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Strong and Δ-convergence theorems for two asymptotically nonexpansive mappings in the intermediate sense in CAT(0) spaces

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Abstract
In this paper, we study strong and Δ-convergence for a newly defined two-step iteration process involving two asymptotically nonexpansive mappings in the intermediate sense which is wider than the class of asymptotically nonexpansive mappings in the setting of CAT(0) spaces. Our results generalize, unify and extend many known results from the existing literature.
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1 Introduction

A metric space (X,d) is a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having non-positive sectional curvature is a CAT(0) space. Fixed point theory in CAT(0) spaces was first studied by Kirk (see [1, 2]). He showed that every nonexpansive (single-valued) mapping defined on a bounded closed convex subset of a complete CAT(0) space always has a fixed point. It is worth mentioning that the results in CAT(0) spaces can be applied to any CAT(k) space with $k \leq 0$ since any CAT(k) space is a CAT(m) space for every $m \geq k$ (see [3]).

The Mann iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

Further, the Ishikawa iteration process is defined by the sequence $\{x_n\}$,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$. This iteration process reduces to the Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

In 2007, Agarwal *et al.* [4] introduced the S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 1, \end{cases} \tag{1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$. Note that (1.3) is independent of (1.2) (and hence (1.1)). They showed that their process is independent of those of Mann and Ishikawa and converges faster than both of these (see [4, Proposition 3.1]).

Schu [5], in 1991, considered the modified Mann iteration process which is a generalization of the Mann iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$.

Tan and Xu [6], in 1994, studied the modified Ishikawa iteration process which is a generalization of the Ishikawa iteration process,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$. This iteration process reduces to the modified Mann iteration process when $\beta_n = 0$ for all $n \geq 1$.

Recently, Agarwal *et al.* [4] introduced the modified S-iteration process in a Banach space,

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.6}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are the sequences in $(0,1)$. Note that (1.6) is independent of (1.5) (and hence of (1.4)). Also (1.6) reduces to (1.3) when $T^n = T$ for all $n \geq 1$.

Very recently, Şahin and Başarir [7] modified iteration process (1.6) in a CAT(0) space as follows.

Let K be a nonempty, closed and convex subset of a complete CAT(0) space X, and let $T: K \to K$ be an asymptotically quasi-nonexpansive mapping with $F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.7}$$

where and throughout the paper $\{\alpha_n\}$, $\{\beta_n\}$ are sequences such that $0 \leq \alpha_n, \beta_n \leq 1$, for all $n \geq 1$. They studied the modified S-iteration process for asymptotically quasi-nonexpansive mappings in a CAT(0) space and established some strong convergence results under some suitable conditions which generalize some results of Khan and Abbas [8].

Inspired and motivated by the work of Şahin and Başarir [7] and some others, we further modify iteration scheme (1.7) for two mappings in a CAT(0) space as follows.

Consider K to be a nonempty closed convex subset of a complete CAT(0) space X and $S, T\colon K \to K$ to be two asymptotically nonexpansive mappings in the intermediate sense with $F = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is a sequence generated iteratively by

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n \oplus \alpha_n S^n y_n, \\ y_n = (1-\beta_n)S^n x_n \oplus \beta_n T^n x_n, \quad n \geq 1, \end{cases} \tag{1.8}$$

where and throughout the paper $\{\alpha_n\}$, $\{\beta_n\}$ are the sequences such that $0 \leq \alpha_n, \beta_n \leq 1$ for all $n \geq 1$.

Remark 1.1 If we take $S = I$, where I is the identity mapping and $\beta_n = 0$ for all $n \geq 1$, then (1.8) reduces to the modified Mann iteration process in a CAT(0) space.

In this paper, we study the newly defined two-step iteration process (1.8) involving two asymptotically nonexpansive mappings in the intermediate sense and investigate the existence and convergence theorems for the above said mappings and iteration scheme in the framework of CAT(0) spaces. Our results generalize, unify and extend several comparable results in the existing literature.

2 Preliminaries and lemmas

In order to prove the main results of this paper, we need the following definitions, concepts and lemmas.

Let (X, d) be a metric space and K be its nonempty subset. Consider $T\colon K \to K$ to be a mapping. A point $x \in K$ is called a fixed point of T if $Tx = x$. We will also denote by F the set of common fixed points of S and T, that is, $F = \{x \in K : Sx = Tx = x\}$.

The concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [9] in 1972. The iterative approximation problems for asymptotically nonexpansive and asymptotically quasi-nonexpansive mappings were studied by many authors in a Banach space and a CAT(0) space (see, *e.g.*, [6, 10–18]).

Definition 2.1 Let (X, d) be a metric space and K be its nonempty subset. Then $T\colon K \to K$ is said to be:

(1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;
(2) asymptotically nonexpansive if there exists a sequence $\{u_n\} \subset [0, \infty)$, with $\lim_{n\to\infty} u_n = 0$, such that $d(T^n x, T^n y) \leq (1 + u_n)d(x, y)$ for all $x, y \in K$ and $n \geq 1$;
(3) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq Ld(x, y)$ for all $x, y \in K$ and $n \geq 1$;
(4) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$.

In 1993, Bruck *et al.* [19] introduced a notion of asymptotically nonexpansive mapping in the intermediate sense. More accurately, a mapping $T: K \to K$ is said to be asymptotically nonexpansive in the intermediate sense provided that T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} \{d(T^n x, T^n y) - d(x,y)\} \le 0.$$

From the above definitions, it follows that an asymptotically nonexpansive mapping must be an asymptotically nonexpansive mapping in the intermediate sense. But the converse does not hold as the following example shows.

Example 2.1 (see [20]) Let $X = \mathbb{R}$, $K = [-\frac{1}{\pi}, \frac{1}{\pi}]$ and $|\lambda| < 1$. For each $x \in K$, define

$$T(x) = \begin{cases} \lambda x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then T is an asymptotically nonexpansive mapping in the intermediate sense but it is not an asymptotically nonexpansive mapping.

Remark 2.1 It is clear that the class of asymptotically nonexpansive mappings includes nonexpansive mappings, whereas the class of asymptotically nonexpansive mappings in the intermediate sense is larger than that of asymptotically nonexpansive mappings.

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y) is a map c from a closed interval $[0,l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t),c(t')) = |t - t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry, and $d(x,y) = l$. The image α of c is called a geodesic (or metric) *segment* joining x and y. We say that X is

(i) a *geodesic space* if any two points of X are joined by a geodesic;
(ii) *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x,y \in X$, which we will denote by $[x,y]$, called the segment joining x to y.

A *geodesic triangle* $\triangle(x_1,x_2,x_3)$ in a geodesic metric space (X,d) consists of three points in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the *edges* of $\triangle$). A *comparison triangle* for the geodesic triangle $\triangle(x_1,x_2,x_3)$ in (X,d) is a triangle $\overline{\triangle}(x_1,x_2,x_3) := \triangle(\overline{x}_1,\overline{x}_2,\overline{x}_3)$ in $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i,\overline{x}_j) = d(x_i,x_j)$ for $i,j \in \{1,2,3\}$. Such a triangle always exists (see [3]).

CAT(0) space A geodesic metric space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following CAT(0) comparison axiom.

Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle} \subset \mathbb{R}^2$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x,y \in \triangle$ and all comparison points $\overline{x},\overline{y} \in \overline{\triangle}$,

$$d(x,y) \le d_{\mathbb{R}^2}(\overline{x},\overline{y}). \tag{2.1}$$

Complete CAT(0) spaces are often called *Hadamard spaces* (see [20]). If x, y_1, y_2 are points of a CAT(0) space and y_0 is the midpoint of the segment $[y_1,y_2]$, which we will denote by

$(y_1 \oplus y_2)/2$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y_1 \oplus y_2}{2}\right) \leq \frac{1}{2}d^2(x, y_1) + \frac{1}{2}d^2(x, y_2) - \frac{1}{4}d^2(y_1, y_2). \tag{2.2}$$

Inequality (2.1) is the (CN) inequality of Bruhat and Tits [21]. The above inequality has been extended in [22] as

$$d^2\bigl(z, \alpha x \oplus (1-\alpha)y\bigr) \leq \alpha d^2(z, x) + (1-\alpha)d^2(z, y) - \alpha(1-\alpha)d^2(x, y) \tag{2.3}$$

for any $\alpha \in [0,1]$ and $x, y, z \in X$.

Let us recall that a geodesic metric space is a CAT(0) space if and only if it satisfies the (CN) inequality (see [3, p.163]). Moreover, if X is a CAT(0) metric space and $x, y \in X$, then for any $\alpha \in [0,1]$, there exists a unique point $\alpha x \oplus (1-\alpha)y \in [x, y]$ such that

$$d\bigl(z, \alpha x \oplus (1-\alpha)y\bigr) \leq \alpha d(z, x) + (1-\alpha)d(z, y) \tag{2.4}$$

for any $z \in X$ and $[x, y] = \{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$.

A subset K of a CAT(0) space X is convex if, for any $x, y \in K$, we have $[x, y] \subset K$.

For the development of our main results, we recall some definitions, and some key results are listed in the form of lemmas.

Lemma 2.1 ([14]) *Let X be a* CAT(0) *space.*

(i) *For $x, y \in X$ and $t \in [0,1]$, there exists a unique point $z \in [x, y]$ such that*

$$d(x, z) = td(x, y) \quad \text{and} \quad d(y, z) = (1-t)d(x, y). \tag{A}$$

We use the notation $(1-t)x \oplus ty$ for the unique point z satisfying (A).

(ii) *For $x, y \in X$ and $t \in [0,1]$, we have*

$$d\bigl((1-t)x \oplus ty, z\bigr) \leq (1-t)d(x, z) + td(y, z).$$

Let $\{x_n\}$ be a bounded sequence in a closed convex subset K of a CAT(0) space X. For $x \in X$, set

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r\bigl(\{x_n\}\bigr) = \inf\bigl\{r\bigl(x, \{x_n\}\bigr) : x \in X\bigr\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A\bigl(\{x_n\}\bigr) = \bigl\{x \in X : r\bigl(\{x_n\}\bigr) = r\bigl(x, \{x_n\}\bigr)\bigr\}.$$

It is known that, in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point; please see [23, Proposition 7].

We now recall the definition of $\triangle$-convergence and weak convergence ($\rightharpoonup$) in a CAT(0) space.

Definition 2.2 ([24]) A sequence $\{x_n\}$ in a CAT(0) space X is said to $\triangle$-converge to $x \in X$ if x is the unique asymptotic center of $\{x_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

In this case we write $\triangle\text{-}\lim_n x_n = x$ and call x the $\triangle$-limit of $\{x_n\}$.

Now, let us recall that a bounded sequence $\{x_n\}$ in X is said to be regular if $r(\{x_n\}) = r(\{u_n\})$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In the Banach space it is known that every bounded sequence has a regular subsequence; please see [25, Lemma 15.2].

Since in a CAT(0) space every regular sequence $\triangle$-converges, we see that every bounded sequence in X has a $\triangle$-convergent subsequence, also it is noticed in [24, p.3690].

Lemma 2.2 ([26]) *Given $\{x_n\} \subset X$ such that $\{x_n\}$ $\triangle$-converges to x and given $y \in X$ with $y \neq x$, then*

$$\limsup_n d(x_n, x) < \limsup_n d(x_n, y).$$

In a Banach space the above condition is known as the Opial property.

Now, recall the definition of weak convergence in a CAT(0) space.

Definition 2.3 ([27]) Let K be a closed convex subset of a CAT(0) space X. A bounded sequence $\{x_n\}$ in K is said to converge weakly to $q \in K$ if and only if $\Phi(q) = \inf_{x\in K} \Phi(x)$, where $\Phi(x) = \limsup_{n\to\infty} d(x_n, x)$.

Note that $\{x_n\} \rightharpoonup q$ if and only if $A_K\{x_n\} = \{q\}$.

Nanjaras and Panyanak [28] established the following relation between $\triangle$-convergence and weak convergence in a CAT(0) space.

Lemma 2.3 ([28, Proposition 3.12]) *Let $\{x_n\}$ be a bounded sequence in a* CAT(0) *space X, and let K be a closed convex subset of X which contains $\{x_n\}$. Then*

(i) $\triangle\text{-}\lim_{x_n} = x$ *implies* $x_n \rightharpoonup x$.

(ii) *The converse of* (i) *is true if $\{x_n\}$ is regular.*

Lemma 2.4 ([22, Lemma 2.8]) *If $\{x_n\}$ is a bounded sequence in a* CAT(0) *space X, with $A(\{x_n\}) = \{x\}$, and $\{u_n\}$ is a subsequence of $\{x_n\}$, with $A(\{u_n\}) = \{u\}$, and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

Lemma 2.5 ([29, Proposition 2.1]) *If K is a closed convex subset of a* CAT(0) *space X and if $\{x_n\}$ is a bounded sequence in K, then the asymptotic center of $\{x_n\}$ is in K.*

Lemma 2.6 ([30]) *Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.7 ([26, Theorem 3.1]) *Let X be a complete* CAT(0) *space, K be a nonempty closed convex subset of X. If $T : K \to K$ is an asymptotically nonexpansive mapping in the intermediate sense, then T has a fixed point.*

Lemma 2.8 ([26, Theorem 3.2]) *Let X be a complete* CAT(0) *space, K be a nonempty closed convex subset of X. If $T: K \to K$ is an asymptotically nonexpansive mapping in the intermediate sense, then* $\mathrm{Fix}(T)$ *is closed and convex.*

Lemma 2.9 (Demiclosed principle [26, Proposition 3.3]) *Let K be a closed convex subset of a complete* CAT(0) *space X and $T: K \to K$ be an asymptotically nonexpansive mapping in the intermediate sense. If $\{x_n\}$ is a bounded sequence in K such that* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ *and $\{x_n\} \rightharpoonup w$, then $Tw = w$.*

Lemma 2.10 ([26, Corollary 3.4]) *Let K be a closed convex subset of a complete* CAT(0) *space X and $T: K \to K$ be an asymptotically nonexpansive mapping in the intermediate sense. If $\{x_n\}$ is a bounded sequence in K $\triangle$-converging to x and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, *then $x \in K$ and $Tx = x$.*

3 The main results

Now, we prove the following lemmas using the newly defined two-step iteration scheme (1.8) needed in the sequel.

Lemma 3.1 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, and let $S, T: K \to K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F = F(S) \cap F(T) \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.8). *Put*

$$c_n = \max\Big\{0, \sup_{x,y\in K, n\geq 1} \big(d\big(S^n x, S^n y\big) - d(x,y)\big)\Big\} \tag{3.1}$$

and

$$d_n = \max\Big\{0, \sup_{x,y\in K, n\geq 1} \big(d\big(T^n x, T^n y\big) - d(x,y)\big)\Big\} \tag{3.2}$$

such that $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$. Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0,1)$. Then:

(i) $\lim_{n\to\infty} d(x_n, p)$ *exists for all $p \in F$.*

(ii) $\lim_{n\to\infty} d(x_n, F)$ *exists.*

Proof Let $p \in F$. From (1.8), (3.1), (3.2) and Lemma 2.1(ii), we have

$$\begin{aligned} d(y_n, p) &= d\big((1-\beta_n)S^n x_n \oplus \beta_n T^n x_n, p\big) \\ &\leq (1-\beta_n) d\big(S^n x_n, p\big) + \beta_n d\big(T^n x_n, p\big) \\ &\leq (1-\beta_n)\big[d(x_n, p) + c_n\big] + \beta_n \big[d(x_n, p) + d_n\big] \\ &\leq d(x_n, p) + c_n + d_n. \end{aligned} \tag{3.3}$$

Again, using (1.8), (3.1)-(3.3) and Lemma 2.1(ii), we obtain

$$\begin{aligned} d(x_{n+1}, p) &= d\big((1-\alpha_n) T^n x_n \oplus \alpha_n S^n y_n, p\big) \\ &\leq (1-\alpha_n) d\big(T^n x_n, p\big) + \alpha_n d\big(T^n y_n, p\big) \end{aligned}$$

$$
\begin{aligned}
&\le (1-\alpha_n)\big[d(x_n,p)+d_n\big]+\alpha_n\big[d(y_n,p)+c_n\big] \\
&= (1-\alpha_n)d(x_n,p)+\alpha_n d(y_n,p)+(1-\alpha_n)d_n+\alpha_n c_n \\
&\le (1-\alpha_n)d(x_n,p)+\alpha_n\big[d(x_n,p)+c_n+d_n\big] \\
&\quad +(1-\alpha_n)d_n+\alpha_n c_n \\
&\le d(x_n,p)+2\alpha_n c_n+d_n. \qquad (3.4)
\end{aligned}
$$

Taking infimum over all $p \in F$, we get

$$
d(x_{n+1},p) \le d(x_n,F)+2\alpha_n c_n+d_n. \qquad (3.5)
$$

Since by the hypothesis of the theorem $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, it follows from Lemma 2.6, relations (3.4) and (3.5) that $\lim_{n\to\infty} d(x_n,p)$ and $\lim_{n\to\infty} d(x_n,F)$ exist. □

Lemma 3.2 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, and let $S,T: K \to K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.8) *and c_n and d_n are taken as in Lemma* 3.1. *Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[l,m]$ for some $l,m \in (0,1)$. If $d(x,Sx) \le d(Tx,Sx)$ for all $x \in K$, then* $\lim_{n\to\infty} d(x_n,Sx_n) = 0$ *and* $\lim_{n\to\infty} d(x_n,Tx_n) = 0$.

Proof Using (1.8) and (2.3), we have

$$
\begin{aligned}
d^2(y_n,p) &= d^2\big((1-\beta_n)S^n x_n \oplus \beta_n T^n x_n, p\big) \\
&\le \beta_n d^2\big(T^n x_n,p\big)+(1-\beta_n)d^2\big(S^n x_n,p\big) \\
&\quad -\beta_n(1-\beta_n)d^2\big(T^n x_n,S^n x_n\big) \\
&\le \beta_n\big[d(x_n,p)+d_n\big]^2+(1-\beta_n)\big[d(x_n,p)+c_n\big]^2 \\
&\quad -\beta_n(1-\beta_n)d^2\big(T^n x_n,S^n x_n\big) \\
&\le d^2(x_n,p)+A_n+B_n-\beta_n(1-\beta_n)d^2\big(T^n x_n,S^n x_n\big), \qquad (3.6)
\end{aligned}
$$

where $A_n = d_n^2+2d_n d(x_n,p)$ and $B_n = c_n^2+2c_n d(x_n,p)$, since by the hypothesis $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, it follows that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$. Again using (1.8), (2.3) and (3.6), we have

$$
\begin{aligned}
d^2(x_{n+1},p) &= d^2\big((1-\alpha_n)T^n x_n \oplus \alpha_n S^n y_n, p\big) \\
&\le \alpha_n d^2\big(S^n y_n,p\big)+(1-\alpha_n)d^2\big(T^n x_n,p\big) \\
&\quad -\alpha_n(1-\alpha_n)d^2\big(S^n y_n, T^n x_n\big) \\
&\le \alpha_n\big[d(y_n,p)+c_n\big]^2+(1-\alpha_n)\big[d(x_n,p)+d_n\big]^2 \\
&\quad -\alpha_n(1-\alpha_n)d^2\big(S^n y_n, T^n x_n\big) \\
&\le \alpha_n\big[d^2(y_n,p)+L_n\big]+(1-\alpha_n)\big[d^2(x_n,p)+A_n\big] \\
&\quad -\alpha_n(1-\alpha_n)d^2\big(S^n y_n, T^n x_n\big)
\end{aligned}
$$

$$\begin{aligned}
&\le \alpha_n d^2(y_n,p) + (1-\alpha_n)d^2(x_n,p) + \alpha_n L_n + (1-\alpha_n)A_n \\
&\quad - \alpha_n(1-\alpha_n)d^2\big(S^n y_n, T^n x_n\big) \\
&\le \alpha_n\big[d^2(x_n,p) + A_n + B_n - \beta_n(1-\beta_n)d^2\big(T^n x_n, S^n x_n\big)\big] \\
&\quad + (1-\alpha_n)d^2(x_n,p) + \alpha_n L_n + (1-\alpha_n)A_n \\
&\quad - \alpha_n(1-\alpha_n)d^2\big(S^n y_n, T^n x_n\big) \\
&\le d^2(x_n,p) - \alpha_n\beta_n(1-\beta_n)d^2\big(T^n x_n, S^n x_n\big) \\
&\quad + A_n + B_n + L_n - \alpha_n(1-\alpha_n)d^2\big(S^n y_n, T^n x_n\big) \\
&= d^2(x_n,p) - \alpha_n\beta_n(1-\beta_n)d^2\big(T^n x_n, S^n x_n\big) \\
&\quad + H_n - \alpha_n(1-\alpha_n)d^2\big(S^n y_n, T^n x_n\big),
\end{aligned} \tag{3.7}$$

where $A_n = d_n^2 + 2d_n d(x_n,p)$, $L_n = c_n^2 + 2c_n d(y_n,p)$ and $H_n = A_n + B_n + L_n$, since by the hypothesis $\sum_{n=1}^{\infty} c_n < \infty$, $\sum_{n=1}^{\infty} d_n < \infty$, $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$, it follows that $\sum_{n=1}^{\infty} L_n < \infty$ and $\sum_{n=1}^{\infty} H_n < \infty$. This implies that

$$\begin{aligned}
d^2\big(T^n x_n, S^n x_n\big) &\le \frac{1}{\alpha_n\beta_n(1-\beta_n)}\big[d^2(x_n,p) - d^2(x_{n+1},p)\big] + \frac{H_n}{\alpha_n\beta_n(1-\beta_n)} \\
&\le \frac{1}{l^2(1-m)}\big[d^2(x_n,p) - d^2(x_{n+1},p)\big] + \frac{H_n}{l^2(1-m)}
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
d^2\big(S^n y_n, T^n x_n\big) &\le \frac{1}{\alpha_n(1-\alpha_n)}\big[d^2(x_n,p) - d^2(x_{n+1},p)\big] + \frac{H_n}{\alpha_n(1-\alpha_n)} \\
&\le \frac{1}{l(1-m)}\big[d^2(x_n,p) - d^2(x_{n+1},p)\big] + \frac{H_n}{l(1-m)}.
\end{aligned} \tag{3.9}$$

Since $H_n \to 0$ as $n \to \infty$ and $d(x_n,p)$ is convergent, therefore on taking limit as $n \to \infty$ in (3.8) and (3.9), we get

$$\lim_{n\to\infty} d\big(T^n x_n, S^n x_n\big) = 0 \tag{3.10}$$

and

$$\lim_{n\to\infty} d\big(S^n y_n, T^n x_n\big) = 0. \tag{3.11}$$

Now

$$\begin{aligned}
d\big(T^n x_n, x_n\big) &\le d\big(T^n x_n, S^n x_n\big) + d\big(S^n x_n, x_n\big) \\
&\le d\big(T^n x_n, S^n x_n\big) + d\big(S^n x_n, T^n x_n\big) \\
&= 2d\big(T^n x_n, S^n x_n\big) \to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{3.12}$$

and

$$d\big(S^n x_n, x_n\big) \le d\big(S^n x_n, T^n x_n\big) + d\big(T^n x_n, x_n\big)$$

by (3.10) and (3.12), we obtain

$$\lim_{n\to\infty} d\bigl(S^n x_n, x_n\bigr) = 0. \tag{3.13}$$

Again note that

$$\begin{aligned} d\bigl(x_{n+1}, T^n x_n\bigr) &= d\bigl((1-\alpha_n)T^n x_n \oplus \alpha_n S^n y_n, T^n x_n\bigr) \\ &\le (1-\alpha_n) d\bigl(T^n x_n, T^n x_n\bigr) + \alpha_n d\bigl(S^n y_n, T^n x_n\bigr) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.14}$$

By (3.12) and (3.14), we have

$$\begin{aligned} d(x_{n+1}, x_n) &\le d\bigl(x_{n+1}, T^n x_n\bigr) + d\bigl(T^n x_n, x_n\bigr) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.15}$$

Let $\rho_n = d(T^n x_n, x_n)$, by (3.12) we have $\rho_n \to 0$ as $n \to \infty$. Now, we have

$$\begin{aligned} d(x_n, Tx_n) &\le d(x_n, x_{n+1}) + d\bigl(x_{n+1}, T^{n+1} x_{n+1}\bigr) + d\bigl(T^{n+1} x_{n+1}, T^{n+1} x_n\bigr) + d\bigl(T^{n+1} x_n, Tx_n\bigr) \\ &\le d(x_n, x_{n+1}) + d\bigl(x_{n+1}, T^{n+1} x_{n+1}\bigr) + d(x_{n+1}, x_n) + d_{n+1} + d\bigl(T^{n+1} x_n, Tx_n\bigr) \\ &\le \rho_{n+1} + 2d(x_n, x_{n+1}) + d_{n+1} + d\bigl(T^{n+1} x_n, Tx_n\bigr) \\ &\to 0 \quad \text{as } n \to \infty \end{aligned} \tag{3.16}$$

by (3.12), (3.15), $d_{n+1} \to 0$ and the uniform continuity of T. Similarly, we can prove that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{3.17}$$

This completes the proof. □

Now we prove the $\triangle$-convergence and strong convergence results.

Theorem 3.1 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, and let $S, T\colon K \to K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.8) *and c_n and d_n are taken as in Lemma* 3.1. *Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0,1)$. Then the sequence $\{x_n\}$ is $\triangle$-convergent to a point of F.*

Proof We first show that $w_w(\{x_n\}) \subseteq F$. Let $v \in w_w(\{x_n\})$, then there exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $A(\{x_n\}) = \{v\}$. By Lemma 2.5, there exists a subsequence $\{w_n\}$ of $\{v_n\}$ such that $\triangle\text{-}\lim_n w_n = w \in K$. By Lemma 2.10, $w \in F(T)$ and $w \in F(S)$ and so $w \in F$. By Lemma 3.1 $\lim_{n\to\infty} d(x_n, F)$ exists, so by Lemma 2.4 we have $v = w$, *i.e.*, $w_w(\{x_n\}) \subseteq F$.

To show that $\{x_n\}$ $\triangle$-converges to a point in F, it is sufficient to show that $w_w(\{x_n\})$ consists of exactly one point.

Let $\{v_n\}$ be a subsequence of $\{x_n\}$ with $A(\{v_n\}) = \{v\}$, and let $A(\{x_n\}) = \{x\}$ for some $v \in w_w(\{x_n\}) \subseteq F$ and $\{d(x_n, w)\}$ converges. By Lemma 2.4, we have $x = w \in F$. Thus $w_w(\{x_n\}) = \{x\}$. This shows that $\{x_n\}$ is $\triangle$-convergent to a point of F. This completes the proof. □

Theorem 3.2 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, and let $S, T: K \to K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.8) *and c_n and d_n are taken as in Lemma* 3.1. *Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0,1)$. If $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F} d(x, p)$, then the sequence $\{x_n\}$ converges strongly to a point in F.*

Proof From (3.5) of Lemma 3.1, we have

$$d(x_{n+1}, p) \leq d(x_n, F) + 2\alpha_n c_n + d_n,$$

where $p \in F$. Since by the hypothesis of the theorem $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, by Lemma 2.6 and $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$ gives that

$$\lim_{n\to\infty} d(x_n, F) = 0. \tag{3.18}$$

Now, we show that $\{x_n\}$ is a Cauchy sequence in K.

From (3.4) and by the hypothesis $0 < l \leq \alpha_n, \beta_n \leq m < 1$, we have

$$\begin{aligned} d(x_{n+p}, q) &\leq d(x_{n+p-1}, q) + 2mc_{n+p-1} + d_{n+p-1} \\ &\leq d(x_{n+p-2}, q) + 2m[c_{n+p-2} + c_{n+p-1}] + d_{n+p-2} + d_{n+p-1} \\ &\leq \cdots \\ &\leq d(x_n, q) + 2m \sum_{k=n}^{n+p-1} c_k + \sum_{k=n}^{n+p-1} d_k \end{aligned} \tag{3.19}$$

for the natural numbers n, p and $q \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, therefore for any $\varepsilon > 0$, there exists a natural number n_0 such that $d(x_n, F) < \varepsilon/12$, $\sum_{k=n}^{n+p-1} c_k < \varepsilon/12m$ and $\sum_{k=n}^{n+p-1} d_k < \varepsilon/6$ for all $n \geq n_0$. So, we can find $p^* \in F$ such that $d(x_{n_0}, p^*) < \varepsilon/6$. Hence, for all $n \geq n_0$ and $p \geq 1$, we have

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d\bigl(x_{n+p}, p^*\bigr) + d\bigl(x_n, p^*\bigr) \\ &\leq d\bigl(x_{n_0}, p^*\bigr) + 2m \sum_{k=n_0}^{n+p-1} c_k + \sum_{k=n_0}^{n+p-1} d_k + d\bigl(x_{n_0}, p^*\bigr) + 2m \sum_{k=n_0}^{n+p-1} c_k + \sum_{k=n_0}^{n+p-1} d_k \\ &= 2d\bigl(x_{n_0}, p^*\bigr) + 4m \sum_{k=n_0}^{n+p-1} c_k + 2 \sum_{k=n_0}^{n+p-1} d_k \\ &< 2\left(\frac{\varepsilon}{6}\right) + 4m\left(\frac{\varepsilon}{12m}\right) + 2\left(\frac{\varepsilon}{6}\right) \\ &= \varepsilon. \end{aligned} \tag{3.20}$$

This proves that $\{x_n\}$ is a Cauchy sequence in K. Thus, the completeness of X implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = q^*$. Since K is closed, therefore $q^* \in K$. Next, we show that $q^* \in F$. Since $\lim_{n\to\infty} d(x_n, F) = 0$, we get $d(q^*, F) = 0$, the closedness of F gives that $q^* \in F$. This completes the proof. □

Theorem 3.3 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, and let $S, T: K \to K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.8) *and c_n and d_n are taken as in Lemma* 3.1. *Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0, 1)$. If S and T satisfy the following conditions*:

(i) $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

(ii) *If the sequence $\{z_n\}$ in K satisfies* $\lim_{n\to\infty} d(z_n, Sz_n) = 0$ *and* $\lim_{n\to\infty} d(z_n, Tz_n) = 0$, *then* $\liminf_{n\to\infty} d(z_n, F) = 0$ *or* $\limsup_{n\to\infty} d(z_n, F) = 0$.

Then the sequence $\{x_n\}$ converges strongly to a point of F.

Proof It follows from the hypothesis that $\lim_{n\to\infty} d(x_n, Sx_n) = 0$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. From (ii), $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$. Therefore, the sequence $\{x_n\}$ must converge strongly to a point in F by Theorem 3.2. This completes the proof. □

Theorem 3.4 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, and let $S, T: K \to K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.8) *and c_n and d_n are taken as in Lemma* 3.1. *Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0, 1)$. If either S or T is semi-compact, then the sequence $\{x_n\}$ converges strongly to a point of F.*

Proof Suppose that T is semi-compact. By Lemma 3.2, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. So there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to p \in K$. Now Lemma 3.2 guarantees that $\lim_{n_j\to\infty} d(x_{n_j}, Tx_{n_j}) = 0$ and so $d(p, Tp) = 0$. Similarly, we can show that $d(p, Sp) = 0$. Thus $p \in F$. By (3.5), we have

$$d(x_{n+1}, p) \leq d(x_n, F) + 2\alpha_n c_n + d_n.$$

Since by the hypothesis $\sum_{n=1}^{\infty} c_n < \infty$ and $\sum_{n=1}^{\infty} d_n < \infty$, by Lemma 2.6, $\lim_{n\to\infty} d(x_n, p)$ exists and $x_{n_j} \to p \in F$ gives that $x_n \to p \in F$. This shows that $\{x_n\}$ converges strongly to a point of F. This completes the proof. □

We recall the following definition.

A mapping $T: K \to K$, where K is a subset of a normed linear space E, is said to satisfy Condition (A) [31] if there exists a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in K$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T) \neq \emptyset\}$.

Now, we modify this definition for two mappings.

Two mappings $S, T: K \to K$, where K is a subset of a normed linear space E, are said to satisfy Condition (B) if there exists a nondecreasing function $f: [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0, \infty)$ such that $a_1\|x - Sx\| + a_2\|x - Tx\| \geq f(d(x, F))$ for all $x \in K$, where $d(x, F) = \inf\{\|x - p\| : p \in F \neq \emptyset\}$ and a_1 and a_2 are two nonnegative real numbers such that $a_1 + a_2 = 1$. It is to be noted that Condition (B) is weaker than the compactness of the domain K.

Remark 3.1 Condition (B) reduces to Condition (A) when $S = T$.

As an application of Theorem 3.2, we establish some strong convergence results as follows.

Theorem 3.5 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, and let $S, T: K \to K$ be two asymptotically nonexpansive mappings in the intermediate sense with $F \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.8) *and c_n and d_n are taken as in Lemma* 3.1. *Suppose that $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0, 1)$. If S and T satisfy Condition* (B), *then the sequence $\{x_n\}$ converges strongly to a point of F.*

Proof By Lemma 3.2, we know that

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.21}$$

From Condition (B) and (3.21), we get

$$\lim_{n\to\infty} f\big(d(x_n, F)\big) \leq a_1 \cdot \lim_{n\to\infty} d(x_n, Sx_n) + a_2 \cdot \lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

i.e., $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since $f: [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(t) > 0$ for all $t \in (0, \infty)$, therefore we obtain

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

The conclusion now follows from Theorem 3.2. This completes the proof. □

Theorem 3.6 *Let K be a nonempty closed convex subset of a complete* CAT(0) *space X, and let $S, T: K \to K$ be two uniformly continuous asymptotically nonexpansive mappings with $F \neq \emptyset$. Suppose that $\{x_n\}$ is defined by the iteration process* (1.8) *and $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[l, m]$ for some $l, m \in (0, 1)$. If $\liminf_{n\to\infty} d(x_n, F) = 0$ or $\limsup_{n\to\infty} d(x_n, F) = 0$, where $d(x, F) = \inf_{p\in F} d(x, p)$, then the sequence $\{x_n\}$ converges strongly to a point in F.*

Proof Since S is uniformly continuous and asymptotically nonexpansive mapping, we know that there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$d\big(S^n x, S^n y\big) \leq k_n d(x, y), \quad \forall x, y \in K, n \geq 1.$$

This implies that

$$d\big(S^n x, S^n y\big) - k_n d(x, y) \leq 0, \quad \forall x, y \in K, n \geq 1.$$

Therefore we have

$$\limsup_{n\to\infty} \Big\{ \sup_{x,y\in K} \big(d\big(S^n x, S^n y\big) - d(x, y)\big) \Big\} \leq 0.$$

This implies that S is an asymptotically nonexpansive mapping in the intermediate sense. By a similar fashion, we can show that T is also an asymptotically nonexpansive mapping in the intermediate sense. Thus the conclusion of Theorem 3.6 follows from Theorem 3.2 immediately. This completes the proof. □

Example 3.1 Let $\mathbb{R}$ be a real line with the usual metric d and $K = [0,1]$. Define two mappings $T, S\colon K \to K$ by

$$T(x) = \begin{cases} kx & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where $0 < k < 1$ and

$$S(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then

$$\begin{aligned} \limsup_{n\to\infty} \bigl\{ \bigl|T^n x - T^n y\bigr| - |x-y| \bigr\} &= \limsup_{n\to\infty} \bigl\{ k^n |x-y| - |x-y| \bigr\} \\ &\leq 0, \end{aligned}$$

because $\lim_{n\to\infty} k^n = 0$ as $0 < k < 1$ for all $x, y \in K$ and $n \geq 1$, and T is uniformly continuous. Thus T is an asymptotically nonexpansive mapping in the intermediate sense. Similarly, we can show that S is an asymptotically nonexpansive mapping in the intermediate sense since it is also uniformly continuous and satisfies the inequality

$$\begin{aligned} \limsup_{n\to\infty} \bigl\{ \bigl|S^n x - S^n y\bigr| - |x-y| \bigr\} &= \limsup_{n\to\infty} \biggl\{ \frac{1}{2^n} |x-y| - |x-y| \biggr\} \\ &\leq 0 \end{aligned}$$

for all $x, y \in K$ and $n \geq 1$. It is clear that $F(T) = F(S) = \{0\}$ and hence $F = F(T) \cap F(S) = \{0\}$, that is, 0 is a common fixed point of T and S. Also T and S both satisfy all the conditions of Theorems 3.3-3.5.

4 Concluding remarks

This work contains our dedicated study to develop and improve methods for solving equations by means of iteration methods. We have introduced our results by using as basic framework the research of Kirk; please see [1, 2], and the inspired work of Şahin and Başarir [7] and some others. This study is motivated by relevant applications for solving many real-world problems which give rise to mathematical models in the sphere of functional analysis.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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