z \in B(x)\}.$$

Proof Theorem 3.4 shows that $K \neq \emptyset$. From Lemma 2.5, it is sufficient to show that $h(K)$ is compact. Since h is u.s.c. and $K \subseteq E$, by Lemma 2.1, we only need to show that K is closed. Let $\{x_\alpha\} \subseteq K$ be a net with $x_\alpha \to x_0$. Then $x_\alpha \in S(x_\alpha)$,

$$F(x_\alpha, y', z') \cap -\operatorname{int} C(x) = \emptyset, \quad \forall y' \in S(x_\alpha), \forall z' \in B(x_\alpha)$$

and so

$$F(x_\alpha, y', z') \subseteq W(x_\alpha), \quad \forall y' \in S(x_\alpha), \forall z' \in B(x_\alpha).$$

Similar to the proof of Theorem 4.1, we have $x_0 \in S(x_0)$,

$$F(x_0, y, z) \subseteq W(x_0), \quad \forall y \in S(x_0), \forall z \in B(x_0)$$

and so

$$F(x_0, y, z) \cap -\operatorname{int} C(x_0) = \emptyset, \quad \forall y \in S(x_0), \forall z \in B(x_0).$$

Thus, K is closed. This completes the proof. □

Corollary 4.2 *Suppose that all the conditions of Corollary* 3.2 *are satisfied. Moreover, assume that* $F(\cdot,\cdot)$ *and* S *are l.s.c. Then there is a solution to the problem*

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x,y) \cap C(x) = \emptyset, \forall y \in S(x)\}.$$

Remark 4.2 When $S(x) = E$ for all $x \in E$, Corollary 4.2 was given by Theorem 6 of Yang and Huang [17] under some different conditions.

Theorem 4.4 *Suppose that all the conditions of Theorem* 3.5 *are satisfied. Moreover, assume that* $F(\cdot,\cdot,\cdot)$ *and* S *are l.s.c. Then there is a solution to the problem*

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x,y,z) \cap -\operatorname{int} C(x) = \emptyset, \forall y \in S(x)\}.$$

Proof It follows from Theorem 3.5 that $K \neq \emptyset$. From Lemma 2.5, it is sufficient to show that $h(K)$ is compact. Since h is u.s.c. and $K \subseteq E$, by Lemma 2.1, we only need to show K is closed. Let $\{x_\alpha\} \subseteq K$ be a net with $x_\alpha \to x_0$. Then $x_\alpha \in S(x_\alpha)$ and there exists $z_\alpha \in B(x_\alpha)$ such that

$$F(x_\alpha, y', z_\alpha) \cap -\operatorname{int} C(x_\alpha) = \emptyset, \quad \forall y' \in S(x_\alpha).$$

Thus, there exists $z_\alpha \in B(x_\alpha)$ such that

$$F(x_\alpha, y', z_\alpha) \subseteq C(x_\alpha), \quad \forall y' \in S(x_\alpha).$$

Similar to the proof of Theorem 4.2, we know that $x_0 \in S(x_0)$ and there exists $z_0 \in B(x_0)$ such that

$$F(x_0, y, z_0) \subseteq W(x_0), \quad \forall y \in S(x_0).$$

Thus, $x_0 \in S(x_0)$ and there exists $z_0 \in B(x_0)$ such that

$$F(x_0, y, z_0) \cap -\operatorname{int} C(x_0) = \emptyset, \quad \forall y \in S(x_0).$$

It follows that K is closed. This completes the proof. □

Theorem 4.5 *Suppose that all the conditions of Theorem* 3.6 *are satisfied. Moreover, assume that* $F(\cdot,\cdot,\cdot)$ *is a u.s.c. mapping with compact values and* S *is l.s.c. Then there is a solution to the problem*

$$\operatorname{wMin}_H h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x, y, z) \nsubseteq -\operatorname{int} C(x), \forall y \in S(x), \forall z \in B(x)\}.$$

Proof Theorem 3.6 shows that $K \neq \emptyset$. From Lemma 2.5, it is sufficient to show that $h(K)$ is compact. Since h is u.s.c. and $K \subseteq E$, by Lemma 2.1, we only need to prove that K is closed. Let $\{x_\alpha\} \subseteq K$ be a net with $x_\alpha \to x_0$. Then $x_\alpha \in S(x_\alpha)$,

$$F(x_\alpha, y', z') \nsubseteq -\operatorname{int} C(x_\alpha), \quad \forall y' \in S(x_\alpha), \forall z' \in B(x_\alpha)$$

and so there exists $v_\alpha \in V$ such that

$$v_\alpha \in F(x_\alpha, y', z') \setminus (-\operatorname{int} C(x_\alpha)).$$

By the lower semi-continuity of S and B, for any $y \in S(x_0)$ and $z \in B(x_0)$, it follows from Lemma 2.3 that there exist $y_\alpha \in S(x_\alpha)$ and $z_\alpha \in B(x_\alpha)$ such that $y_\alpha \to y$ and $z_\alpha \to z$. Since $F(\cdot,\cdot,\cdot)$ is a u.s.c. mapping with compact values, Lemma 2.2 shows that there exists a subnet of $\{v_\alpha\}$, denoted again by $\{v_\alpha\}$, such that $v_\alpha \to v_0 \in F(x_0, y, z)$. On the other hand, the fact that $v_\alpha \notin -\operatorname{int} C(x_\alpha)$ shows that $v_\alpha \in W(x_\alpha)$. Now the closedness of W shows that $v_0 \in W(x_0)$ and so $v_0 \notin -\operatorname{int} C(x_0)$. Moreover, the closedness of $E \setminus G_0$ shows that $x_0 \in S(x_0)$. Thus,

$$F(x_0, y, z) \nsubseteq -\operatorname{int} C(x_0)$$

for all $y \in S(x_0)$ and $z \in B(x_0)$ and so K is closed. This completes the proof. □

Corollary 4.3 *Suppose that all the conditions of Corollary* 3.3 *are satisfied. Moreover, assume that $F(\cdot,\cdot)$ is u.s.c. and S is l.s.c. Then there is a solution to the problem*

$$\operatorname{wMin}_H h(K),$$

where

$$K = \bigl\{x \in E : x \in S(x), F(x, y) \nsubseteq C(x), \forall y \in S(x)\bigr\}.$$

Remark 4.3 When $S(x) = E$ for all $x \in E$, Corollary 4.3 was given by Theorem 8 of Yang and Huang [17] under some different conditions.

Theorem 4.6 *Suppose that all the conditions of Theorem* 3.7 *are satisfied. Moreover, assume that $F(\cdot,\cdot,\cdot)$ is a u.s.c. mapping with compact values and S is l.s.c. Then there is a solution to the problem*

$$\operatorname{wMin}_H h(K),$$

where

$$K = \bigl\{x \in E : x \in S(x), \exists z \in B(x), F(x, y, z) \nsubseteq -\operatorname{int} C(x), \forall y \in S(x)\bigr\}.$$

Proof Theorem 3.7 shows that $K \neq \emptyset$. By Lemma 2.5, it is sufficient to prove that $h(K)$ is compact. Since h is u.s.c. and $K \subseteq E$, from Lemma 2.1, we only need to show that K is closed. Let $\{x_\alpha\} \subseteq K$ be a net with $x_\alpha \to x_0$. Then $x_\alpha \in S(x_\alpha)$ and there exists $z_\alpha \in B(x_\alpha)$ such that

$$F\bigl(x_\alpha, y', z_\alpha\bigr) \nsubseteq -\operatorname{int} C(x_\alpha), \quad \forall y' \in S(x_\alpha).$$

Thus, there exists $v_\alpha \in V$ such that

$$v_\alpha \in F\bigl(x_\alpha, y', z_\alpha\bigr) \setminus \bigl(-\operatorname{int} C(x_\alpha)\bigr).$$

Since B is a u.s.c. mapping with compact values, it follows from Lemma 2.2 that there exists a subnet of $\{z_\alpha\}$, denoted again by $\{z_\alpha\}$, such that $z_\alpha \to z_0 \in B(x_0)$. By the lower semi-continuity of S, for any $y \in S(x_0)$, Lemma 2.3 shows that there exists $y_\alpha \in S(x_\alpha)$ such that $y_\alpha \to y$. Since $F(\cdot,\cdot,\cdot)$ is a u.s.c. mapping with compact values, Lemma 2.2 implies that there exists a subnet of $\{v_\alpha\}$, denoted again by $\{v_\alpha\}$, such that $v_\alpha \to v_0 \in F(x_0, y, z_0)$. Similar to the proof of Theorem 4.5, we can prove that K is closed. This completes the proof. □

Theorem 4.7 *Suppose that all the conditions of Theorem* 3.8 *are satisfied. Moreover, assume that* $F(\cdot,\cdot,\cdot)$ *is a u.s.c. mapping with compact values and* S *is l.s.c. Then there is a solution to the problem*

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x,y,z) \cap C(x) \neq \emptyset, \forall y \in S(x), \forall z \in B(x)\}.$$

Proof Theorem 3.8 shows that $K \neq \emptyset$. From Lemma 2.5, it is sufficient to show that $h(K)$ is compact. Since h is u.s.c. and $K \subseteq E$, by Lemma 2.1, we only need to show K is closed. Let $\{x_\alpha\} \subseteq K$ be a net with $x_\alpha \to x_0$. Then $x_\alpha \in S(x_\alpha)$,

$$F(x_\alpha, y', z') \cap C(x_\alpha) \neq \emptyset, \quad \forall y' \in S(x_\alpha), z \in B(x_\alpha)$$

and so there exists $v_\alpha \in V$ such that

$$v_\alpha \in F(x_\alpha, y', z') \cap C(x_\alpha).$$

By the lower semi-continuity of S and B, for any $y \in S(x_0)$ and $z \in B(x_0)$, Lemma 2.3 shows that there exist $y_\alpha \in S(x_\alpha)$ and $z_\alpha \in B(x_\alpha)$ such that $y_\alpha \to y$ and $z_\alpha \to z$. Since $F(\cdot,\cdot,\cdot)$ is u.s.c. with compact values, by Lemma 2.2, there exists a subnet of $\{v_\alpha\}$, denoted again by $\{v_\alpha\}$, such that $v_\alpha \to v_0 \in F(x_0,y,z)$. Now the closedness of C with $v_\alpha \in C(x_\alpha)$ shows that $v_0 \in C(x_0)$ and so

$$F(x_0,y,z) \cap C(x_0) \neq \emptyset, \quad \forall y \in S(x_0), \forall z \in B(x_0).$$

Moreover, the closedness of $E \setminus G_0$ shows that $x_0 \in S(x_0)$. Thus, K is closed. This completes the proof. □

Corollary 4.4 *Suppose that all the conditions of Corollary* 3.4 *are satisfied. Moreover, assume that* $F(\cdot,\cdot)$ *and* S *are l.s.c. Then there is a solution to the problem*

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), F(x,y) \cap C(x) \neq \emptyset, \forall y \in S(x)\}.$$

Remark 4.4 When $S(x) = E$ for all $x \in E$, Corollary 4.4 was given by Theorem 7 of Yang and Huang [17] under some different conditions.

Theorem 4.8 *Suppose that all the conditions of Theorem* 3.9 *are satisfied. Moreover, assume that* $F(\cdot,\cdot,\cdot)$ *is a u.s.c. mapping with compact values and* S *is l.s.c. Then there is a solution to the problem*

$$\mathrm{wMin}_H\, h(K),$$

where

$$K = \{x \in E : x \in S(x), \exists z \in B(x), F(x,y,z) \cap C(x) \neq \emptyset, \forall y \in S(x)\}.$$

Proof Theorem 3.8 shows that $K \neq \emptyset$. From Lemma 2.5, it is sufficient to prove that $h(K)$ is compact. Since h is u.s.c. and $K \subseteq E$, by Lemma 2.1, we only need to show K is closed. Let $\{x_\alpha\} \subseteq K$ be a net with $x_\alpha \to x_0$. Then $x_\alpha \in S(x_\alpha)$ and there exists $z_\alpha \in B(x_\alpha)$ such that

$$F(x_\alpha, y', z_\alpha) \cap C(x_\alpha) \neq \emptyset, \quad \forall y' \in S(x_\alpha).$$

Thus, there exists $v_\alpha \in V$ such that

$$v_\alpha \in F(x_\alpha, y', z_\alpha) \cap C(x_\alpha).$$

Since B is a u.s.c. mapping with compact values, it follows from Lemma 2.2 that there exists a subnet of $\{z_\alpha\}$, denoted again by $\{z_\alpha\}$, such that $z_\alpha \to z_0 \in B(x_0)$. By the lower semi-continuity of S, for any $y \in S(x_0)$, Lemma 2.3 implies that there exists $y_\alpha \in S(x_\alpha)$ such that $y_\alpha \to y$. Since $F(\cdot,\cdot,\cdot)$ is a u.s.c. mapping with compact values, by Lemma 2.2, there exists a subnet of $\{v_\alpha\}$, denoted again by $\{v_\alpha\}$, such that $v_\alpha \to v_0 \in F(x_0, y, z_0)$. Now the closedness of C with $v_\alpha \in C(x_\alpha)$ shows that $v_0 \in C(x_0)$ and so there exists $z_0 \in B(x_0)$ such that

$$F(x_0, y, z_0) \cap C(x_0) \neq \emptyset, \quad \forall y \in S(x_0).$$

Moreover, the closedness of $E \setminus G_0$ shows that $x_0 \in S(x_0)$. Therefore, K is closed. This completes the proof. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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