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Approximation of common solutions for variational inequalities and fixed point of strict pseudo-contractions in q-uniformly smooth Banach spaces

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Abstract
In the present paper, we introduce a general iterative algorithm for finding a common element of the set of common fixed points of an infinite family of strict pseudo-contractions and the set of solutions of the variational inequalities for finite family of strongly accretive mappings in a q-uniformly smooth Banach space. Furthermore, we prove strong convergence of the iterative sequence under suitable conditions. Our results generalize some recent results.
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1 Introduction

Throughout this paper, we always assume that X is a real Banach space with the dual X^*. Let C be a subset of X, and T be a self-mapping of C. We use $F(T)$ to denote the fixed points of T. For $q > 1$, the generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by

$$J_q(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\},$$

where $\langle \cdot, \cdot\rangle$ denotes the duality pairing between X and X^*. In particular, $J_q = J_2$ is called the normalized duality mapping and $J_q(x) = \|x\|^{q-2}J_2(x)$ for $x \neq 0$. If $X := H$ is a real Hilbert space, then $J = I$ where I is the identity mapping. It is well known that if X is smooth, then J_q is single-valued, which is denoted by j_q [1].

Let $U = \{x \in X : \|x\| = 1\}$. A Banach space X is said to be strictly convex if $\frac{\|x+y\|}{2} \leq 1$ for all $x, y \in X$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also called uniformly convex if $\lim \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$, $\{y_n\}$ in X such that $\|x_n\| = \|y_n\| = 1$ and $\lim \|\frac{x_n+y_n}{2}\| = 1$. A Banach space X is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1}$$

exists for all $x, y \in U$. In this case X is smooth. Also, we define a function $\rho_X : [0, \infty) \to [0, \infty)$ called the modulus of smoothness of X as follows:

$$\rho_X(t) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1 : x \in U, \|y\| < t\right\}.$$

A Banach space X is said to be uniformly smooth if $\frac{\rho_X(t)}{t} \to 0$ as $t \to 0$. Suppose that $q > 1$, then X is said to be q-uniformly smooth if there exists $c > 0$ such that $\rho_X(t) \leq ct^q$. It is easy to see that if X is q-uniformly smooth, then $q \leq 2$ and X is uniformly smooth.

Let C be a nonempty, closed, and convex subset of a Banach space X and D be a nonempty subset of C, then a mapping $Q : C \to D$ is said to be sunny provided

$$Q\big(Qx + t(x - Qx)\big) = Qx,$$

whenever $Qx + t(x - Qx) \in C$ for $x \in C$, and $t \geq 0$. A mapping $Q : C \to D$ is called a retraction if $Qx = x$ for all $x \in D$. Furthermore, Q is a sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive.

A subset D of C is called a sunny nonexpansive retraction of C if there exists a sunny nonexpansive retraction from C onto D. In real Hilbert space, a sunny nonexpansive retraction Q_C coincides with the metric projection from X onto C.

Definition 1.1 A mapping $T : C \to C$ is said to be:

(i) λ-strictly pseudo contractive [2], if for all $x, y \in C$ there exist $\lambda > 0$ and $j_q(x - y) \in J_q(x - y)$ such that

$$\big\langle Tx - Ty, j_q(x - y)\big\rangle \leq \|x - y\|^q - \lambda\big\|(I - T)x - (I - T)y\big\|^q,$$

or equivalently

$$\big\langle (I - T)x - (I - T)y, j_q(x - y)\big\rangle \geq \lambda\big\|(I - T)x - (I - T)y\big\|^q.$$

(ii) L-Lipschitzian if for all $x, y \in C$, there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|.$$

If $0 < L < 1$, then T is a contraction, and if $L = 1$, then T is a nonexpansive mapping.

Remark 1.2 Let C be a nonempty subset of a real Hilbert space H and $T : C \to C$ be a mapping. Then T is said to be k-strictly pseudocontractive [2], if for all $x, y \in C$, there exists constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\| + k\big\|(I - T)x - (I - T)y\big\|^2.$$

Definition 1.3 A mapping $F : C \to X$ is said to be accretive if for all $x, y \in C$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\big\langle Fx - Fy, j_q(x - y)\big\rangle \geq 0.$$

For some $\eta > 0$, $F : C \to X$ is said to be η-strongly accretive if for all $x, y \in C$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Fx - Fy, j_q(x - y)\rangle \geq \eta\|x - y\|^q.$$

For some $\mu > 0$, the mapping $F : C \to X$ is said to be μ-inverse strongly accretive if for all $x, y \in C$ there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle Fx - Fy, j_q(x - y)\rangle \geq \mu\|Fx - Fy\|^q.$$

Note that if $X := H$ is a real Hilbert space, accretive and strongly accretive operators coincide with monotone and strongly monotone operators, respectively.

Let C be a nonempty, closed, and convex subset of X, and $A : C \to X$ be a mapping. The classical variational inequality problem is to find $x^* \in C$ such that

$$\langle Ax^*, j_q(x - x^*)\rangle \geq 0, \quad \forall x \in C, \tag{2}$$

where $j_q(x - x^*) \in J_q(x - x^*)$. The solution set of a variational inequality is denoted by $VI(C, A)$. If $X =: H$ is a real Hilbert space, the variational inequality problem reduces to find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{3}$$

For more details of the variational inequality and its applications, we recommend the reader [3, 4]. On the other hand, we note that the iterative approximations of fixed points for nonexpansive mappings have been extensively studied by many authors [5–9].

In order to find the common element of the solution set of a variational inclusion (3) and the set of fixed points of a nonexpansive mapping, Takahashi and Toyoda [10] introduced the following iterative scheme in a Hilbert space H. Starting with an arbitrary point $x_1 = x \in H$, define sequences $\{x_n\}$ by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)SP_C(x_n - \lambda_n Ax_n), \tag{4}$$

where $A : H \to H$ is an α-inverse-strongly monotone mapping, $S : C \to C$ is a nonexpansive mapping and $\{\alpha_n\}$ is a sequence in $[0,1]$. Under mild conditions, they obtained a weak convergence theorem.

On the other hand, Aoyama *et al.* [11] considered the following algorithm in a uniformly convex and 2-uniformly smooth Banach spaces. For $x_1 = x \in C$,

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Q_C(x_n - \lambda_n Ax_n), \tag{5}$$

where $Q_C : X \to C$ is a sunny nonexpansive retraction, and A is a β-Lipschitzian and η-inverse strongly accretive operator. They proved that $\{x_n\}$ generated by (5) converges weakly to a unique element z of $VI(C, A)$.

Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth uniformly convex Banach space X. Assume the mapping $A_m : C \to X$ be a μ_m-inverse-strongly accretive mapping for each $1 \leq m \leq r$, where r is a positive integer. Let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a family of λ-strict pseudo-contractions with $0 < \lambda < 1$. Define a mapping $S_n x := (1-\gamma_n)x + \gamma_n T_n x$ for all $x \in C$ and $n \geq 1$.

In this paper, motivated by the works mentioned above, we consider the following iteration:

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n) \sum_{m=1}^{r} \eta_n^m Q_C(x_n - \lambda_m A_m x_n), \\ x_{n+1} = Q_C[\beta_n \gamma f x_n + (I - \beta_n \mu F) S_n y_n], \quad n \geq 1, \end{cases} \tag{6}$$

and we prove that the proposed iterative algorithm is strongly convergent under some mild conditions imposed on the algorithm parameters. The results proved in this paper represent a refinement and improvement of the previously found results in the earlier and recent literature.

2 Preliminaries

In order to prove our main results, we need the following lemmas.

Lemma 2.1 [12, 13] *Let C be a closed convex subset of a smooth Banach space X. Let D be a nonempty subset of C. Let $Q : C \to D$ be a retraction and J be the normalized duality mapping on X. Then the following are equivalent*:

(a) *Q is sunny and nonexpansive.*
(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle$, $\forall x, y \in C$.
(c) $\langle x - Qx, J(y - Qx)\rangle \leq 0$, $\forall x \in C$, $y \in D$.
(d) $\langle x - Qx, J_q(y - Qx)\rangle \leq 0$, $\forall x \in C$, $y \in D$.

Lemma 2.2 [14] *Let C be a closed convex subset of a strictly convex Banach space X. Let T_1 and T_2 be two nonexpansive mappings from C into itself with $F(T_1) \cap F(T_2) \neq \emptyset$. Define a mapping S by*

$$Sx = kT_1 x + (1-k)T_2 x, \quad \forall x \in C,$$

where k is a constant in $(0,1)$. Then S is nonexpansive and $F(S) = F(T_1) \cap F(T_2)$.

Lemma 2.3 [15] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} = (1-a_n)s_n + a_n b_n + c_n,$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ satisfy the restrictions:

(i) $\lim_{n\to\infty} a_n = 0$, $\sum_{n=1}^{\infty} a_n = \infty$,
(ii) $c_n \geq 0$, $\sum_{n=1}^{\infty} c_n < \infty$,
(iii) $\limsup_{n\to\infty} b_n \leq 0$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.4 [16] *Suppose that* $q > 1$. *Then the following inequality holds*:

$$ab \leq \frac{1}{q}a^q + \left(\frac{q-1}{q}\right)b^{\frac{q}{q-1}},$$

for arbitrary positive real numbers a, b.

Lemma 2.5 [17] *Let* X *be a real* q-*uniformly smooth Banach space, then there exists a constant* $C_q > 0$ *such that*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q,$$

for all $x, y \in X$. *In particular, if* X *is real* 2-*uniformly smooth Banach space, then there exists a best smooth constant* $K > 0$ *such that*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x)\rangle + 2K\|y\|^2$$

for all $x, y \in C$.

Lemma 2.6 [18] *Let* X *a real smooth and uniformly convex Banach space and let* $r > 0$. *Then there exists a strictly increasing, continuous, and convex function* $g : [0, 2r] \to R$ *such that* $g(0) = 0$ *and* $g(\|x-y\|) \leq \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$, *for all* $x, y \in B_r$, *where* $B_r = \{z \in X : \|z\| \leq r\}$.

Definition 2.7 [11] Let T_n be a family of mappings from a subset C of a Banach space X into itself with $\bigcap_{n=1}^{\infty} F(T_n) \neq \emptyset$. We say that $\{T_n\}$ satisfies the AKTT-condition if for each bounded subset B of C,

$$\sum_{n=1}^{\infty} \sup_{\omega \in B} \|T_{n+1}\omega - T_n\omega\| < \infty. \tag{7}$$

Lemma 2.8 [11] *Suppose that* $\{T_n\}$ *satisfies the AKTT-condition such that*:

(i) *For each* $x \in C$, $\{T_n x\}$ *is converge strongly to some point in* C.

(ii) *Let the mapping* $T : C \to C$ *defined by* $Tx = \lim_{n\to\infty} T_n x$, *for all* $x \in C$.

Then $\lim_{n\to\infty} \sup_{\omega \in B} \|T\omega - T_n\omega\| = 0$, *for each bounded subset* B *of* C.

Lemma 2.9 [7, 8] *Let* C *be a closed and convex subset of a smooth Banach space* X. *Suppose that* $\{T_n\}_{n=1}^{\infty} : C \to X$ *is a family of* λ-*strictly pseudocontractive mappings*; $\{\mu_m\}_{m=1}^{\infty}$ *is a real sequence in* $(0, 1)$ *such that* $\sum_{n=1}^{\infty} \mu_m = 1$. *Then the following conclusions hold*:

(i) *A mapping* $G : C \to X$ *defined by* $G := \sum_{n=1}^{\infty} \mu_n T_n$ *is a* λ-*strictly pseudocontractive mapping.*

(ii) $F(G) = \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 2.10 [19] *Let* C *be a nonempty, closed, and convex subset of a real* q-*uniformly smooth Banach space* X *which admits weakly sequentially continuous generalized duality mapping* j_q *from* X *into* X^*. *Let* $T : C \to C$ *be a nonexpansive mapping. Then, for all* $\{x_n\} \subset C$, *if* $x_n \rightharpoonup x$ *and* $x_n - Tx_n \to 0$, *then* $x = Tx$.

Lemma 2.11 [19] *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space X. Let $F : C \to E$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$. Let $0 < \mu < (\frac{q\eta}{C_q k^q})^{\frac{1}{q-1}}$ and $\tau = \mu(\eta - \frac{C_q \mu^{q-1} k^q}{q})$. Then for $t \in (0, \min\{1, \frac{1}{\tau}\})$, the mapping $S : C \to E$ defined by $S := (I - t\mu F)$ is a contraction with a constant $1 - t\tau$.*

Lemma 2.12 [19] *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $F : C \to X$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$, $f : C \to X$ be an L-Lipschitzian mapping with a constant $L \geq 0$ and $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $0 < \mu < (\frac{q\eta}{C_q k^q})^{\frac{1}{q-1}}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \frac{C_q \mu^{q-1} k^q}{q})$. Then $\{x_t\}$ defined by*

$$x_t = Q_C\bigl[t\gamma f x_t + (I - t\mu F)Sx_t\bigr] \tag{8}$$

has the following properties:

(i) $\{x_t\}$ *is bounded for each* $t \in (0, \min\{1, \frac{1}{\tau}\})$.

(ii) $\lim_{t\to 0} \|x_t - Sx_t\| = 0$.

(iii) $\{x_t\}$ *defines a continuous curve from* $(0, \min\{1, \frac{1}{\tau}\})$ *into* C.

Lemma 2.13 [13] *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q from X into X^*. Let Q_C be a sunny nonexpansive retraction from X onto C. Let $F : C \to X$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$, $f : C \to X$ be an L-Lipschitzian mapping with a constant $L \geq 0$, and $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Suppose that $0 < \mu < (\frac{q\eta}{C_q k^q})^{\frac{1}{q-1}}$ and $0 \leq \gamma L < \tau$, where $\tau = \mu(\eta - \frac{C_q \mu^{q-1} k^q}{q})$. For each $t \in (0, \min\{1, \frac{1}{\tau}\})$, let $\{x_t\}$ be defined by* (8), *then $\{x_t\}$ converges strongly to $x^* \in F(S)$ as $t \to 0$, in which x^* is the unique solution of the variational inequality*

$$\bigl\langle (\mu F - \gamma V)x^*, j_q\bigl(x^* - p\bigr)\bigr\rangle \leq 0, \quad \forall p \in F(S). \tag{9}$$

Lemma 2.14 [20] *Let X be a Banach space and J be a normality duality mapping. Then for any given $x, y \in X$, the following inequality holds*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\bigl\langle y, j(x + y)\bigr\rangle,$$

for all $j(x + y) \in J(x + y)$.

3 Main results

Theorem 3.1 *Let C be a nonempty, closed, and convex subset of a real q-uniformly smooth, uniformly convex Banach space X. Let Q_C be a sunny nonexpansive retraction from X onto C. Assume that the mapping $A_m : C \to H$ is a μ_m-inverse-strongly accretive mapping for each $1 \leq m \leq r$, where r is a positive integer. Let $F : C \to X$ be a k-Lipschitzian and η-strongly accretive operator with constants $k, \eta > 0$, $f : C \to X$ be an L-Lipschitzian mapping with a constant $L \geq 0$. Suppose that $0 < \mu < (\frac{q\eta}{C_q k^q})^{\frac{1}{q-1}}$ and $0 \leq \gamma L < \tau$, where*

$\tau = \mu(\eta - \frac{C_q\mu^{q-1}k^q}{q})$. *Let* $\{T_n\}_{n=1}^{\infty} : C \to C$ *be a family of* λ*-strict pseudo-contractions with* $0 < \lambda < 1$. *Define a mapping* $S_n x := (1-\gamma_n)x + \gamma_n T_n x$, *for all* $x \in C$ *and* $n \geq 1$. *Assume that* $F := (\bigcap_{m=1}^{r} VI(C, A_m)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \neq \emptyset$. *Let* $\{x_n\}$ *be a sequence generated by the following iterative algorithm*:

$$\begin{cases} x_1 \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\sum_{m=1}^{r} \eta_n^m Q_C(x_n - \lambda_m A_m x_n), \\ x_{n+1} = Q_C[\beta_n \gamma f x_n + (I - \beta_n \mu F)S_n y_n], \quad n \geq 1, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\eta_n^1\}, \{\eta_n^2\}, \ldots$ *and* $\{\eta_n^r\}$ *are sequences in* $(0,1)$ *and* λ_m *is a real number such that* $0 < \lambda_m < (\frac{q\mu_m}{C_q})^{\frac{1}{q-1}}$, *for each* $1 \leq m \leq r$. *Assume that the above control sequences satisfy the following restrictions*:

(i) $\sum_{m=1}^{r} \eta_n^m = 1$, $\forall n \geq 1$, $\sum_{n=1}^{\infty} |\eta_{n+1}^m - \eta_n^m| < \infty$.

(ii) $\lim_{n\to\infty} \eta_n^m = \eta^m \in (0,1)$, *for each* m, *where* $1 \leq m \leq r$.

(iii) $\sum_{n=1}^{\infty} \beta_n = \infty$, $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

(iv) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\liminf_{n\to\infty} \alpha_n > 0$.

(v) $0 \leq \gamma_n \leq \delta$, $\delta = \min\{1, (\frac{q\lambda}{C_q})^{\frac{1}{q-1}}\}$, *and* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

Suppose in addition that $\{T_n\}_{n=0}^{\infty}$ *satisfies the AKTT-condition. Let* $T : C \to C$ *be the mapping defined by* $Tx = \lim_{n\to\infty} T_n x$ *for all* $x \in C$ *and suppose that* $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. *Then the sequence* $\{x_n\}$ *converges strongly to* $x^* \in F$ *as* $n \to \infty$, *in which* x^* *is the unique solution of the variational inequality,*

$$\langle (\mu F - \gamma f)x^*, j_q(x^* - p)\rangle \leq 0, \quad \forall p \in F(S).$$

Proof We divide the proof into several steps.

Step 1. We show that $I - \lambda_m A_m$ is nonexpansive for each m. Indeed, from Lemma 2.4, for all $x, y \in C$ we have

$$\begin{aligned} &\|(I-\lambda_m A_m)x - (I-\lambda_m A_m)y\|^q \\ &\quad = \|(x-y) - \lambda_m(A_m x - A_m y)\|^q \\ &\quad \leq \|x-y\|^q - q\lambda_m\langle A_m x - A_m y, j_q(x-y)\rangle + C_q\lambda_m^q\|A_m x - A_m y\|^q \\ &\quad \leq \|x-y\|^q - q\mu_m\lambda_m\|A_m x - A_m y\|^q + C_q\lambda_m^q\|A_m x - A_m y\|^q \\ &\quad \leq \|x-y\|^q - \lambda_m(q\mu_m - C_q\lambda_m^{q-1})\|A_m x - A_m y\|^q. \end{aligned}$$

It is clear that if $0 < \lambda_m \leq (\frac{q\mu_m}{C_q})^{\frac{1}{q-1}}$, then $I - \lambda_m A_m$ is nonexpansive for each $1 \leq m \leq r$.

Now, for each $1 \leq m \leq r$, put

$$k_n^m = Q_C(x_n - \lambda_m A_m x_n), \qquad z_n = \sum_{m=1}^{r} \eta_n^m k_n^m.$$

Let $x^* \in F$, we have

$$\begin{aligned} \|k_n^m - x^*\| &= \|Q_C(x_n - \lambda_m A_m x_n) - Q_C(x^* - \lambda_n A_m x^*)\| \\ &\leq \|x_n - x^*\| \quad \forall m, 1 \leq m \leq r. \end{aligned}$$

On the other hand we have

$$\begin{aligned}\|y_n - x^*\| &= \left\|\alpha_n x_n + (1-\alpha_n)\sum_{m=1}^{r}\eta_n^m k_n^m - x^*\right\| \\ &\leq \alpha_n\|x_n - x^*\| + (1-\alpha_n)\sum_{m=1}^{r}\eta_n^m\|x_n - x^*\| \\ &= \alpha_n\|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\| = \|x_n - x^*\|. \end{aligned} \tag{10}$$

From (10) and the fact that S_n is nonexpansive [19] we have

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|Q_C(\beta_n\gamma f x_n + (I - \beta_n\mu F)S_n y_n) - Q_C x^*\| \\ &\leq \|\beta_n\gamma f x_n + (I - \beta_n\mu F)S_n y_n - x^*\| \\ &= \|\beta_n(\gamma f x_n - \mu F x^*) + (I - \beta_n\mu F)(S_n y_n - x^*)\| \\ &\leq \beta_n\|\gamma f x_n - \mu F x^*\| + (1 - \beta_n\tau)\|S_n y_n - x^*\| \\ &\leq \beta_n\gamma\|f x_n - f x^*\| + \beta_n\|\gamma f x^* - \mu F x^*\| + (1 - \beta_n\tau)\|y_n - x^*\| \\ &\leq \beta_n L\gamma\|x_n - x^*\| + \beta_n\|\gamma f x^* - \mu F x^*\| + (1 - \beta_n\tau)\|x_n - x^*\| \\ &\leq (1 - \beta_n(\tau - L\gamma))\|x_n - x^*\| + \beta_n\|\gamma f x^* - \mu F x^*\| \\ &\leq \max\{\|x_n - x^*\|, (\tau - \gamma L)^{-1}\|\gamma f x^* - \mu F x^*\|\}. \end{aligned}$$

By induction, we find that

$$\|x_{n+1} - x^*\| \leq \max\{\|x_0 - x^*\|, (\tau - \gamma L)^{-1}\|\gamma f x^* - \mu F x^*\|\}.$$

This shows that $\{x_n\}$ is bounded. Hence by (10), $\{y_n\}$ is also bounded.

Step 2: We show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since

$$\|k_{n+1}^m - k_n^m\| = \|Q_C(I - \lambda_m A_m)x_{n+1} - Q_C(I - \lambda_m A_m)x_n\| \leq \|x_{n+1} - x_n\| \quad \forall 1 \leq m \leq r.$$

On the other hand, we have

$$\begin{aligned}\|z_{n+1} - z_n\| &= \left\|\sum_{m=1}^{r}\eta_{n+1}^m k_{n+1}^m - \sum_{m=1}^{r}\eta_n^m k_n^m\right\| \\ &\leq \left\|\sum_{m=1}^{r}\eta_{n+1}^m k_{n+1}^m - \sum_{m=1}^{r}\eta_{n+1}^m k_n^m\right\| + \left\|\sum_{m=1}^{r}\eta_{n+1}^m k_n^m - \sum_{m=1}^{r}\eta_n^m k_n^m\right\| \\ &\leq \sum_{m=1}^{r}\eta_{n+1}^m\|k_{n+1}^m - k_n^m\| + \sum_{m=1}^{r}|\eta_{n+1}^m - \eta_n^m|\|k_n^m\| \\ &\leq \|x_{n+1} - x_n\| + M\sum_{m=1}^{r}|\eta_{n+1}^m - \eta_n^m|, \end{aligned} \tag{11}$$

where M is an appropriate constant such that

$$M = \max\{\sup\{\|P_C(I - \lambda_m A_m)x_n\| : n \geq 1\} : 1 \leq m \leq r\}.$$

Observe that

$$y_{n+1} - y_n = (\alpha_{n+1} - \alpha_n)(x_{n+1} - z_n) + \alpha_n(x_{n+1} - x_n) + (1 - \alpha_{n+1})(z_{n+1} - z_n).$$

It follows from (11) that

$$\begin{aligned}
\|y_{n+1} - y_n\| &\le |\alpha_{n+1} - \alpha_n|\|x_{n+1} - z_n\| + \alpha_{n+1}\|x_{n+1} - x_n\| + (1 - \alpha_{n+1})\|z_{n+1} - z_n\| \\
&\le |\alpha_{n+1} - \alpha_n|\|x_{n+1} - z_n\| + \alpha_{n+1}\|x_{n+1} - x_n\| \\
&\quad + (1 - \alpha_{n+1})\left(\|x_{n+1} - x_n\| + M\sum_{m=1}^{r}\left|\eta_{n+1}^m - \eta_n^m\right|\right) \\
&\le |\alpha_{n+1} - \alpha_n|\|x_{n+1} - z_n\| + \|x_{n+1} - x_n\| + M\sum_{m=1}^{r}\left|\eta_{n+1}^m - \eta_n^m\right|. \qquad (12)
\end{aligned}$$

Note that

$$\begin{aligned}
\|S_{n+1}y_{n+1} - S_n y_n\| &\le \|S_{n+1}y_{n+1} - S_{n+1}y_n\| + \|S_{n+1}y_n - S_n y_n\| \\
&\le \|y_{n+1} - y_n\| + \left\|(1 - \gamma_{n+1})y_n + \gamma_{n+1}T_n y_n - \left[(1 - \gamma_n)y_n + \gamma_n T_n y_n\right]\right\| \\
&\le \|y_{n+1} - y_n\| + \left\|(\gamma_{n+1} - \gamma_n)(T_{n+1}y_n - y_n) + \gamma_n(T_{n+1}y_n - T_n y_n)\right\| \\
&\le \|y_{n+1} - y_n\| + |\gamma_{n+1} - \gamma_n|\|T_{n+1}y_n - y_n\| + \gamma_n\|T_{n+1}y_n - T_n y_n\| \\
&\le |\alpha_{n+1} - \alpha_n|\|x_{n+1} - z_n\| + \|x_{n+1} - x_n\| + M\sum_{m=1}^{r}\left|\eta_{n+1}^m - \eta_n^m\right| \\
&\quad + |\gamma_{n+1} - \gamma_n|\|T_{n+1}y_n - y_n\| + \gamma_n\|T_{n+1}y_n - T_n y_n\|. \qquad (13)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad = \left\|Q_C\left(\beta_n\gamma f x_n + (I - \beta_n\mu F)S_n y_n\right) - Q_C\left(\beta_{n-1}\gamma f x_{n-1} + (I - \beta_{n-1}\mu F)S_{n-1}y_{n-1}\right)\right\| \\
&\quad \le \left\|\beta_n\gamma f x_n + (I - \beta_n\mu F)S_n y_n - \left(\beta_{n-1}\gamma f x_{n-1} + (I - \beta_{n-1}\mu F)S_{n-1}y_{n-1}\right)\right\| \\
&\quad \le \left\|\beta_n\gamma(f x_n - f x_{n-1}) + (\beta_n - \beta_{n-1})\gamma f x_{n-1}\right. \\
&\qquad \left. + (I - \beta_n\mu F)(S_n y_n - S_{n-1}y_{n-1}) + (\beta_n - \beta_{n-1})\mu F S_{n-1}y_{n-1}\right\| \\
&\quad \le \beta_n\gamma L\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\left(\gamma\|f x_{n-1}\| + \mu\|F S_{n-1}y_{n-1}\|\right) \\
&\qquad + (1 - \beta_n\tau)\|S_n y_n - S_{n-1}y_{n-1}\|. \qquad (14)
\end{aligned}$$

Substituting (13) into (14), we obtain

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \beta_n\gamma L\|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|\left(\gamma\|f x_{n-1}\| + \mu\|F S_{n-1}y_{n-1}\|\right) \\
&\qquad + (1 - \beta_n\tau)\left(|\alpha_n - \alpha_{n-1}|\|x_n - z_{n-1}\| + \|x_n - x_{n-1}\| + M\sum_{m=1}^{r}\left|\eta_n^m - \eta_{n-1}^m\right|\right.
\end{aligned}$$

$$+ |\gamma_n - \gamma_{n-1}| \|T_n y_{n-1} - y_{n-1}\| + \gamma_{n-1} \|T_n y_{n-1} - T_{n-1} y_{n-1}\| \Bigg)$$

$$\leq \big(1 - \beta_n(\tau - \gamma L)\big) \|x_n - x_{n-1}\| + \Bigg(|\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}|$$

$$+ \sum_{m=1}^{r} \big|\eta_n^m - \eta_{n-1}^m\big| \Bigg) M_1 + \|T_n y_{n-1} - T_{n-1} y_{n-1}\|, \tag{15}$$

where $M_1 = \sup_{n \geq 0}\{\gamma \|fx_{n-1}\| + \mu \|FS_{n-1}y_{n-1}\|, \|x_n - z_{n-1}\|, \|T_n y_{n-1} - y_{n-1}\|, M\}$.

Since $\{T_n\}_{n=1}^{\infty}$ satisfies the AKTT-condition, we deduce that

$$\sum_{n=0}^{\infty} \|T_n y_{n-1} - T_{n-1} y_{n-1}\| \leq \sum_{n=0}^{\infty} \sup_{\omega \in \{y_{n-1}\}} \|T_n \omega - T_{n-1} \omega\| < \infty. \tag{16}$$

From (14), (16), and Lemma 2.3, we deduce that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{17}$$

We observe that

$$\begin{aligned}
\|S_n y_n - x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - S_n y_n\| \\
&= \|x_{n+1} - x_n\| + \big\| Q_C\big(\beta_n \gamma f x_n + (I - \beta_n \mu F) S_n y_n\big) - S_n y_n \big\| \\
&= \|x_{n+1} - x_n\| + \big\| \big(\beta_n \gamma f x_n + (I - \beta_n \mu F) S_n y_n\big) - S_n y_n \big\| \\
&= \|x_{n+1} - x_n\| + \beta_n \|\gamma f x_n - \mu F S_n y_n\|.
\end{aligned}$$

From the condition (iii) and (17), we have

$$\lim_{n \to \infty} \|S_n y_n - x_n\| = 0. \tag{18}$$

Step 3. We prove that $\lim_{n \to \infty} \|T_n x_n - x_n\| = 0$.

From Lemma 2.5, we have

$$\begin{aligned}
\big\|k_n^m - x^*\big\|^q &= \big\| Q_C(x_n - \lambda_m A_m x_n) - Q_c\big(x^* - \lambda_m A_m x^*\big) \big\|^q \\
&\leq \big\| (I - \lambda_m A_m) x_n - (I - \lambda_m A_m) x^* \big\|^q \\
&\leq \big\| x_n - x^* \big\|^q - \lambda_m \big(q \mu_m - C_q \lambda_m^{q-1}\big) \big\| A_m x_n - A_m x^* \big\|^q
\end{aligned}$$

and

$$\begin{aligned}
\big\|z_n - x^*\big\|^q &= \left\| \sum_{m=1}^{r} \eta_n^m k_n^m - x^* \right\|^q \leq \sum_{m=1}^{r} \eta_n^m \big\|k_n^m - x^*\big\|^q \\
&\leq \sum_{m=1}^{r} \eta_n^m \Big(\big\|x_n - x^*\big\|^q - \lambda_m \big(q \mu_m - C_q \lambda_m^{q-1}\big) \big\|A_m x_n - A_m x^*\big\|^q \Big) \\
&= \big\|x_n - x^*\big\|^q - \sum_{m=1}^{r} \eta_n^m \lambda_m \big(q \mu_m - C_q \lambda_m^{q-1}\big) \big\|A_m x_n - A_m x^*\big\|^q.
\end{aligned}$$

By the convexity of $\|\cdot\|$, for all $q > 1$, and Lemma 2.5, we obtain

$$\begin{aligned}
&\|x_{n+1} - x^*\|^q \\
&\quad = \|Q_C(\beta_n\gamma fx_n + (I - \beta_n\mu F)S_ny_n) - x^*\|^q \\
&\quad \le \|(\beta_n\gamma fx_n + (I - \beta_n\mu F)S_ny_n) - x^*\|^q \\
&\quad = \|\beta_n(\gamma fx_n - \mu FS_ny_n) + S_ny_n - x^*\|^q \\
&\quad \le \|S_ny_n - x^*\|^q + q\langle\beta_n(\gamma fx_n - \mu FS_ny_n), J_q(S_ny_n - x^*)\rangle + C_q\|\beta_n(\gamma fx_n - \mu FS_ny_n)\|^q \\
&\quad \le \|y_n - x^*\|^q + q\beta_n\|\gamma fx_n - \mu FS_ny_n\|\|S_ny_n - x^*\|^{q-1} + C_q\beta_n^q\|\gamma fx_n - \mu FS_ny_n\|^q \\
&\quad \le \|\beta_nx_n + (1 - \beta_n)z_n - x^*\|^q + \beta_nM_2 \\
&\quad \le \|\beta_n(x_n - x^*) + (1 - \beta_n)(z_n - x^*)\|^q + \beta_nM_2 \\
&\quad \le \beta_n\|x_n - x^*\|^q + (1 - \beta_n)\|z_n - x^*\|^q + \beta_nM_2, \\
&\quad \le \beta_n\|x_n - x^*\|^q + (1 - \beta_n)\Biggl[\|x_n - x^*\|^q - \sum_{m=1}^{r}\eta_n^m\lambda_m(q\mu_m \\
&\qquad - C_q\lambda_m^{q-1})\|A_mx_n - A_mx^*\|^q\Biggr] + \beta_nM_2, \\
&\quad \le \|x_n - x^*\|^q - (1 - \beta_n)\sum_{m=1}^{r}\eta_n^m\lambda_m(q\mu_m - C_q\lambda_m^{q-1})\|A_mx_n - A_mx^*\|^q + \beta_nM_2,
\end{aligned}$$

where

$$M_2 = \sup_{n\ge0}\{q\|\gamma fx_n - \mu FS_ny_n\|\|S_ny_n - x^*\|^{q-1} + C_q\beta_n^{q-1}\|\gamma fx_n - \mu FS_ny_n\|^q\} < \infty.$$

By the fact that $a^r - b^r \le ra^{r-1}(a - b)$, $\forall r \ge 1$, we get

$$\begin{aligned}
&(1 - \beta_n)\sum_{m=1}^{r}\eta_n^m\lambda_m(q\mu_m - C_q\lambda_m^{q-1})\|A_mx_n - A_mx^*\|^q \\
&\quad \le \|x_n - x^*\|^q - \|x_{n+1} - x^*\|^q + \beta_nM_2 \\
&\quad \le q\|x_n - x^*\|^{q-1}(\|x_n - x^*\| - \|x_{n+1} - x^*\|) + \beta_nM_2 \\
&\quad \le q\|x_n - x^*\|^{q-1}\|x_n - x_{n+1}\| + \beta_nM_2.
\end{aligned}$$

Since $0 < \lambda_m < (\frac{q\mu_m}{C_q})^{\frac{1}{q-1}}$, from (17) and (iii) and the fact that $\{x_n\}$ is bounded we have

$$\lim_{n\to\infty}\|A_mx_n - A_mx^*\| = 0, \quad \forall m, 1 \le m \le r. \tag{19}$$

Setting $r_m = \sup\{\|x_n - x^*\|, \|k_n^m - x^*\|\}$, we have from Lemmas 2.1 and 2.6

$$\begin{aligned}
\|k_n^m - x^*\|^2 &= \|Q_C(I - \lambda_mA_m)x_n - Q_C(I - \lambda_mA_m)x^*\|^2 \\
&\le \langle x_n - \lambda_mA_mx_n - (x^* - \lambda_mA_mx^*), j(k_n^m - x^*)\rangle
\end{aligned}$$

$$\begin{aligned}&\leq \langle x_n - x^*, j(k_n^m - x^*)\rangle + \lambda_m\langle A_m x^* - A_m x_n, j(k_n^m - x^*)\rangle \\ &\leq \frac{1}{2}\left[\|x_n - x^*\|^2 + \|k_n^m - x^*\|^2 - g_m\left(\|x_n - x^* - k_n^m + x^*\|\right)\right] \\ &\quad + \lambda_m\langle A_m x^* - A_m x_n, j(k_n^m - x^*)\rangle,\end{aligned}$$

where $g_m : [0, 2r_m) \to [0, \infty)$ is a continuous, strictly increasing, and convex function such that $g_m(0) = 0$ for all $1 \leq m \leq r$. Hence, we have

$$\|k_n^m - x^*\|^2 \leq \|x_n - x^*\|^2 - g_m\left(\|x_n - k_n^m\|\right) + 2\lambda_m\|A_m x^* - A_m x_n\|\|k_n^m - x^*\| \tag{20}$$

for all m, with $1 \leq m \leq r$. On the other hand, we have

$$\|z_n - x_n\|^2 \leq \left\|\sum_{m=1}^r \eta_n^m k_n^m - x_n\right\|^2 \leq \sum_{m=1}^r \eta_n^m \|k_n^m - x_n\|^2.$$

Since g_m is increasing and convex by using (20) we have

$$\begin{aligned}&g_m\left(\|z_n - x_n\|^2\right) \\ &\quad\leq g_m\left(\sum_{m=1}^r \eta_n^m\|k_n^m - x_n\|^2\right) \leq \sum_{m=1}^r \eta_n^m g_m\left(\|k_n^m - x_n\|^2\right) \\ &\quad\leq \sum_{m=1}^r \eta_n^m\left[\|x_n - x^*\|^2 - \|k_n^m - x^*\|^2 + 2\lambda_m\|A_m x^* - A_m x_n\|\|k_n^m - x^*\|\right] \\ &\quad= \|x_n - x^*\|^2 - \sum_{m=1}^r \eta_n^m\|k_n^m - x^*\|^2 + 2\sum_{m=1}^r \eta_n^m\lambda_m\|A_m x^* - A_m x_n\|\|k_n^m - x^*\|.\end{aligned}$$

Thus we have

$$\sum_{m=1}^r \eta_n^m\|k_n^m - x^*\|^2 \leq \|x_n - x^*\|^2 - g_m\left(\|z_n - x_n\|^2\right) + 2\sum_{m=1}^r \eta_n^m\lambda_m\|A_m x^* - A_m x_n\|\|k_n^m - x^*\|.$$

Thanks to Lemma 2.5 we have

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\quad= \left\|Q_C\left(\beta_n\gamma f x_n + (I - \beta_n\mu F)S_n y_n\right) - x^*\right\|^2 \\ &\quad\leq \left\|\left(\beta_n\gamma f x_n + (I - \beta_n\mu F)S_n y_n\right) - x^*\right\|^2 \\ &\quad= \left\|\beta_n(\gamma f x_n - \mu F S_n y_n) + S_n y_n - x^*\right\|^2 \\ &\quad\leq \|S_n y_n - x^*\|^2 + 2\left\langle\beta_n(\gamma f x_n - \mu F S_n y_n), j_q\left(\beta_n(\gamma f x_n - \mu F S_n y_n) + S_n y_n - x^*\right)\right\rangle \\ &\quad\leq \|y_n - x^*\|^2 + \beta_n M_3 \\ &\quad= \|\beta_n x_n + (1 - \beta_n)z_n - x^*\|^2 + \beta_n M_3 \\ &\quad\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\|z_n - x^*\|^2 + \beta_n M_3\end{aligned}$$

$$
\begin{aligned}
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\left(\left\|\sum_{m=1}^{r} \eta_n^m k_n^m - x^*\right\|\right)^2 + \beta_n M_3 \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\sum_{m=1}^{r} \eta_n^m \|k_n^m - x^*\|^2 + \beta_n M_3 \\
&\le \beta_n \|x_n - x^*\|^2 + (1-\beta_n)\Bigg(\|x_n - x^*\|^2 - g_m\big(\|z_n - x_n\|^2\big) \\
&\quad + 2\sum_{m=1}^{r} \eta_n^m \lambda_m \|A_m x^* - A_m x_n\| \|k_n^m - x^*\|\Bigg) + \beta_n M_3 \\
&\le \|x_n - x^*\|^2 - (1-\beta_n) g_m\big(\|z_n - x_n\|^2\big) + 2(1-\beta_n)\sum_{m=1}^{r} \eta_n^m \lambda_m \\
&\quad \times \|A_m x^* - A_m x_n\| \|k_n^m - x^*\| + \beta_n M_3,
\end{aligned}
$$

where $M_3 = \sup_{n\ge 0}\{2\langle \gamma f x_n - \mu F S_n y_n, j_q(\beta_n(\gamma f x_n - \mu f S_n y_n) + S_n y_n - x^*)\rangle\}$.

This in turn implies that

$$
\begin{aligned}
(1-\beta_n) g_m\big(\|z_n - x_n\|^2\big) &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2(1-\beta_n)\sum_{m=1}^{r} \eta_n^m \lambda_m \|A_m x^* - A_m x_n\| \|k_n^m - x^*\| + \beta_n M_3 \\
&\le \|x_n - x_{n+1}\|\big(\|x_n - x^*\| + \|x_{n+1} - x^*\|\big) \\
&\quad + 2(1-\beta_n)\sum_{m=1}^{r} \eta_n^m \lambda_m \|A_m x^* - A_m x_n\| \|k_n^m - x^*\| + \beta_n M_3.
\end{aligned}
$$

In view of (ii), (iii), (17), and (19) we have

$$
\lim_{n\to\infty} g_m\big(\|z_n - x_n\|^2\big) = 0.
$$

By the properties of g_m, we get

$$
\lim_{n\to\infty} \|z_n - x_n\|^2 = 0. \tag{21}
$$

On the other hand,

$$
\begin{aligned}
\|S_n x_n - x_n\| &\le \|S_n x_n - S_n y_n\| + \|S_n y_n - x_n\| \\
&\le \|x_n - y_n\| + \|S_n y_n - x_n\| \\
&\le \|x_n - z_n\| + \|z_n - y_n\| + \|S_n y_n - x_n\| \\
&= \|x_n - z_n\| + \beta_n \|x_n - z_n\| + \|S_n y_n - x_n\|.
\end{aligned}
$$

It follows from (21), (18), and (iii) that

$$
\lim_{n\to\infty} \|S_n x_n - x_n\| = 0. \tag{22}
$$

Next, we show that $\|x_n - Sx_n\| \to 0$ as $n \to \infty$. For any bounded subset B of C, we observe that

$$\begin{aligned}
\sup \|S_{n+1}\omega - S_n\omega\| &= \sup_{\omega\in B}\big\|\gamma_{n+1}\omega + (1-\gamma_{n+1})T_{n+1}\omega - \big(\gamma_n\omega + (1-\gamma_n)T_n\omega\big)\big\| \\
&\leq |\gamma_{n+1}-\gamma_n|\sup_{\omega\in B}|\omega| + (1-\gamma_{n+1})\sup_{\omega\in B}\|T_{n+1}\omega - T_n\omega\| \\
&\quad + |\gamma_{n+1}-\gamma_n|\sup_{\omega\in B}\|T_n\omega\| \\
&\leq |\gamma_{n+1}-\gamma_n|M_3 + \sup_{\omega\in B}\|T_{n+1}\omega - T_n\omega\|,
\end{aligned}$$

where $M_3 = \sup_{n\geq 1}\{\|\omega\|, \|T_n\omega\|\}$. By (v) and the fact that $\{T_n\}$ satisfies the AKTT-condition, we have

$$\sum_{n=1}^{\infty}\sup_{\omega\in B}\|S_{n+1}\omega - S_n\omega\| < \infty,$$

that is, $\{S_n\}$ satisfies the AKTT-condition. Now we define the nonexpansive mapping $S : C \to C$ by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$. Since $\{\gamma_n\}$ is bounded, there exists a subsequence $\{\gamma_{n_i}\}$ of $\{\gamma_n\}$ such that $\gamma_{n_i} \to \nu$ as $i \to \infty$. It follows that

$$Sx = \lim_{i\to\infty} S_{n_i}x = \lim_{i\to\infty}\big[\gamma_{n_i}x + (1-\gamma_{n_i})T_{n_i}x\big] = \nu x + (1-\nu)Tx, \quad \forall x \in C.$$

That is $F(S) = F(T)$. Hence $F(S) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{n=1}^{\infty} F(S_n)$. On the other hand we have

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - S_nx_n\| + \|S_nx_n - Sx_n\| \\
&\leq \|x_n - S_nx_n\| + \sup_{\omega\in\{x_n\}}\|S_n\omega - S\omega\|.
\end{aligned}$$

This implies by Lemma 2.8 and (22) that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{23}$$

Now we define a mapping $h : C \to C$ by

$$hx = \sum_{m=1}^{r}\eta^m P_C(I - \lambda_m A_m)x, \quad \forall x \in C,$$

where $\eta^m = \lim_{n\to\infty}\eta_n^m$. From Lemma 2.9, h is nonexpansive such that

$$F(h) = \bigcap_{m=1}^{r} F\big(P_C(I-\lambda_m A_m)\big) = \bigcap_{m=1}^{r} VI(C, A_m) = \Omega.$$

Next, we define a mapping $U : C \to C$ by $Ux = \delta Sx + (1-\delta)hx$, where $\delta \in (0,1)$ is a constant. Then by Lemma 2.2, U is a nonexpansive and

$$F(U) = F(S) \cap F(h) = \bigcap_{n=1}^{\infty} F(T_n) \cap \Omega = F = F(T) \cap \Omega.$$

Note that

$$\begin{aligned}\|x_n - hx_n\| &\le \|x_n - z_n\| + \|z_n - hx_n\| \\ &\le \|x_n - z_n\| + \left\| \sum_{n=1}^{m} \eta_n^m P_C(I - \lambda_m A_m)x_n - \sum_{m=1}^{r} \eta^m P_C(I - \lambda_m A_m)x_n \right\| \\ &\le \|x_n - z_n\| + M \sum_{m=1}^{r} \left|\eta_n^m - \eta^m\right|.\end{aligned}$$

In view of restriction (ii), we find from (21) that

$$\lim_{n\to\infty} \|x_n - hx_n\| = 0. \tag{24}$$

Setting $x_t = Q_C[t\gamma fx_t + (I - t\mu F)Ux_t]$, it follows from Lemma 2.13 that $\{x_t\}$ converges strongly to a point $x^* \in F(U) = F$, in which x^* is the unique solution of the variational inequality (9). From (23) and (24), we have

$$\begin{aligned}\|x_n - Ux_n\| &= \left\|\delta(x_n - Sx_n) + (1-\delta)(x_n - hx_n)\right\| \\ &\le \delta\|x_n - Sx_n\| + (1-\delta)\|x_n - hx_n\| \to 0.\end{aligned}$$

Step 4. We show that

$$\limsup \left\langle (\gamma f - \mu F)x^*, j_q\left(x_n - x^*\right)\right\rangle \le 0,$$

where x^* is a solution of the variational inequality (9). To show this, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \left\langle (\gamma f - \mu F)x^*, j_q\left(x_n - x^*\right)\right\rangle = \lim_{j\to\infty} \left\langle (\gamma f - \mu F)x^*, j_q\left(x_{n_j} - x^*\right)\right\rangle.$$

By reflexivity of a Banach space X and since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to z. Without loss of generality, we can assume that $x_{n_j} \rightharpoonup z$. Since $\|x_n - Ux_n\| \to 0$ by step 3, we obtain $z = Uz$ and we have $z \in F(U)$. Since Banach space X has a weakly sequentially continuous generalized duality mapping, we obtain

$$\begin{aligned}\limsup_{n\to\infty} \left\langle (\gamma f - \mu F)x^*, j_q\left(x_n - x^*\right)\right\rangle &= \lim_{j\to\infty} \left\langle (\gamma f - \mu F)x^*, j_q\left(x_{n_j} - x^*\right)\right\rangle \\ &= \left\langle (\gamma f - \mu F)x^*, j_q\left(z - x^*\right)\right\rangle \le 0.\end{aligned}$$

Step 5. Finally, we show that $\lim_{n\to\infty} \|x_n - x^*\| = 0$. Setting $h_n = \beta_n\gamma fx_n + (I - \beta_n\mu F)S_n y_n$, $\forall n \ge 1$. Then we can rewrite $x_{n+1} = Q_C h_n$. It follows from Lemmas 2.1 and 2.4 that

$$\begin{aligned}&\left\|x_{n+1} - x^*\right\|^q \\ &\quad= \left\langle Q_C h_n - h_n, j_q\left(x_{n+1} - x^*\right)\right\rangle + \left\langle h_n - x^*, j_q\left(x_{n+1} - x^*\right)\right\rangle \\ &\quad\le \left\langle h_n - x^*, j_q\left(x_{n+1} - x^*\right)\right\rangle\end{aligned}$$

$$
\begin{aligned}
&= \beta_n \langle \gamma f x_n - \mu F x^*, j_q(x_{n+1} - x^*) \rangle + \langle (I - \beta_n \mu F)(S_n y_n - x^*), j_q(x_{n+1} - x^*) \rangle \\
&= \beta_n \langle \gamma (f x_n - f x^*), j_q(x_{n+1} - x^*) \rangle + \beta_n \langle \gamma f x^* - \mu F x^*, j_q(x_{n+1} - x^*) \rangle \\
&\quad + \langle (I - \beta_n \mu F)(S_n y_n - x^*), j_q(x_{n+1} - x^*) \rangle \\
&\leq \beta_n \gamma L \| x_n - x^* \| \| x_{n+1} - x^* \|^{q-1} + \beta_n \langle \gamma f x^* - \mu F x^*, j_q(x_{n+1} - x^*) \rangle \\
&\quad + (1 - \beta_n \tau) \| y_n - x^* \| \| x_{n+1} - x^* \|^{q-1} \\
&\leq \beta_n \gamma L \| x_n - x^* \| \| x_{n+1} - x^* \|^{q-1} + \beta_n \langle \gamma f x^* - \mu F x^*, j_q(x_{n+1} - x^*) \rangle \\
&\quad + (1 - \beta_n \tau) \| x_n - x^* \| \| x_{n+1} - x^* \|^{q-1} \\
&= \big(1 - (\tau - \gamma L)\beta_n\big) \| x_n - x^* \| \| x_{n+1} - x^* \|^{q-1} + \beta_n \langle \gamma f x^* - \mu F x^*, j_q(x_{n+1} - x^*) \rangle \\
&\leq \big(1 - (\tau - \gamma L)\beta_n\big) \left[\frac{1}{q} \| x_n - x^* \|^q + \frac{q-1}{q} \| x_{n+1} - x^* \|^{q-1} \right] \\
&\quad + \beta_n \langle \gamma f x^* - \mu F x^*, j_q(x_{n+1} - x^*) \rangle,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\| x_{n+1} - x^* \|^q &\leq \frac{1 - (\tau - \gamma L)\beta_n}{1 + (q-1)(\tau - \gamma)\beta_n} \| x_n - x^* \|^q \\
&\quad + \frac{q\beta_n}{1 + (q-1)(\tau - \gamma L)\beta_n} + \langle \gamma f x^* - \mu F x^*, j_q(x_{n+1} - x^*) \rangle \\
&\leq \big(1 - (\tau - \gamma L)\beta_n\big) \| x_n - x^* \|^q \\
&\quad + \frac{q\beta_n}{1 + (q-1)(\tau - \gamma L)\beta_n} + \langle \gamma f x^* - \mu F x^*, j_q(x_{n+1} - x^*) \rangle.
\end{aligned}
$$

Put $a_n = \beta_n(\tau - \gamma L)$ and $b_n = \frac{q}{(1+(q-1)(\tau-\gamma L)\beta_n)(\tau-\gamma L)} + \langle \gamma f x^* - \mu F x^*, j_q(x_{n+1} - x^*) \rangle$. Applying Lemma 2.3, we obtain $x_n \to x^*$ as $n \to \infty$. This completes the proof. □

Remark 3.2 Theorem 3.1 improves and extends Theorem 2.1; see Cho and Kang [21]. Especially, our results extend the above results from Hilbert space to a more general q-uniformly smooth Banach space.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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