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Viscosity approximation method with Meir-Keeler contractions for common zero of accretive operators in Banach spaces

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Abstract
The purpose of this paper is to introduce a new iteration by the combination of the viscosity approximation with Meir-Keeler contractions and proximal point algorithm for finding common zeros of a finite family of accretive operators in a Banach space with a uniformly Gâteaux differentiable norm. The results of this paper improve and extend corresponding well-known results by many others.
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1 Introduction

Let E be a real Banach space and let J be the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E,$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot\rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued. In the sequel, we denote the single-valued normalized duality mapping by j. For an operator $A : E \to 2^E$, we define its domain, range, and graph as follows:

$$D(A) = \{x \in E : Ax \neq \emptyset\},$$
$$R(A) = \bigcup\{Az : z \in D(A)\},$$

and

$$G(T) = \{(x, y) \in E \times E : x \in D(A), y \in Ax\},$$

respectively. The inverse A^{-1} of A is defined by

$$x \in A^{-1}y, \quad \text{if and only if} \quad y \in Ax.$$

An operator A is said to be accretive if, for each $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle u-v, j(x-y)\rangle \geq 0,$$

for all $u \in Ax$ and $v \in Ay$. We denote by I the identity operator on E. An accretive operator A is said to be maximal accretive if there is no proper accretive extension of A and A is said to be m-accretive if $R(I+\lambda A) = E$, for all $\lambda > 0$. If A is m-accretive, then it is maximal, but generally, the converse is not true. If A is accretive, then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda^A : R(I+\lambda A) \to D(A)$ by

$$J_\lambda^A = (I+\lambda A)^{-1}.$$

It is called the resolvent of A which is denoted by J^A when $\lambda = 1$.

Let $A : E \to 2^E$ be an m-accretive operator. It is well known that many problems in nonlinear analysis and optimization can be formulated as the problem: Find $x \in E$ such that

$$0 \in A(x).$$

One popular method of solving the equation $0 \in A(x)$, where A is a maximal monotone operator in a Hilbert space H, is the proximal point algorithm. The proximal point algorithm generates, for any starting point $x_0 = x \in E$, a sequence $\{x_n\}$ by the rule

$$x_{n+1} = J_{r_n}^A(x_n), \tag{1.1}$$

for all $n \in \mathbb{N}$, where $\{r_n\}$ is a regularization sequence of positive real numbers, $J_{r_n}^A = (I + r_n A)^{-1}$ is the resolvent of A, and $\mathbb{N}$ is the set of all natural numbers. Some of them deal with the weak convergence theorem of the sequence $\{x_n\}$ generated by (1.1) and others proved strong convergence theorems by imposing assumptions on A.

Note that algorithm (1.1) can be rewritten as

$$x_{n+1} - x_n + r_n A(x_{n+1}) \ni 0, \tag{1.2}$$

for all $n \in \mathbb{N}$. This algorithm was first introduced by Martinet [1]. If $\psi : H \to \mathbb{R} \cup \{\infty\}$ is proper lower semicontinuous convex function, then the algorithm reduces to

$$x_{n+1} = \operatorname*{argmin}_{y \in H} \left\{ \psi(y) + \frac{1}{2r_n}\|x_n - y\|^2 \right\},$$

for all $n \in \mathbb{N}$. Moreover, Rockafellar [2] has given a more practical method which is an inexact variant of the method:

$$x_n + e_n \ni x_{n+1} + r_n A x_{n+1}, \tag{1.3}$$

for all $n \in \mathbb{N}$, where $\{e_n\}$ is regarded as an error sequence and $\{r_n\}$ is a sequence of positive regularization parameters. Note that the algorithm (1.3) can be rewritten as

$$x_{n+1} = J_{r_n}^A(x_n + e_n), \tag{1.4}$$

for all $n \in \mathbb{N}$. This method is called inexact proximal point algorithm. It was shown in Rockafellar [2] that if $e_n \to 0$ quickly enough such that $\sum_{n=1}^{\infty} \|e_n\| < \infty$, then $x_n \rightharpoonup z \in H$ with $0 \in Az$.

Further, Rockafellar [2] posed the open question of whether the sequence generated by (1.1) converges strongly or not. In 1991, Güler [3] gave an example showing that Rockafellar's proximal point algorithm does not converge strongly.

An example of the authors, Bauschke *et al.* [4] also showed that the proximal algorithm only converges weakly but not strongly.

When A is maximal monotone in a Hilbert space H, Lehdili and Moudafi [5] obtained the convergence of the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = J_{r_n}^{A_n}(x_n), \tag{1.5}$$

where $A_n = \mu_n I + A$, $\mu > 0$ is viewed as a Tikhonov regularization of A. Next, in 2006, Xu [6] and in 2009, Song and Yang [7] used the technique of nonexpansive mappings to get convergence theorems for $\{x_n\}$ defined by the perturbed version of algorithm (1.4) in the form

$$x_{n+1} = J_{r_n}^{A}\big(t_n u + (1 - t_n)x_n + e_n\big). \tag{1.6}$$

Note that algorithm (1.6) can be rewritten as

$$r_n A(x_{n+1}) + x_{n+1} \ni t_n u + (1 - t_n)x_n + e_n, \quad n \geq 0. \tag{1.7}$$

In [8], Tuyen was studied an extension the results of Xu [6], when A is an m-accretive operator in a uniformly smooth Banach space E which has a weakly sequentially continuous normalized duality mapping j from E to E^* (*cf.* [9]). At that time, in [10], Sahu and Yao also extended the results of Xu [6] for the zero of an accretive operator in a Banach space which has a uniformly Gâteaux differentiable norm by combining the prox-Tikhonov method and the viscosity approximation method. They introduced the iterative method to define the sequence $\{x_n\}$ as follows:

$$x_{n+1} = J_{r_n}^{A}\big((1 - \alpha_n)x_n + \alpha_n f(x_n)\big), \tag{1.8}$$

for all $n \in \mathbb{N}$, where A is an accretive operator such that $S = A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset C \subset \bigcap_{t>0} R(I + tA)$, and f is a contractive mapping on C.

Zegeye and Shahzed [11] studied the convergence problem of finding a common zero of a finite family of m-accretive operators (*cf.* [12, 13]). More precisely, they proved the following result.

Theorem 1.1 [11] *Let E be a strictly convex and reflexive Banach space with a uniformly Gâteaux differentiable norm, K be a nonempty, closed, and convex subset of E and $A_i : K \to E$ be an m-accretive operator, for each $i = 1, 2, \ldots, N$ with*

$$\bigcap_{i=1}^{N} A_i^{-1}0 \neq \emptyset.$$

For any $u, x_0 \in K$, let $\{x_n\}$ be a sequence in K generated by the algorithm:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) S_N(x_n), \quad \forall n \geq 0, \tag{1.9}$$

where $S_N := a_0 I + a_1 J^{A_1} + a_2 J^{A_2} + \cdots + a_N J^{A_N}$ with $J^{A_i} = (I + A_i)^{-1}$ for $0 < a_i < 1$, $i = 0, 1, 2, \ldots, N$, $\sum_{i=0}^{N} a_i = 1$, and $\{\alpha_n\}$ is a real sequence which satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$.

If every nonempty, bounded, closed, and convex subset of E has the fixed point property for nonexpansive mapping, then $\{x_n\}$ converges strongly to a common solution of the equations $A_i(x) = 0$ for $i = 1, 2, \ldots, N$.

Motivated by Xu [6] and Zegeye and Shahzed [11], Tuyen [14] introduced an iterative algorithm as follows:

$$\begin{cases} x_0 \in C, \\ x_{n+1} = S_N(\alpha_n f(x_n) + (1 - \alpha_n) x_n), \quad \forall n \geq 0, \end{cases} \tag{1.10}$$

where $S_N := a_0 I + a_1 J^{A_1} + a_2 J^{A_2} + \cdots + a_N J^{A_N}$ with $a_0, a_1, \ldots, a_N$ in $(0,1)$ such that $\sum_{i=0}^{N} a_i = 1$ and $\{\alpha_n\} \subset (0,1)$ is a real sequence of positive numbers. The result of Tuyen [14] is given by the following.

Theorem 1.2 [14] *Let E be a strictly convex and reflexive Banach space which has a weakly continuous duality mapping J_φ with gauge φ. Let C be a nonempty, closed, and convex subset of E and f be a contraction mapping of C into itself with the contractive coefficient $c \in (0,1)$. Let $A_i : C \to E$ be an m-accretive operator, for each $i = 1, 2, \ldots, N$ with*

$$\bigcap_{i=1}^{N} A_i^{-1} 0 \neq \emptyset.$$

Let $J^{A_i} = (I + A_i)^{-1}$ for $i = 1, 2, \ldots, N$. For any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by algorithm (1.10). *If the sequence $\{\alpha_n\}$ satisfies the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$,

then $\{x_n\}$ converges strongly to a common solution of the equations $A_i(x) = 0$ for $i = 1, 2, \ldots, N$.

In this paper, we combine the proximal point method [9] and the viscosity approximation method [15] with Meir-Keeler contractions to get strong convergence theorems for the problem of finding a common zero of a finite family of accretive operators in Banach spaces. We also give some applications of our results for the convex minimization problem and the variational inequality problem in Hilbert spaces.

2 Preliminaries

Let E be a real Banach space and $M \subseteq E$. We denote by $F(T)$ the set of all fixed points of the mapping $T : M \to M$.

Recall that a mapping $\phi : (X, d) \to (X, d)$ from the metric space (X, d) into itself is said to be a Meir-Keeler contraction, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x, y) < \varepsilon + \delta$ implies

$$d(\phi x, \phi y) < \varepsilon,$$

for all $x, y \in X$. We know that if (X, d) is a complete metric space, then ϕ has a unique fixed point [16]. In the sequel, we always use Σ_M to denote the collection of all Meir-Keeler contractions on M and S_E to denote the unit sphere $S_E = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be strictly convex if $x, y \in S_E$ with $x \neq y$, and, for all $t \in (0, 1)$,

$$\|(1 - t)x + ty\| < 1.$$

A Banach space E is said to be smooth provided the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each x and y in S_E. In this case, the norm of E is said to be Gâteaux differentiable. It is said to be uniformly Gâteaux differentiable if for each $y \in S_E$, this limit is attained uniformly for $x \in S_E$. It is well known that every uniformly smooth Banach space has a uniformly Gâteaux differentiable norm.

A closed convex subset C of a Banach space E is said to have the fixed point property for nonexpansive mappings if every nonexpansive mapping of a nonempty, closed, and convex subset M of C into itself has a fixed point in M.

A subset C of a Banach space E is called a retract of E if there is a continuous mapping P from E onto C such that $Px = x$, for all $x \in C$. We call such P a retraction of E onto C. It follows that if P is a retraction, then $Py = y$, for all y in the range of P. A retraction P is said to be sunny if $P(Px + t(x - Px)) = Px$, for all $x \in E$ and $t \geq 0$. If a sunny retraction P is also nonexpansive, then C is said to be a sunny nonexpansive retract of E.

An accretive operator A defined on a Banach space E is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$, for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A. We know that for an accretive operator A which satisfies the range condition, $A^{-1}0 = F(J_\lambda^A)$, for all $\lambda > 0$.

Let f be a continuous linear functional on l_∞. We use $f_n(x_{n+m})$ to denote

$$f(x_{m+1}, x_{m+2}, \dots, x_{m+n}, \dots),$$

for $m = 0, 1, 2, \dots$. A continuous linear functional f on l_∞ is called a Banach limit if $\|f\| = f(e) = 1$ and $f_n(x_n) = f_n(x_{n+1})$ for each $x = (x_1, x_2, \dots)$ in l_∞. Fix any Banach limit and denote it by LIM. Note that $\|LIM\| = 1$, and, for all $\{x_n\} \in l_\infty$,

$$\liminf_{n \to \infty} x_n \leq LIM_n x_n \leq \limsup_{n \to \infty} x_n. \tag{2.1}$$

The following lemmas play crucial roles for the proof of main theorems in this paper.

Lemma 2.1 [17] *Let ϕ be a Meir-Keeler contraction on a convex subset C of a Banach space E. Then for each $\varepsilon > 0$, there exists $r \in (0,1)$ such that, for all $x, y \in C$, $\|x-y\| \geq \varepsilon$ implies*

$$\|\phi x - \phi y\| \leq r\|x-y\|. \tag{2.2}$$

Remark 2.2 From Lemma 2.1, for each $\varepsilon > 0$, there exists $r \in (0,1)$ such that

$$\|\phi x - \phi y\| \leq \max\{\varepsilon, r\|x-y\|\}, \tag{2.3}$$

for all $x, y \in C$.

Lemma 2.3 [17] *Let C be a convex subset of a Banach space E. Let T be a nonexpansive mapping on C and ϕ be a Meir-Keeler contraction on C. Then, for each $t \in (0,1)$, a mapping $x \mapsto (1-t)Tx + t\phi x$ is also a Meir-Keeler contraction on C.*

Lemma 2.4 [18] *Let C be a convex subset of a smooth Banach space E, D a nonempty subset of C, and P a retraction from C onto D. Then the following statements are equivalent*:

(i) *P is sunny nonexpansive.*
(ii) *$\langle x - Px, j(z - Px)\rangle \leq 0$, for all $x \in C$, $z \in D$.*
(iii) *$\langle x - y, j(Px - Py)\rangle \geq \|Px - Py\|^2$, for all $x, y \in C$.*

We can easily prove the following lemma from Lemma 1 in [19].

Lemma 2.5 [19] *Let E be a Banach space with a uniformly Gâteaux differentiable norm, C a nonempty, closed, and convex subset of E and $\{x_n\}$ a bounded sequence in E. Let LIM be a Banach limit and $y \in C$ such that*

$$LIM_n\|x_n - y\|^2 = \inf_{x \in C} LIM_n\|x_n - x\|^2.$$

Then $LIM_n\langle x - y, j(x_n - y)\rangle \leq 0$, for all $x \in C$.

Lemma 2.6 [20] *Let $\{a_n\}$, $\{b_n\}$, $\{\sigma_n\}$ be sequences of positive numbers satisfying the inequality*:

$$a_{n+1} \leq (1-b_n)a_n + \sigma_n, \quad b_n < 1.$$

If $\sum_{n=0}^{\infty} b_n = +\infty$ and $\lim_{n\to\infty} \sigma_n/b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 [21] *Let E be a Banach space with a uniformly Gâteaux differentiable norm and let C be a nonempty, closed, and convex subset of E with fixed point property for nonexpansive self-mappings. Let $A : D(A) \subset E \to 2^E$ be an accretive operator such that $A^{-1}0 \neq \emptyset$ and $\overline{D(A)} \subset \bigcap_{t>0} R(I + tA)$. Then $A^{-1}0$ is a sunny nonexpansive retract of C.*

Lemma 2.8 [11] *Let C be a nonempty, closed, and convex subset of a strictly convex Banach space E. Let $A_i : C \to E$ be an m-accretive operator for each $i = 1, 2, \ldots, N$ with $\bigcap_{i=1}^{N} N(A_i) \neq \emptyset$. Let $a_0, a_1, \ldots, a_N$ be real numbers in $(0,1)$ such that $\sum_{i=0}^{N} a_i = 1$ and let $S_N := a_0 I + a_1 J^{A_1} + a_2 J^{A_2} + \cdots + a_N J^{A_N}$, where $J^{A_i} := (I + A_i)^{-1}$. Then S_N is a nonexpansive mapping and $F(S_N) = \bigcap_{i=1}^{N} N(A_i)$.*

3 Main results

Now, we are in a position to introduce and prove the main theorems.

Propositon 3.1 *Let E be a reflexive Banach space with a uniformly Gâteaux differentiable norm and let C be a closed convex subset of E which has the fixed point property for nonexpansive mappings. Let T be a nonexpansive mapping on C. Then for each $\phi \in \Sigma_C$ and every $t \in (0,1)$, there exists a unique fixed point $v_t \in C$ of the Meir-Keeler contraction $C \ni v_t \mapsto t\phi v_t + (1-t)Tv_t$, such that $\{v_t\}$ converges strongly to $x^* \in F(T)$ as $t \to 0$ which solves the variational inequality*:

$$\langle x^* - \phi x^*, j(x^* - x)\rangle \leq 0, \tag{3.1}$$

for all $x \in F(T)$.

Proof By Lemma 2.3, the mapping $C \ni v \mapsto t\phi v + (1-t)Tv$ is a Meir-Keeler contraction on C. So, there is a unique $v_t \in C$ which satisfies

$$v_t = t\phi v_t + (1-t)Tv_t.$$

Now we show that $\{v_t\}$ is bounded. Indeed, take a $p \in F(T)$ and a number $\varepsilon > 0$.

Case 1. Let $\|v_t - p\| \leq \varepsilon$. Then we can see easily that $\{v_t\}$ is bounded.

Case 2. Let $\|v_t - p\| \geq \varepsilon$. Then, by Lemma 2.4, there exists $r \in (0,1)$ such that

$$\|\phi v_t - \phi p\| \leq r\|v_t - p\|.$$

So, we have

$$\begin{aligned}\|v_t - p\| &= \big\|t\phi v_t + (1-t)Tv_t - p\big\| \\ &\leq t\|\phi v_t - \phi p\| + t\|\phi p - p\| + (1-t)\|v_t - p\| \\ &\leq rt\|v_t - p\| + t\|\phi p - p\| + (1-t)\|v_t - p\|.\end{aligned}$$

Therefore,

$$\|v_t - p\| \leq \frac{\|\phi p - p\|}{1-r}.$$

Hence, we conclude that $\{v_t\}$ is bounded and $\{\phi v_t\}$, $\{Tv_t\}$ are also bounded.

By the boundedness of $\{v_t\}$, $\{\phi v_t\}$, and $\{Tv_t\}$, we have

$$\|v_t - Tv_t\| = t\|\phi v_t - Tv_t\| \to 0 \quad \text{as } t \to 0.$$

Assume $t_n \to 0$. Set $v_n := v_{t_n}$ and define $\varphi : C \to \mathbb{R}^+$ by

$$\varphi(x) = LIM_n\|v_n - x\|^2,$$

for all $x \in C$ and let

$$M = \Big\{y \in C : \varphi(y) = \inf_{x \in C} \varphi(x)\Big\}.$$

Since E is reflexive, $\varphi(x) \to \infty$ as $\|x\| \to \infty$, and φ is a continuous convex function, from Barbu and Precupanu [22], we know that M is a nonempty subset of C. By Takahashi [23], we see that M is also closed, convex, and bounded.

For all $x \in M$, from $\|v_n - Tv_n\| \to 0$ as $n \to \infty$, we have

$$\begin{aligned}
\varphi(Tx) &= LIM_n\|v_n - Tx\|^2 \\
&\leq LIM_n\big(\|v_n - Tv_n\| + \|Tv_n - Tx\|\big)^2 \\
&\leq LIM_n\|Tv_n - Tx\|^2 \\
&\leq LIM_n\|v_n - x\|^2 \\
&= \varphi(x).
\end{aligned}$$

So, M is invariant under T, *i.e.*, $T(M) \subset M$. By assumption, we have $M \cap F(T) \neq \emptyset$. Let $x^* \in M \cap F(T)$. By Lemma 2.7, we obtain

$$LIM_n\langle x - x^*, j(v_n - x^*)\rangle \leq 0, \tag{3.2}$$

for all $x \in C$. In particular,

$$LIM_n\langle \phi x^* - x^*, j(v_n - x^*)\rangle \leq 0. \tag{3.3}$$

Suppose that $LIM_n\|v_n - x^*\|^2 \geq \varepsilon > 0$. By (2.1),

$$\limsup_{n\to\infty}\|v_n - x^*\|^2 \geq \varepsilon.$$

So, there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that, for all $k \geq 1$,

$$\|v_{n_k} - x^*\| \geq \varepsilon_0,$$

where $\varepsilon_0 \in (0, \sqrt{\varepsilon})$. By Lemma 2.3, there is $r_0 \in (0,1)$ such that

$$\|\phi v_{n_k} - \phi x^*\| \leq r\|v_{n_k} - x^*\|.$$

From

$$\langle Tv_{n_k} - v_{n_k}, j(v_{n_k} - x^*)\rangle \leq 0,$$

for all $k \geq 1$, we have

$$\begin{aligned}
\|v_{n_k} - x^*\|^2 &= t\langle \phi v_{n_k} - x^*, j(v_{n_k} - x^*)\rangle + (1-t)\langle Tv_{n_k} - x^*, j(v_{n_k} - x^*)\rangle \\
&\leq t\langle \phi v_{n_k} - x^*, j(v_{n_k} - x^*)\rangle + (1-t)\|v_{n_k} - x^*\|^2,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|v_{n_k} - x^*\|^2 &\leq \langle \phi v_{n_k} - x^*, j(v_{n_k} - x^*)\rangle \\
&\leq \langle \phi v_{n_k} - x, j(v_{n_k} - x^*)\rangle + \langle \phi x - x^*, j(v_{n_k} - x^*)\rangle,
\end{aligned}$$

for all $x \in C$. So, from (3.2), we get

$$\begin{aligned} LIM_n \| v_{n_k} - x^* \|^2 &\leq LIM_n \langle \phi v_{n_k} - x, j(v_{n_k} - x^*) \rangle + LIM_n \langle \phi x - x^*, j(v_{n_k} - x^*) \rangle \\ &\leq LIM_n \| \phi v_{n_k} - x \| \| v_{n_k} - x^* \|, \end{aligned}$$

for all $x \in C$. In particular,

$$\begin{aligned} LIM_n \| v_{n_k} - x^* \|^2 &\leq LIM_n \| \phi v_{n_k} - \phi x^* \| \| v_{n_k} - x^* \| \\ &\leq r_0 LIM_n \| v_{n_k} - x^* \|^2, \end{aligned}$$

which is a contradiction. Hence, $LIM_n \| v_n - x^* \| = 0$ and there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ such that $v_{n_k} \to x^*$ as $k \to \infty$.

Assume that $\{v_{n_l}\}$ is another subsequence of $\{v_n\}$ such that $v_{n_l} \to y^*$ with $y^* \neq x^*$. It is easy to see that $y^* \in F(T)$. By Lemma 2.3, there exists $r_1 \in (0,1)$ such that

$$\| \phi x^* - \phi y^* \| \leq r_1 \| x^* - y^* \|. \tag{3.4}$$

Observe that

$$\begin{aligned} &\left| \langle v_n - \phi v_n, j(v_n - y^*) \rangle - \langle x^* - \phi x^*, j(x^* - y^*) \rangle \right| \\ &\quad \leq \left| \langle v_n - \phi v_n, j(v_n - y^*) \rangle - \langle x^* - \phi x^*, j(v_n - y^*) \rangle \right| \\ &\qquad + \left| \langle x^* - \phi x^*, j(v_n - y^*) \rangle - \langle x^* - \phi x^*, j(x^* - y^*) \rangle \right| \\ &\quad \leq \| v_n - \phi v_n - (x^* - \phi x^*) \| \| v_n - y^* \| + \left| \langle x^* - \phi x^*, j(v_n - y^*) - j(x^* - y^*) \rangle \right|, \end{aligned}$$

for all $n \in \mathbb{N}$. Since $v_{n_k} \to x^*$ and j is norm to weak* uniformly continuous, we obtain

$$\langle x^* - \phi x^*, j(x^* - y^*) \rangle \leq 0.$$

Similarly, we have

$$\langle y^* - \phi y^*, j(y^* - x^*) \rangle \leq 0.$$

Adding the above two inequalities yields

$$\langle x^* - y^* - (\phi x^* - \phi y^*), j(x^* - y^*) \rangle \leq 0,$$

and combining with (3.4) implies that

$$\| x^* - y^* \| \leq r_1 \| x^* - y^* \|,$$

which is a contradiction. Hence $\{v_{t_n}\}$ converges strongly to x^*.

Now, we prove that the net $\{v_t\}$ converges strongly to x^* as $t \to 0$. We assume that there is another subsequence $\{s_n\}$ with $s_n \in (0,1)$, for all n and $s_n \to 0$ as $n \to \infty$ such that $v_{s_n} \to z^*$ as $n \to \infty$. Then we have $z^* \in F(T)$. For each t and $z \in F(T)$, we have

$$\langle v_t - \phi v_t, j(v_t - z) \rangle = \frac{1-t}{t} \langle T v_t - v_t, j(v_t - z) \rangle \leq 0.$$

So, we obtain

$$\langle v_{t_n} - \phi v_{t_n}, j(v_{t_n} - z^*)\rangle \le 0$$

and similarly, we have

$$\langle v_{s_n} - \phi v_{s_n}, j(v_{s_n} - x^*)\rangle \le 0,$$

which implies that

$$\langle x^* - \phi x^*, j(x^* - z^*)\rangle \le 0$$

and

$$\langle z^* - \phi z^*, j(z^* - x^*)\rangle \le 0.$$

Thus, we have $x^* = z^*$. Therefore, $\{v_t\}$ converges strongly to x^* and it is easy to see that x^* solves the variational inequality

$$\langle x^* - \phi x^*, j(x^* - x)\rangle \le 0,$$

for all $x \in F(T)$. This completes the proof. □

Remark 3.2 Let Q be a sunny nonexpansive retraction from C onto $F(T)$. By the uniqueness of Q, inequality (3.1) and Lemma 2.4, we obtain $Q\phi x^* = x^*$.

Proposition 3.3 *Let C be a closed convex subset of a reflexive Banach space E with a uniformly Gâteaux differentiable norm and let T be a nonexpansive mapping on C with $F(T) \neq \emptyset$. Assume $\{x_n\}$ is a bounded sequence such that $x_n - Tx_n \to 0$ as $n \to \infty$. Let $x_t = t\phi x_t + (1-t)Tx_t$, for all $t \in (0,1)$, where $\phi \in \Sigma_C$. Assume that $x^* = \lim_{t\to 0} x_t$ exists. Then we have*

$$\limsup_{n\to\infty} \langle (\phi - I)x^*, j(x_n - x^*)\rangle \le 0. \tag{3.5}$$

Proof Set $M = \sup\{\|x_n - x_t\| : t \in (0,1), n \ge 0\}$. Then we have

$$\begin{aligned}
\|x_t - x_n\|^2 &= t\langle \phi x_t - x_n, j(x_t - x_n)\rangle + (1-t)\langle Tx_t - x_n, j(x_t - x_n)\rangle \\
&= t\langle \phi x_t - x_t, j(x_t - x_n)\rangle + (1-t)\langle Tx_t - Tx_n, j(x_t - x_n)\rangle \\
&\quad + (1-t)\langle Tx_n - x_n, j(x_t - x_n)\rangle \\
&\le t\langle \phi x_t - x_t, j(x_t - x_n)\rangle + t\|x_t - x_n\|^2 \\
&\quad + (1-t)\|x_t - x_n\|^2 + M\|x_n - Tx_n\|,
\end{aligned}$$

which implies that

$$\langle \phi x_t - x_t, j(x_n - x_t)\rangle \le \frac{M}{t}\|x_n - Tx_n\|.$$

Fix t and letting $n \to \infty$ yields

$$\limsup_{n\to\infty}\langle(\phi - I)x^*, j(x_n - x^*)\rangle \le 0.$$

This completes the proof. □

Now, let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and C a closed convex subset of E which has the fixed point property for nonexpansive mappings. Let $A_i : E \to 2^E$ be an accretive operator, for each $i = 1, 2, \ldots, N$ such that

$$S = \bigcap_{i=1}^{N} A_i^{-1}0 \ne \emptyset$$

and

$$\overline{D(A_i)} \subset C \subset \bigcap_{r>0} R(I + rA_i),$$

for all $i = 1, 2, \ldots, N$.

For each $\phi \in \Sigma_C$, we study the strong convergence of the sequence $\{z_n\}$ defined by

$$\begin{cases} z_0 \in C, \\ z_{n+1} = S_N(\alpha_n\phi z_n + (1-\alpha_n)z_n), \quad \forall n \ge 0, \end{cases} \tag{3.6}$$

where $S_N := a_0I + a_1J^{A_1} + a_2J^{A_2} + \cdots + a_NJ^{A_N}$ with $a_0, a_1, \ldots, a_N$ are real numbers in $(0,1)$ such that $\sum_{i=0}^{N} a_i = 1$ and $\{\alpha_n\} \subset (0,1)$ is a real sequence of positive numbers, under the conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$,

(C2) $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$ or $\lim_{n\to\infty}\frac{|\alpha_n-\alpha_{n-1}|}{\alpha_n} = 0$.

Then we have the following theorem.

Theorem 3.4 *If the sequence $\{\alpha_n\}$ satisfies the conditions* (C1)-(C2), *then the sequence $\{x_n\}$ generated by*

$$x_{n+1} = S_N(\alpha_n u + (1-\alpha_n)x_n), \quad \forall n \ge 0, \tag{3.7}$$

converges strongly to Qu, where $u \in C$ and Q is a sunny nonexpansive retraction from C onto S.

Proof By Lemma 2.8, we have $F(S_N) = \bigcap_{i=1}^{N} A_i^{-1}0 \ne \emptyset$. Now, for each $p \in F(S_N)$, we have

$$\begin{aligned} \|x_{n+1} - p\| &= \|S_N(\alpha_n u + (1-\alpha_n)x_n) - S_N(p)\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|u - p\| \\ &\le \max\{\|x_n - p\|, \|u - p\|\} \\ &\vdots \\ &\le \max\{\|x_0 - p\|, \|u - p\|\}. \end{aligned} \tag{3.8}$$

Hence $\{x_n\}$ is bounded. Suppose that $\max\{\sup\|x_n\|, \|u\|\} \leq K$. It follows that

$$\begin{aligned}\|x_{n+1} - S_N(x_n)\| &= \|S_N(\alpha_n f(x_n) + (1-\alpha_n)x_n) - S_N(x_n)\| \\ &\leq \alpha_n \|f(x_n) - x_n\| \to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.9}$$

From (1.10), we get

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|S_N(\alpha_n u + (1-\alpha_n)x_n) - S_N(\alpha_{n-1}u + (1-\alpha_{n-1})x_{n-1})\| \\ &\leq (1-\alpha_n)\|x_n - x_{n-1}\| + \alpha_n\beta_n,\end{aligned}$$

where $\beta_n = 2K\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n}$.

We consider two cases of condition (C2).

First, suppose that $\sum_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$. Then

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n)\|x_n - x_{n-1}\| + \sigma_n,$$

where $\sigma_n = 2K|\alpha_n - \alpha_{n-1}|$. So, we have $\sum_{n=1}^{\infty}\sigma_n < \infty$.

Second, suppose that $\lim_{n\to\infty}\frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n} = 0$. Then

$$\|x_{n+1} - x_n\| \leq (1-\alpha_n)\|x_n - x_{n-1}\| + \sigma_n,$$

where $\sigma_n = \alpha_n\beta_n$. So, we have $\sigma_n = o(\alpha_n)$.

For any case, we have $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, from Lemma 2.6. By (3.9) we obtain

$$\|x_n - S_N x_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - S_N x_n\| \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

Let $y_n = \alpha_n u + (1-\alpha_n)x_n$. Then we have

$$\|y_n - x_n\| = \alpha_n\|u - x_n\| \to 0 \quad \text{as } n \to \infty,$$

it follows that

$$\begin{aligned}\|y_n - S_N y_n\| &\leq \|y_n - x_n\| + \|x_n - S_N x_n\| + \|S_N x_n - S_N y_n\| \\ &\leq 2\|y_n - x_n\| + \|x_n - S_N x_n\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

For each $t \in (0,1)$, let $x_t = tu + (1-t)S_N x_t$. Apply Proposition 3.1 with $\phi x = u$, for all $x \in C$, we know that $\{x_t\}$ converges strongly to $x^* \in F(S_N)$, satisfying $Qu = x^*$. It follows from Proposition 3.3 that

$$\limsup_{n\to\infty}\langle u - x^*, j(y_n - x^*)\rangle \leq 0.$$

Observe that

$$\begin{aligned}\|y_n - x^*\|^2 &= \langle \alpha_n u + (1-\alpha_n)x_n - x^*, j(y_n - x^*)\rangle \\ &\leq (1-\alpha_n)\|x_n - x^*\|\|y_n - x^*\| + \alpha_n\langle u - x^*, j(y_n - x^*)\rangle \\ &\leq \frac{(1-\alpha_n)}{2}(\|x_n - x^*\|^2 + \|y_n - x^*\|^2) + \alpha_n\langle u - x^*, j(y_n - x^*)\rangle.\end{aligned}$$

Hence, we have

$$\|y_n - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, j(y_n - x^*)\rangle.$$

Next, we have

$$\|x_{n+1} - x^*\|^2 \le (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n\langle u - x^*, j(y_n - x^*)\rangle. \tag{3.11}$$

From Lemma 2.6, we have the desired result. That is, the sequence $\{x_n\}$ converges strongly to $Qu = x^*$. This completes the proof. □

The following is a strong convergence theorem for the sequence $\{z_n\}$ in (3.6).

Theorem 3.5 *If the sequence* $\{\alpha_n\}$ *satisfies the conditions* (C1)-(C2), *then the sequence* $\{z_n\}$ *generated by* (3.6) *converges strongly to* $x^* \in S$, *which satisfies* $Q\phi x^* = x^*$, *where* Q *is a sunny nonexpansive retraction from* C *onto* S.

Proof Let x^* be a unique fixed point of $Q\phi$, that is, $Q\phi x^* = x^*$. Let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = S_N\big(\alpha_n\phi x^* + (1-\alpha_n)x_n\big), \quad \text{for all } n \ge 0.$$

By Theorem 3.4, $x_n \to Q\phi x^* = x^*$ as $n \to \infty$.

Now, we prove that $\|z_n - x_n\| \to 0$ as $n \to \infty$. Assume that

$$\limsup_{n\to\infty} \|z_n - x_n\| > 0.$$

Then we choose ε with $\varepsilon \in (0, \limsup_{n\to\infty} \|z_n - x_n\|)$. By Lemma 2.3, there exists $r \in (0,1)$ satisfying (2.2). We also choose $n_1 \in \mathbb{N}$ such that

$$\frac{r\|x_n - x^*\|}{1-r} < \varepsilon,$$

for all $n \ge n_1$. We divide this into the following two cases:

(i) There exists $n_2 \in \mathbb{N}$ satisfying $n_2 \ge n_1$ and $\|z_{n_2} - x_{n_2}\| \le \varepsilon$.

(ii) $\|z_n - x_n\| > \varepsilon$, for all $n \ge n_1$.

In the case of (i), we have

$$\begin{aligned}
\|z_{n_2+1} - x_{n_2+1}\| &\le (1-\alpha_{n_2})\|z_{n_2} - x_{n_2}\| + \alpha_{n_2}\|\phi z_{n_2} - \phi x^*\| \\
&\le (1-\alpha_{n_2})\|z_{n_2} - x_{n_2}\| + \alpha_{n_2}\max\big\{r\|z_{n_2} - x^*\|, \varepsilon\big\} \\
&\le \max\bigg\{(1-\alpha_{n_2} + r\alpha_{n_2})\|z_{n_2} - x_{n_2}\| + \alpha_{n_2}(1-r)\frac{r\|x_n - x^*\|}{1-r}, \\
&\qquad (1-\alpha_{n_2})\|z_{n_2} - x_{n_2}\| + \alpha_{n_2}\varepsilon\bigg\} \\
&\le \varepsilon.
\end{aligned}$$

By induction, we can show that $\|z_n - x_n\| \le \varepsilon$, for all $n \ge n_2$. This is a contradiction to the fact that $\varepsilon < \limsup_{n\to\infty} \|z_n - x_n\|$.

In the case of (ii), for each $n \ge n_1$, we have

$$\begin{aligned}\|z_{n+1} - x_{n+1}\| &\le (1-\alpha_n)\|z_n - x_n\| + \alpha_n \|\phi z_n - \phi x^*\| \\ &\le (1-\alpha_n)\|z_n - x_n\| + \alpha_n\|\phi z_n - \phi x_n\| + \alpha_n \|\phi x_n - \phi x^*\| \\ &\le \bigl[1 - \alpha_n(1-r)\bigr]\|z_n - x_n\| + \alpha_n \|\phi x_n - \phi x^*\|.\end{aligned}$$

So, by Lemma 2.1, we get $\lim_{n\to\infty}\|z_n - x_n\| = 0$. This is a contradiction. Therefore $\lim_{n\to\infty}\|z_n - x_n\| = 0$. Thus we obtain

$$\lim_{n\to\infty}\|z_n - x^*\| \le \lim_{n\to\infty}\|z_n - x_n\| + \lim_{n\to\infty}\|x_n - x^*\| = 0.$$

Hence $\{z_n\}$ convergence strongly to $Q\phi x^* = x^*$. This completes the proof. □

Corollary 3.6 *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a closed convex subset of E which has the fixed point property for nonexpansive mappings. Let $A_i : E \to 2^E$ be an m-accretive operator, for each $i = 1, 2, \dots, N$ such that*

$$S = \bigcap_{i=1}^{N} A_i^{-1}0 \neq \emptyset.$$

For each $\phi \in \Sigma_C$, let $\{z_n\}$ be a sequence generated by (3.6). *If the sequence $\{\alpha_n\}$ satisfies the conditions* (C1)-(C2), *then the sequence $\{z_n\}$ converges strongly to $x^* \in S$ which satisfies $Q\phi x^* = x^*$, where Q is a sunny nonexpansive retraction from C onto S.*

Proof Since for each $i = 1, 2, \dots, N$, A_i is an m-accretive operator, the condition $\overline{D(A_i)} \subset C \subset \bigcap_{r>0} R(I + rA_i)$ is satisfied, for all $i = 1, 2, \dots, N$. By the assumption and Theorem 3.5, we have $z_n \to x^*$ as $n \to \infty$ which satisfies $Q\phi x^* = x^*$. This completes the proof. □

Remark 3.7 Corollary 3.6 is a generalization of the results of Tuyen [14], Zegeye and Shahzad [11] and Jung [24].

Remark 3.8 If we take $r = 1$, then we may take $S_1 := J^A = (I + A)^{-1}$ and strict convexity of E and the real constants a_i, $i = 0, 1$, may not be needed.

Corollary 3.9 *Let E be a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm and let C be a closed convex subset of E which has the fixed point property for nonexpansive mappings. Let $A : E \to 2^E$ be an m-accretive operator such that $S = A^{-1}0 \neq \emptyset$. For each $\phi \in \Sigma_C$, let $\{z_n\}$ be a sequence defined by*

$$\begin{cases} z_0 \in C, \\ z_{n+1} = J^A(\alpha_n \phi z_n + (1-\alpha_n) z_n), \end{cases} \tag{3.12}$$

for all $n \ge 0$. If the sequence $\{\alpha_n\}$ satisfies the conditions (C1)-(C2), *then the sequence $\{z_n\}$ converges strongly to $x^* \in S$ which satisfies $Q\phi x^* = x^*$, where Q is a sunny nonexpansive retraction from C onto S.*

Remark 3.10 Corollary 3.9 is a generalization of the results of Tuyen in [8].

4 Applications

In this section, we give some applications in the framework of Hilbert spaces. We first apply Corollary 3.9 to the convex minimization problem.

Theorem 4.1 *Let H be a Hilbert space and let $f : H \to (-\infty, \infty]$ be a proper lower semi-continuous convex function such that $(\partial f)^{-1}0 \neq \emptyset$ for a subdifferential mapping ∂f of f. Let $\{x_n\}$ be a sequence defined as follows*:

$$\begin{cases} x_0 \in H, \\ y_n = \alpha_n \phi x_n + (1 - \alpha_n)x_n, \\ x_{n+1} = \operatorname{argmin}_{z\in H}\{f(z) + \frac{1}{2}\|z - y_n\|^2\}, \end{cases} \tag{4.1}$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence positive real numbers and $\phi \in \Sigma_H$. If the sequence $\{\alpha_n\}$ satisfies the conditions (C1)-(C2), *then the sequence $\{x_n\}$ converges strongly to x^* in* $(\partial f)^{-1}0$.

Proof By the Rockafellar theorem [25] (*cf.* [26]), the subdifferential mapping ∂f is maximal monotone in H. So,

$$x_{n+1} = \operatorname*{argmin}_{z\in H}\left\{f(z) + \frac{1}{2}\|z - y_n\|^2\right\}$$

is equivalent to $\partial f(x_{n+1}) + x_{n+1} \ni y_n$. Using Corollary 3.9, $\{x_n\}$ converges strongly to an element x^* in $(\partial f)^{-1}0$. This completes the proof. □

We next apply Proposition 3.3 to the variational inequality problem. Let C be a nonempty, closed, and convex subset of a Hilbert space H and let $A : C \to H$ be a single-valued monotone operator which is hemicontinuous. Then a point $u \in C$ is said to be a solution of the variational inequality for A if

$$\langle y - u, Au\rangle \geq 0, \tag{4.2}$$

for all $y \in C$. We denote by $VI(C, A)$ the set of all solutions of the variational inequality (4.2) for A. We also denote by $N_C(x)$ the normal cone for C at a point $x \in C$, that is,

$$N_C(x) = \bigl\{z \in H : \langle y - x, z\rangle \leq 0, \text{for all } y \in C\bigr\}.$$

Theorem 4.2 *Let C be a nonempty, closed, and convex subset of a Hilbert space H and let $A : C \longrightarrow H$ be a single-valued monotone operator and hemicontinuous such that $VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence defined as follows*:

$$\begin{cases} x_0 \in H, \\ y_n = \alpha_n \phi x_n + (1 - \alpha_n)x_n, \\ x_{n+1} = VI(C, A + I - y_n), \end{cases} \tag{4.3}$$

for all $n \geq 0$, where $\{\alpha_n\}$ is a sequence of positive real numbers and $\phi \in \Sigma_H$. If the sequence $\{\alpha_n\}$ satisfies the conditions (C1)-(C2), *then the sequence $\{x_n\}$ converges strongly to x^* in* $VI(C,A)$.

Proof Define a mapping $T \subset H \times H$ by

$$Tx = \begin{cases} Ax + N_C(x), & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

By the Rockafellar theorem [27], we know that T is maximal monotone and $T^{-1}0 = VI(C,A)$.

Note that

$$x_{n+1} = VI(C, A + I - y_n)$$

if and only if

$$\langle y - x_{n+1}, Ax_{n+1} + x_{n+1} - y_n \rangle \geq 0,$$

for all $y \in C$, that is,

$$-Ax_{n+1} - x_{n+1} + y_n \in N_C(x_{n+1}).$$

This implies that

$$x_{n+1} = J^T\big(\alpha_n \phi x_n + (1 - \alpha_n)x_n\big).$$

Using Corollary 3.9, $\{x_n\}$ converges strongly to an element x^* in $VI(C,A)$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved final manuscript.

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