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Generalized probabilistic metric spaces and fixed point theorems

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Abstract

In this paper, we introduce a new concept of probabilistic metric space, which is a generalization of the Menger probabilistic metric space, and we investigate some topological properties of this space and related examples. Also, we prove some fixed point theorems, which are the probabilistic versions of Banach's contraction principle. Finally, we present an example to illustrate the main theorems.

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1 Introduction and preliminaries

Let $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^+$ be the set of all nonnegative real numbers, Δ denote the set of all probability distribution functions, *i.e.*, $\Delta = \{F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0,1] : F$ is left continuous and nondecreasing on $\mathbb{R}, F(-\infty) = 0$ and $F(+\infty) = 1\}$.

Definition 1.1 ([1]) A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *continuous t-norm* if T satisfies the following conditions:

(1) T is commutative and associative, *i.e.*, $T(a,b) = T(b,a)$ and $T(a,T(b,c)) = T(T(a,b),c)$, for all $a,b,c \in [0,1]$;
(2) T is continuous;
(3) $T(a,1) = a$ for all $a \in [0,1]$;
(4) $T(a,b) \le T(c,d)$ whenever $a \le c$ and $b \le d$ for all $a,b,c,d \in [0,1]$.

From the definition of T it follows that $T(a,b) \le \min\{a,b\}$ for all $a,b \in [0,1]$.

Two simple examples of continuous t-norm are $T_M(a,b) = \min\{a,b\}$ and $T_p(a,b) = ab$ for all $a,b \in [0,1]$.

In 1942, Menger [2] developed the theory of metric spaces and proposed a generalization of metric spaces called Menger probabilistic metric spaces (briefly, Menger PM-space).

Definition 1.2 A *Menger PM-space* is a triple (X,F,T), where X is a nonempty set, T is a continuous t-norm and F is a mapping from $X \times X \to \mathcal{D}$ ($F_{x,y}$ denotes the value of F at the pair (x,y)) satisfying the following conditions:

(PM-1) $F_{x,y}(t) = 1$ for all $x,y \in X$ and $t > 0$ if and only $x = y$;
(PM-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x,y \in X$ and $t > 0$;
(PM-3) $F_{x,z}(t+s) \ge T(F_{x,y}(t), F_{y,z}(s))$ for all $x,y,z \in X$ and $t,s \ge 0$.

The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. Since Menger, many authors have considered fixed point theory in PM-spaces and its applications as a part of probabilistic analysis (see [1, 3–14]).

In 1963, Gähler [15] investigated the concept of 2-metric spaces and he claimed that a 2-metric is a natural generalization of an ordinary metric space (for more detailed results, see the books [16, 17]). But some authors pointed out that there are no relations between 2-metric spaces and ordinary metric spaces [18]. Later, Dhage [19] introduced a new class of generalized metrics called D-metric spaces. However, as pointed out in [20], the D-metric is also not satisfactory.

Recently, Mustafa and Sims [21] introduced a new class of metric spaces called generalized metric spaces or G-metric spaces as follows.

Definition 1.3 ([21]) Let X be a nonempty set and $G : X \times X \times X :\rightarrow \mathbb{R}^+$ be a function satisfying the following conditions:

(G1) $G(x,y,z) = 0$ if $x = y = z$ for all $x,y,z \in X$;
(G2) $0 < G(x,x,y)$ for all $x,y \in X$ with $x \neq y$;
(G3) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z \in X$ with $z \neq y$;
(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots$ for all $x,y,z \in X$;
(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$.

Then G is called a *generalized metric* or a *G-metric* on X and the pair (X,G) is a *G-metric space*.

It was proved that the G-metric is a generalization of ordinary metric (see [21]). Recently, some authors studied G-metric spaces and obtained fixed point theorems on G-metric spaces [22–24]. Similar work can be found in [25–29].

It is well known that the notion of a PM-space corresponds to the situation that we may know probabilities of possible values of the distance although we do not know exactly the distance between two points. This idea leads us to seek a probabilistic version of G-metric spaces defined by Mustafa and Sims [21].

Definition 1.4 A *Menger probabilistic G-metric space* (shortly, *PGM-space*) is a triple (X,G^*,T), where X is a nonempty set, T is a continuous t-norm and G^* is a mapping from $X \times X \times X$ into $\mathcal{D}$ ($G^*_{x,y,z}$ denotes the value of G^* at the point (x,y,z)) satisfying the following conditions:

(PGM-1) $G^*_{x,y,z}(t) = 1$ for all $x,y,z \in X$ and $t > 0$ if and only if $x = y = z$;
(PGM-2) $G^*_{x,x,y}(t) \geq G^*_{x,y,z}(t)$ for all $x,y \in X$ with $z \neq y$ and $t > 0$;
(PGM-3) $G^*_{x,y,z}(t) = G^*_{x,z,y}(t) = G^*_{y,x,z}(t) = \cdots$ (symmetry in all three variables);
(PGM-4) $G^*_{x,y,z}(t+s) \geq T(G^*_{x,a,a}(s), G^*_{a,y,z}(t))$ for all $x,y,z,a \in X$ and $s,t \geq 0$.

Remark 1.5 Golet introduced a concept of probabilistic 2-metric (or 2-Menger space) [30] based on 2-metric [15] defined by Gähler. In the concept of probabilistic 2-metric, a 2-t-norm is used. Our definition of a Menger probabilistic G-metric space is different from the one of Golet. The metric of Golet is not continuous in two arguments although it is continuous in any one of its three arguments. But G^* is continuous in any two arguments as shown in Theorem 2.5.

Example 1.6 Let H denote the specific distribution function defined by

$$H(t) = \begin{cases} 0, & t \leq 0, \\ 1, & t > 0, \end{cases}$$

and D be a distribution function defined by

$$D(t) = \begin{cases} 0, & t \leq 0, \\ 1 - e^{-t}, & t > 0. \end{cases}$$

For any $t > 0$, define a function $G^* : X \times X \times X \to \mathbb{R}^+$ by

$$G^*_{x,y,z}(t) = \begin{cases} H(t), & x = y = z, \\ D(\frac{t}{G(x,y,z)}), & \text{otherwise}, \end{cases}$$

where G is a G-metric as in Definition 1.3. Set $T = \min$. Then G^* is a probabilistic G-metric.

Proof It is easy to see that G^* satisfies (PGM-1)-(PGM-3). Next we show $G^*(x,y,z)(s+t) \geq T(G^*(x,a,a)(s), G^*(a,y,y)(t))$ for all $x,y,z,a \in X$ and all $s,t > 0$. In fact, we only need show that D satisfies

$$D\left(\frac{t+s}{G(x,y,z)}\right) \geq \min\left\{D\left(\frac{s}{G(x,a,a)}\right), D\left(\frac{t}{D(a,y,z)}\right)\right\}. \tag{1.1}$$

Since $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$, we have

$$\frac{s+t}{G(x,y,z)} \geq \frac{s+t}{G(x,a,a) + G(a,y,z)}. \tag{1.2}$$

Furthermore, we have

$$\begin{aligned} \max\left\{\frac{s}{G(x,a,a)}, \frac{t}{G(a,y,z)}\right\} &\geq \frac{s+t}{G(x,a,a) + G(a,y,z)} \\ &\geq \min\left\{\frac{s}{G(x,a,a)}, \frac{t}{G(a,y,z)}\right\}, \end{aligned} \tag{1.3}$$

which, from (1.2) and (1.3), shows that

$$\frac{s+t}{G(x,y,z)} \geq \min\left\{\frac{s}{G(x,a,a)}, \frac{t}{G(a,y,z)}\right\}.$$

This implies (1.1) since D is nondecreasing. □

Example 1.7 Let (X, F, T) be a PM-space. Define a function $G^* : X \times X \times X \to \mathbb{R}^+$ by

$$G^*_{x,y,z}(t) = \min\{F_{x,y}(t), F_{y,z}(t), F_{x,z}(t)\}$$

for all $x, y, z, \in X$ and $t > 0$. Then G^* is a probabilistic G-metric.

Proof It is obvious that G^* satisfies (PGM-1), (PGM-2), and (PGM-3). To prove that G^* satisfies (PGM-4), we need to show that, for all $x, y, z, a \in X$ and all $s, t \geq 0$,

$$G^*_{x,y,z}(s+t) \geq T\big(G^*_{x,a,a}(s), G^*_{a,y,z}(t)\big), \tag{1.4}$$

i.e.,

$$\begin{aligned}&\min\big\{F_{x,y}(s+t), F_{y,z}(s+t), F_{x,z}(s+t)\big\}\\ &\quad\geq T\big(F_{x,a}(s), \min\big\{F_{a,y}(t), F_{a,z}(t), F_{y,z}(t)\big\}\big).\end{aligned} \tag{1.5}$$

Now, from

$$\begin{aligned}F_{x,y}(s+t) &\geq T\big(F_{x,a}(s), F_{a,y}(t)\big)\\ &\geq T\big(F_{x,a}(s), \min\big\{F_{a,y}(t), F_{a,z}(t), F_{y,z}(t)\big\}\big),\\ F_{x,z}(s+t) &\geq T\big(F_{x,a}(s), F_{a,z}(t)\big)\\ &\geq T\big(F_{x,a}(s), \min\big\{F_{a,y}(t), F_{a,z}(t), F_{y,z}(t)\big\}\big)\end{aligned}$$

and

$$\begin{aligned}F_{y,z}(s+t) &\geq F_{y,z}(t)\\ &\geq \min\big\{F_{a,y}(t), F_{a,z}(t), F_{y,z}(t)\big\}\\ &\geq T\big(F_{x,a}(s), \min\big\{F_{a,y}(t), F_{a,z}(t), F_{y,z}(t)\big\}\big),\end{aligned}$$

we conclude that (1.5), *i.e.*, (1.4) holds. Therefore, G^* satisfies (PGM-4) and hence G^* is a probabilistic G-metric. □

Example 1.8 Let (X, F, T_M) be a PM-space. Define a function $G^* : X \times X \times X \to \mathbb{R}^+$ by

$$G^*_{x,y,z}(t) = \min\big\{F_{x,y}(t/3), F_{y,z}(t/3), F_{x,z}(t/3)\big\}$$

for all $x, y, z, \in X$ and $t > 0$. Then (X, G^*, T_M) is a PGM-space.

Proof In fact, the proofs of (PGM-1)-(PGM-3) are immediate. Now, we show that G^* satisfies (PGM-4). It follows that

$$\begin{aligned}&G^*_{x,y,z}(t+s)\\ &\quad= \min\left\{F_{x,y}\left(\frac{t+s}{3}\right), F_{y,z}\left(\frac{t+s}{3}\right), F_{x,z}\left(\frac{t+s}{3}\right)\right\}\\ &\quad\geq \min\left\{\min\left\{F_{x,a}\left(\frac{t}{3}\right), F_{a,y}\left(\frac{s}{3}\right)\right\}, F_{y,z}\left(\frac{s}{3}\right), \min\left\{F_{x,a}\left(\frac{t}{3}\right), F_{a,z}\left(\frac{s}{3}\right)\right\}\right\}\\ &\quad= \min\left\{\min\left\{F_{x,a}\left(\frac{t}{3}\right), F_{a,a}\left(\frac{t}{3}\right), F_{x,a}\left(\frac{t}{3}\right)\right\}, \min\left\{F_{a,y}\left(\frac{s}{3}\right), F_{y,z}\left(\frac{s}{3}\right), F_{a,z}\left(\frac{s}{3}\right)\right\}\right\}\\ &\quad= \min\big\{G^*_{x,a,a}(t), G^*_{a,y,z}(s)\big\}.\end{aligned}$$

Thus (X, G, T_M) is a PGM-space. □

The following remark shows that the PGM-space is a generalization of the Menger PM-space.

Remark 1.9 For any function $G^* : X \times X \times X \to \mathbb{R}^+$, the function $F : X \times X \to \mathbb{R}^+$ defined by

$$F_{x,y}(t) = \min\bigl\{G^*_{x,y,y}(t), G^*_{y,x,x}(t)\bigr\}$$

is a probabilistic metric. It is easy to see that F satisfies the conditions (PM-1) and (PM-2).

Next, we show F satisfies (PM-3). Indeed, for any $s, t \geq 0$ and $x, y, z \in X$, we have

$$\begin{aligned}
&F_{x,y}(s+t) = \min\bigl\{G^*_{x,y,y}(s+t), G^*_{y,x,x}(s+t)\bigr\}, \\
&T\bigl(F_{x,z}(s), F_{z,y}(t)\bigr) = T\bigl(\min\bigl\{G^*_{x,z,z}(s), G^*_{z,x,x}(s)\bigr\}, \min\bigl\{G^*_{z,y,y}(t), G^*_{y,z,z}(t)\bigr\}\bigr).
\end{aligned}$$

It follows from (PGM-4) that

$$\begin{aligned}
&\min\bigl\{G^*_{x,y,y}(s+t), G^*_{y,x,x}(s+t)\bigr\} \\
&\quad \geq \min\bigl\{T\bigl(G^*_{x,z,z}(s), G^*_{z,y,y}(t)\bigr), T\bigl(G^*_{y,z,z}(t), G^*_{z,x,x}(s)\bigr)\bigr\}.
\end{aligned}$$

Since

$$G^*_{x,z,z}(s) \geq \min\bigl\{G^*_{x,z,z}(s), G^*_{z,x,x}(s)\bigr\}, \qquad G^*_{z,y,y}(t) \geq \min\bigl\{G^*_{z,y,y}(t), G^*_{y,z,z}(t)\bigr\}$$

and

$$G^*_{y,z,z}(t) \geq \min\bigl\{G^*_{z,y,y}(t), G^*_{y,z,z}(t)\bigr\}, \qquad G^*_{z,x,x}(s) \geq \min\bigl\{G^*_{x,z,z}(s), G^*_{z,x,x}(s)\bigr\},$$

it follows from (PGM-4) that

$$\begin{aligned}
G^*_{x,y,y}(s+t) &\geq T\bigl(G^*_{x,z,z}(s), G^*_{z,y,y}(t)\bigr) \\
&\geq T\bigl(\min\bigl\{G^*_{x,z,z}(s), G^*_{z,x,x}(s)\bigr\}, \min\bigl\{G^*_{z,y,y}(t), G^*_{y,z,z}(t)\bigr\}\bigr) \\
&= T\bigl(F_{z,x}(s), F_{z,y}(t)\bigr)
\end{aligned}$$

and

$$\begin{aligned}
G^*_{y,x,x}(s+t) &\geq T\bigl(G^*_{y,z,z}(t), G^*_{z,x,x}(s)\bigr) \\
&\geq T\bigl(\min\bigl\{G^*_{z,y,y}(t), G^*_{y,z,z}(t)\bigr\}, \min\bigl\{G^*_{x,z,z}(s), G^*_{z,x,x}(s)\bigr\}\bigr) \\
&= T\bigl(F_{z,y}(t), F_{z,x}(s)\bigr).
\end{aligned}$$

Therefore, we have

$$F_{x,y}(s+t) = \min\bigl\{G^*_{x,y,y}(s+t), G^*_{y,x,x}(s+t)\bigr\} \geq T\bigl(F_{x,z}(s), F_{z,y}(t)\bigr).$$

This shows that F satisfies (PM-3).

2 Topology, convergence, and completeness

In this section, we first introduce the concept of neighborhoods in the PGM-spaces. For the concept of neighborhoods in PM-spaces, we refer the readers to [1, 3].

Definition 2.1 Let (X, G^*, T) be a PGM-space and x_0 be any point in X. For any $\epsilon > 0$ and δ with $0 < \delta < 1$, an (ϵ, δ)-*neighborhood* of x_0 is the set of all points y in X for which $G^*_{x_0,y,y}(\epsilon) > 1 - \delta$ and $G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta$. We write

$$N_{x_0}(\epsilon, \delta) = \{y \in X : G^*_{x_0,y,y}(\epsilon) > 1 - \delta, G^*_{y,x_0,x_0}(\epsilon) > 1 - \delta\}.$$

This means that $N_{x_0}(\epsilon, \delta)$ is the set of all points y in X for which the probability of the distance from x_0 to y being less than ϵ is greater than $1 - \delta$.

Lemma 2.2 *If* $\epsilon_1 \le \epsilon_2$ *and* $\delta_1 \le \delta_2$, *then* $N_{x_0}(\epsilon_1, \delta_1) \subset N_{x_0}(\epsilon_2, \delta_2)$.

Proof Suppose that $z \in N_{x_0}(\epsilon_1, \delta_1)$, so $G^*_{x_0,z,z}(\epsilon_1) > 1 - \delta_1$ and $G^*_{z,x_0,x_0}(\epsilon_1) > 1 - \delta_1$. Since F is monotone, we have

$$G^*_{x_0,z,z}(\epsilon_2) \ge G^*_{x_0,z,z}(\epsilon_1) \ge 1 - \delta_1 \ge 1 - \delta_2$$

and

$$G^*_{z,x_0,x_0}(\epsilon_2) \ge G^*_{z,x_0,x_0}(\epsilon_1) \ge 1 - \delta_1 \ge 1 - \delta_2.$$

Therefore, by the definition, $z \in N_{x_0}(\epsilon_2, \delta_2)$. This completes the proof. □

Theorem 2.3 *Let* (X, G^*, T) *be a Menger PGM-space. Then* (X, G^*, T) *is a Hausdorff space in the topology induced by the family* $\{N_{x_0}(\epsilon, \delta)\}$ *of* (ϵ, δ)-*neighborhoods.*

Proof We show that the following four properties are satisfied:

(A) For any $x_0 \in X$, there exists at least one neighborhood, N_{x_0}, of x_0 and every neighborhood of x_0 contains x_0.

(B) If $N^1_{x_0}$ and $N^2_{x_0}$ are neighborhoods of x_0, then there exists a neighborhood of x_0, $N^3_{x_0}$, such that $N^3_{x_0} \subset N^1_{x_0} \cap N^2_{x_0}$.

(C) If N_{x_0} is a neighborhood of x_0 and $y \in N_{x_0}$, then there exists a neighborhood of y, N_y, such that $N_y \subset N_{x_0}$.

(D) If $x_0 \ne y$, then there exist disjoint neighborhoods, N_{x_0} and N_y, such that $x_0 \in N_{x_0}$ and $y \in N_y$.

Now, we prove that (A)-(D) hold.

(A) For any $\epsilon > 0$ and $0 < \delta < 1$, $x_0 \in N_{x_0}(\epsilon, \delta)$ since $G^*_{x_0,x_0,x_0}(\epsilon) = 1$ for any $\epsilon > 0$.

(B) For any $\epsilon_1, \epsilon_2 > 0$ and $0 < \delta_1, \delta_2 < 1$, let

$$N^1_{x_0}(\epsilon_1, \delta_1) = \{y \in X : G^*_{x_0,y,y}(\epsilon_1) > 1 - \delta_1, G^*_{y,x_0,x_0}(\epsilon_1) > 1 - \delta_1\}$$

and

$$N^2_{x_0}(\epsilon_2, \delta_2) = \{y \in X : G^*_{x_0,y,y}(\epsilon_2) > 1 - \delta_2, G^*_{y,x_0,x_0}(\epsilon_2) > 1 - \delta_2\}$$

be the neighborhoods of x_0. Consider

$$\begin{aligned} N_{x_0}^3 = \bigl\{ & y \in X : G^*_{x_0,y,y}\bigl(\min\{\epsilon_1,\epsilon_2\}\bigr) > 1 - \min\{\delta_1,\delta_2\}, \\ & \text{and } G^*_{y,x_0,x_0}\bigl(\min\{\epsilon_1,\epsilon_2\}\bigr) > 1 - \min\{\delta_1,\delta_2\}\bigr\}. \end{aligned}$$

Clearly, $x_0 \in N_{x_0}^3$ and, since $\min\{\epsilon_1,\epsilon_2\} \leq \epsilon_1$ and $\min\{\delta_1,\delta_2\} \leq \delta_1$, by Lemma 2.2, $N_{x_0}^3 \subset N_{x_0}^1(\epsilon_1,\delta_1)$ and $N_{x_0}^3 \subset N_{x_0}^2(\epsilon_2,\delta_2)$, so

$$N_{x_0}^3 \subset N_{x_0}^1(\epsilon_1,\delta_1) \cap N_{x_0}^2(\epsilon_2,\delta_2).$$

(C) Let $N_{x_0} = \{z \in X : G^*_{x_0,z,z}(\epsilon_1) > 1-\delta_1, G^*_{z,x_0,x_0}(\epsilon_1) > 1-\delta_1\}$ be the neighborhood of x_0. Since $y \in N_{x_0}$,

$$G^*_{x_0,y,y}(\epsilon_1) > 1-\delta_1, \qquad G^*_{y,x_0,x_0}(\epsilon_1) > 1-\delta_1.$$

Now, $G^*_{x_0,y,y}$ is left-continuous at ϵ_1, so there exist $\epsilon_0 < \epsilon_1$ and $\delta_0 < \delta_1$ such that

$$G^*_{x_0,y,y}(\epsilon_0) > 1-\delta_0 > 1-\delta_1, \qquad G^*_{y,x_0,x_0}(\epsilon_0) > 1-\delta_0 > 1-\delta_1.$$

Let $N_y = \{z \in X : G^*_{y,z,z}(\epsilon_2) > 1-\delta_2, G^*_{z,y,y}(\epsilon_2) > 1-\delta_2\}$, where $0 < \epsilon_2 < \epsilon_1 - \epsilon_0$ and δ_2 is chosen such that

$$T(1-\delta_0, 1-\delta_2) > 1-\delta_1.$$

Such a δ_2 exists since T is continuous, $T(a,1) = a$ for all $a \in [0,1]$ and $1-\delta_0 > 1-\delta_1$.

Now, suppose that $s \in N_y$, so that

$$G^*_{y,s,s}(\epsilon_2) > 1-\delta_2, \qquad G^*_{s,y,y}(\epsilon_2) > 1-\delta_2.$$

Then, since G^* is monotone, it follows from (PGM-4) that

$$\begin{aligned} G^*_{x_0,s,s}(\epsilon_1) &\geq T\bigl(G^*_{x_0,y,y}(\epsilon_0), G^*_{y,s,s}(\epsilon_1-\epsilon_0)\bigr) \geq T\bigl(G^*_{x_0,y,y}(\epsilon_0), G^*_{y,s,s}(\epsilon_2)\bigr) \\ &\geq T(1-\delta_0, 1-\delta_2) > 1-\delta_1. \end{aligned}$$

Similarly, we also have $G^*_{s,x_0,x_0}(\epsilon_1) > 1-\delta_1$. This shows $s \in N_{x_0}$ and hence $N_y \subset N_{x_0}$.

(D) Let $y \neq x_0$. Then there exist $\epsilon > 0$ and a_1, a_2 with $0 \leq a_1, a_2 < 1$ such that $G^*_{x_0,y,y}(\epsilon) = a_1$ and $G^*_{y,x_0,x_0}(\epsilon) = a_2$. Let

$$N_{x_0} = \bigl\{z : G^*_{x_0,z,z}(\epsilon/2) > b_1, G^*_{z,x_0,x_0}(\epsilon/2) > b_1\bigr\}$$

and

$$N_y = \bigl\{z : G^*_{y,z,z}(\epsilon/2) > b_2, G^*_{z,y,y}(\epsilon/2) > b_2\bigr\},$$

where b_1 and b_2 are chosen such that $0 < b_1, b_2 < 1$, $T(b_1,b_2) > a$, where $a = \max\{a_1,a_2\}$. Such b_1 and b_2 exist since T is continuous, monotone, and $T(1,1) = 1$.

Now, suppose that there exists a point $s \in N_{x_0} \cap N_y$ such that

$$G^*_{x_0,s,s}(\epsilon/2) > b_1, \qquad G^*_{s,x_0,x_0}(\epsilon/2) > b_1, \qquad G^*_{y,s,s}(\epsilon/2) > b_2, \qquad G^*_{s,y,y}(\epsilon/2) > b_2.$$

Then, by (PGM-4), we have

$$a_1 = G^*_{x_0,y,y}(\epsilon) \geq T\big(G^*_{x_0,s,s}(\epsilon/2), G^*_{s,y,y}(\epsilon/2)\big) \geq T(b_1, b_2) > a \geq a_1$$

and

$$a_2 = G^*_{y,x_0,x_0}(\epsilon) \geq T\big(G^*_{y,s,s}(\epsilon/2), G^*_{s,x_0,x_0}(\epsilon/2)\big) \geq T(b_2, b_1) > a \geq a_2,$$

which are contradictions. Therefore, N_{x_0} and N_y are disjoint. This completes the proof. □

Next, we give the definition of convergence of sequences in PGM-spaces.

Definition 2.4

(1) A sequence $\{x_n\}$ in a PGM-space (X, G^*, T) is said to be *convergent* to a point $x \in X$ (write $x_n \to x$) if, for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $x_n \in N_x(\epsilon, \delta)$ whenever $n > M_{\epsilon,\delta}$.

(2) A sequence $\{x_n\}$ in a PGM-space (X, G^*, T) is called a *Cauchy sequence* if, for any $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $G^*_{x_n,x_m,x_l}(\epsilon) > 1 - \delta$ whenever $m, n, l > M_{\epsilon,\delta}$.

(3) A PGM-space (X, G^*, T) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

Theorem 2.5 *Let (X, G^*, T) be a PGM-space. Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences in X and $x, y, z \in X$. If $x_n \to x$, $y_n \to y$ and $z_n \to z$ as $n \to \infty$, then, for any $t > 0$, $G^*_{x_n,y_n,z_n}(t) \to G^*_{x,y,z}(t)$ as $n \to \infty$.*

Proof For any $t > 0$, there exists $\delta > 0$ such that $t > 2\delta$. Then, by (PGM-4), we have

$$\begin{aligned} G^*_{x_n,y_n,z_n}(t) &\geq G^*_{x_n,y_n,z_n}(t-\delta) \\ &\geq T\big(G^*_{x_n,x,x}(\delta/3), G^*_{x,y_n,z_n}(t - 4\delta/3)\big) \\ &\geq T\big(G^*_{x_n,x,x}(\delta/3), T\big(G^*_{y_n,y,y}(\delta/3), G^*_{y,x,z_n}(t - 5\delta/3)\big)\big) \\ &\geq T\big(G^*_{x_n,x,x}(\delta/3), T\big(G^*_{y_n,y,y}(\delta/3), T\big(G^*_{z,z,z_n}(\delta/3), G^*_{x,y,z}(t - 2\delta)\big)\big)\big) \end{aligned}$$

and

$$\begin{aligned} G^*_{x,y,z}(t) &\geq G^*_{x,y,z}(t-\delta) \\ &\geq T\big(G^*_{x,x_n,x_n}(\delta/3), G^*_{x_n,y,z}(t - 4\delta/3)\big) \\ &\geq T\big(G^*_{x,x_n,x_n}(\delta/3), T\big(G^*_{y,y_n,y_n}(\delta/3), G^*_{y_n,x_n,z}(t - 5\delta/3)\big)\big) \\ &\geq T\big(G^*_{x,x_n,x_n}(\delta/3), T\big(G^*_{y,y_n,y_n}(\delta/3), T\big(G^*_{z,z_n,z_n}(\delta/3), G^*_{x_n,y_n,z_n}(t - 2\delta)\big)\big)\big). \end{aligned}$$

Letting $n \to \infty$ in the above two inequalities and noting that T is continuous, we have

$$\lim_{n\to\infty} G^*_{x_n,y_n,z_n}(t) \geq G^*_{x,y,z}(t-2\delta)$$

and

$$G^*_{x,y,z}(t) \geq \lim_{n\to\infty} G^*_{x_n,y_n,z_n}(t-2\delta).$$

Letting $\delta \to 0$ in above two inequalities, since G^* is left-continuous, we conclude that

$$\lim_{n\to\infty} G^*_{x_n,y_n,z_n}(t) = G^*_{x,y,z}(t)$$

for any $t > 0$. This completes the proof. □

3 Fixed point theorems

In [31], Sehgal extended the notion of a Banach contraction mapping to the setting of Menger PM-spaces. Later on, Sehgal and Bharucha-Raid [32] proved a fixed point theorem for a mapping under the contractive condition in a complete Menger PM-space. Before proving our fixed point theorems, we first introduce a new concept of contraction in PGM-spaces, which is a corresponding version of Sehgal's contraction in PM-spaces.

Definition 3.1 Let (X, G^*, T) be a PGM-space. A mapping $f : X \to X$ is said to be *contractive* if there exists a constant $\lambda \in (0,1)$ such that

$$G^*_{fx,fy,fz}(t) \geq G^*_{x,y,z}(t/\lambda) \tag{3.1}$$

for all $x, y, z \in X$ and $t > 0$.

The mapping f satisfying the condition (3.1) is called a *λ-contraction*.

Let T be a given t-norm. Then (by associativity) a family of mappings $T^n : [0,1] \to [0,1]$ for each $n \geq 1$ is defined as follows:

$$T^1(t) = T(t,t), \qquad T^2(t) = T\big(t, T^1(t)\big), \qquad \ldots, \qquad T^n(t) = T\big(t, T^{n-1}(t)\big), \qquad \ldots$$

for any $t \in [0,1]$.

Definition 3.2 ([33]) A t-norm T is said to be *of Hadzić-type* if the family of functions $\{T^n(t)\}_{n=1}^{\infty}$ is equicontinuous at $t = 1$, that is, for any $\epsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that

$$t > 1-\delta \quad \Longrightarrow \quad T^n(t) > 1-\epsilon$$

for each $n \geq 1$.

The t-norm $T = \min$ is a trivial example of t-norm of Hadzić-type.

Lemma 3.3 *Let (X, G^*, T) be a Menger PGM-space with T of Hadžić-type and $\{x_n\}$ be a sequence in X. Suppose that there exists $\lambda \in (0,1)$ satisfying*

$$G^*_{x_n,x_{n+1},x_{n+1}}(t) \geq G^*_{x_{n-1},x_n,x_n}(t/\lambda)$$

for any $n \geq 1$ and $t > 0$. Then $\{x_n\}$ is a Cauchy sequence in X.

Proof Since $G^*_{x_n,x_{n+1},x_{n+1}}(t) \geq G^*_{x_{n-1},x_n,x_n}(t/\lambda)$, by induction, we have

$$G^*_{x_n,x_{n+1},x_{n+1}}(t) \geq G^*_{x_0,x_1,x_1}\left(t/\lambda^n\right).$$

Since X is a Menger PGM-space, we have $G^*_{x_0,x_1,x_1}(t/\lambda^n) \to 1$ as $n \to \infty$, so

$$\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(t) = 1 \tag{3.2}$$

for any $t > 0$.

Now, let $n \geq 1$ and $t > 0$. We show, by induction, that, for any $k \geq 0$,

$$G^*_{x_n,x_{n+k},x_{n+k}}(t) \geq T^k\left(G^*_{x_n,x_{n+1},x_{n+1}}(t - \lambda t)\right). \tag{3.3}$$

For $k = 0$, since $T(a,b)$ is a real number, $T^0(a,b) = 1$ for all $a, b \in [0,1]$. Hence, $G^*_{x_n,x_n,x_n}(t) = 1 = T^0(G^*_{x_n,x_{n+1},x_{n+1}})(t - \lambda t)$, which implies that (3.3) holds for $k = 0$. Assume that (3.3) holds for some $k \geq 1$. Then, since T is monotone, it follows from (PGM-4) that

$$\begin{aligned}
G^*_{x_n,x_{n+k+1},x_{n+k+1}}(t) &= G^*_{x_n,x_{n+k+1},x_{n+k+1}}(t - \lambda t + \lambda t) \\
&\geq T\left(G^*_{x_n,x_{n+1},x_{n+1}}(t - \lambda t), G^*_{x_{n+1},x_{n+k+1},x_{n+k+1}}(\lambda t)\right) \\
&\geq T\left(G^*_{x_n,x_{n+1},x_{n+1}}(t - \lambda t), G^*_{x_n,x_{n+k},x_{n+k}}(t)\right) \\
&\geq T\left(G^*_{x_n,x_{n+1},x_{n+1}}(t - \lambda t), T^k\left(G^*_{x_n,x_{n+1},x_{n+1}}(t - \lambda t)\right)\right) \\
&= T^{k+1}\left(G^*_{x_n,x_{n+1},x_{n+1}}(t - \lambda t)\right),
\end{aligned}$$

so we have the conclusion.

Now, we show that $\{x_n\}$ is a Cauchy sequence in X, *i.e.*, $\lim_{m,n,l\to\infty} G^*_{x_n,x_m,x_l}(t) = 1$ for any $t > 0$. To this end, we first prove that $\lim_{n,m\to\infty} G^*_{x_n,x_m,x_m}(t) = 1$ for any $t > 0$. Let $t > 0$ and $\epsilon > 0$ be given. By hypothesis, $T^n : n \geq 1$ is equicontinuous at 1 and $T^n(1) = 1$, so there exists $\delta > 0$ such that, for any $a \in (1 - \delta, 1]$,

$$T^n(a) > 1 - \epsilon \tag{3.4}$$

for all $n \geq 1$. From (3.2), it follows that $\lim_{n\to\infty} G^*_{x_n,x_{n+1},x_{n+1}}(t - \lambda t) = 1$. Hence there exists $n_0 \in \mathbb{N}$ such that $G^*_{x_n,x_{n+1},x_{n+1}}(t - \lambda t) \in (1 - \delta, 1]$ for any $n \geq n_0$. Hence, by (3.3) and (3.4), we conclude that $G^*_{x_n,x_{n+k},x_{n+k}}(t) > 1 - \epsilon$ for any $k \geq 0$. This shows $\lim_{n,m\to\infty} G^*_{x_n,x_m,x_m}(t) = 1$ for any $t > 0$. By (GPM-4), we have

$$G^*_{x_n,x_m,x_l}(t) \geq T\big(G^*_{x_n,x_n,x_m}(t/2), G^*_{x_n,x_n,x_l}(t/2)\big),$$
$$G^*_{x_n,x_n,x_m}(t/2) \geq T\big(G^*_{x_n,x_m,x_m}(t/4), G^*_{x_n,x_m,x_m}(t/4)\big),$$
$$G^*_{x_n,x_n,x_l}(t/2) \geq T\big(G^*_{x_n,x_l,x_l}(t/4), G^*_{x_n,x_l,x_l}(t/4)\big).$$

Therefore, by the continuity of T, we conclude that

$$\lim_{m,n,l\to\infty} G^*_{x_n,x_m,x_l}(t) = 1$$

for any $t > 0$. This shows that the sequence $\{x_n\}$ is a Cauchy sequence in X. This completes the proof. □

From Example 1.6 and Lemma 3.3 we get the following corollary.

Corollary 3.4 ([33]) *Let (X, F, T) be a PM-space with T of Hadžić-type and $\{x_n\} \subset X$ be a sequence. If there exists a constant $\lambda \in (0,1)$ such that*

$$F_{x_n,x_{n+1}}(t) \geq F_{x_{n-1},x_n}(t/\lambda), \quad n \geq 1, t > 0,$$

then $\{x_n\}$ is a Cauchy sequence.

Proof Define $G^*_{x,y,z}(t) = \min\{F_{x,y}(t), F_{y,z}(t), F_{x,z}(t)\}$ for all $x, y, z \in X$ and all $t > 0$. Example 1.6 shows that (X, G^*, T) is a PGM-space. Since $G^*_{x_n,x_{n+1},x_{n+1}}(t) = F_{x_n,x_{n+1}}(t)$ and $G^*_{x_{n-1},x_n,x_n}(t/\lambda) = F_{x_{n-1},x_n}(t/\lambda)$, $F_{x_n,x_{n+1}}(t) \geq F_{x_{n-1},x_n}(t/\lambda)$ implies $G^*_{x_n,x_{n+1},x_{n+1}}(t) \geq G^*_{x_{n-1},x_n,x_n}(t/\lambda)$ for all $n \geq 1$ and $t > 0$. By Lemma 3.3 we conclude that $\{x_n\}$ is a Cauchy sequence in the sense of PGM-space (X, G^*, T). That is, for every $\epsilon > 0$ and $0 < \delta < 1$, there exists a positive integer $M_{\epsilon,\delta}$ such that $G^*_{x_n,x_m,x_l}(\epsilon) > 1 - \delta$ for all $m, n, l > M_{\epsilon,\delta}$. By the definition of G^*, we have

$$\min\big\{F_{x_n,x_m}(\epsilon), F_{x_m,x_l}(\epsilon), F_{x_n,x_l}(\epsilon)\big\} > 1 - \delta, \quad m, l, n > M_{\epsilon,\delta}.$$

This shows that $\{x_n\}$ is a Cauchy sequence in the sense of PM-space (X, F, T). □

Theorem 3.5 *Let (X, G^*, T) be a complete Menger PGM-space with T of Hadžić-type. Let $\lambda \in (0,1)$ and $f : X \to X$ be a λ-contraction. Then, for any $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to a unique fixed point of T.*

Proof Take an arbitrary point x_0 in X. Construct a sequence $\{x_n\}$ by $x_{n+1} = f^n x_0$ for all $n \geq 0$. By (3.1), for any $t > 0$, we have

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(t) &= G^*_{fx_{n-1},fx_n,fx_n}(t) \\ &\geq G^*_{x_{n-1},x_n,x_n}(t/\lambda). \end{aligned}$$

Lemma 3.3 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x \in X$ such that $x_n \to x$ as $n \to \infty$. By (3.1), it follows that

$$G^*_{fx,fx_n,fx_n}(t) \geq G^*_{x,x_n,x_n}(t/\lambda).$$

Letting $n \to \infty$, since $x_n \to x$ and $fx_n \to x$ as $n \to \infty$, we have

$$G^*_{fx,x,x}(t) = 1$$

for any $t > 0$. Hence $x = fx$.

Next, suppose that y is another fixed point of f. Then, by (3.1), we have

$$G^*_{x,y,y}(t) = G^*_{fx,fy,fy}(t) \geq G^*_{x,y,y}(t/\lambda) \geq \cdots \geq G^*_{x,y,y}\big(t/\lambda^n\big).$$

Letting $n \to \infty$, since X is a Menger PGM-space, $G^*_{x,y,y}(t/\lambda^n) \to 1$ as $n \to \infty$, so

$$G^*_{x,y,y}(t) = 1$$

for any $t > 0$, which implies that $x = y$. Therefore, f has a unique fixed point in X. This completes the proof. □

Theorem 3.6 *Let (X, G^*, T) be a complete Menger PGM-space with T of Hadžić-type. Let $f : X \to X$ be a mapping satisfying*

$$G^*_{fx,fy,fz}(\lambda t) \geq \frac{1}{3}\big[G^*_{x,fx,fx}(t) + G^*_{y,fy,fy}(t) + G^*_{z,fz,fz}(t)\big] \tag{3.5}$$

for all $x, y, z \in X$, where $\lambda \in (0,1)$. Then, for any $x_0 \in X$, the sequence $\{f^n x_0\}$ converges to a unique fixed point of f.

Proof Take an arbitrary point x_0 in X. Construct a sequence $\{x_n\}$ by $x_{n+1} = f^n x_0$ for all $n \geq 0$. By (3.5), for any $t > 0$, we have

$$\begin{aligned} G^*_{x_n,x_{n+1},x_{n+1}}(\lambda t) &= G^*_{fx_{n-1},fx_n,fx_n}(\lambda t) \\ &\geq \frac{1}{3}\big[G^*_{x_{n-1},fx_{n-1},fx_{n-1}}(t) + 2G^*_{x_n,fx_n,fx_n}(t)\big] \\ &\geq \frac{1}{3}\big[G^*_{x_{n-1},fx_{n-1},fx_{n-1}}(t) + 2G^*_{x_n,fx_n,fx_n}(\lambda t)\big] \\ &= \frac{1}{3}\big[G^*_{x_{n-1},x_n,x_n}(t) + 2G^*_{x_n,x_{n+1},x_{n+1}}(\lambda t)\big]. \end{aligned}$$

This shows that

$$G^*_{x_n,x_{n+1},x_{n+1}}(t) \geq G^*_{x_{n-1},x_n,x_n}(t/\lambda).$$

Lemma 3.3 shows that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a point $x \in X$ such that $x_n \to x$ as $n \to \infty$. By (3.5), it follows that

$$G^*_{fx,fx,fx_n}(t) \geq \frac{1}{3}\big[2G^*_{x,fx,fx}(t/\lambda) + G^*_{x_n,fx_n,fx_n}(t/\lambda)\big].$$

Letting $n \to \infty$, since $x_n \to x$ and $fx_n \to x$ as $n \to \infty$, we have, for any $t > 0$,

$$G^*_{fx,fx,x}(t) \geq \frac{1}{3}\big[2G^*_{x,fx,fx}(t/\lambda) + G^*_{x,x,x}(t/\lambda)\big] \geq \frac{1}{3}\big[2G^*_{x,fx,fx}(t) + G^*_{x,x,x}(t/\lambda)\big]$$

i.e.,

$$G^*_{fx,fx,x}(t) \geq G^*_{x,x,x}(t/\lambda) = 1.$$

Hence $x = Tx$.

Next, suppose that y is another fixed point of f. Then, by (3.5), we have, for any $y > 0$,

$$\begin{aligned} G^*_{x,y,y}(t) &= G^*_{fx,fy,fy}(t) \\ &\geq \frac{1}{3}\left[G^*_{x,fx,fx}(t/\lambda) + 2G^*_{y,fy,fy}(t/\lambda)\right] \\ &= 1. \end{aligned}$$

This shows that $x = y$. Therefore, f has a unique fixed point in X. This completes the proof. □

Finally, we give the following example to illustrate Theorem 3.5 and Theorem 3.6.

Example 3.7 Set $X = [0,\infty)$ and $T(a,b) = \min\{a,b\}$ for all $a,b \in [0,1]$. Define a function $G^* : X^3 \times [0,\infty) \to [0,\infty)$ by

$$G^*_{x,y,z}(t) = \frac{t}{t + G(x,y,z)}$$

for all $x,y,z \in X$, where $G(x,y,z) = |x-y| + |y-z| + |z-x|$. Then G is a G-metric (see [24]). It is easy to check that G^* satisfies (PGM-1)-(PGM-3). Since $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x,y,z,a \in X$, we have

$$\begin{aligned} \frac{t+s}{s+t+G(x,y,z)} &\geq \frac{t+s}{s+t+G(x,a,a)+G(a,y,z)} \\ &\geq \min\left\{\frac{s}{s+G(x,a,a)}, \frac{t}{t+G(a,y,z)}\right\}. \end{aligned}$$

This shows that G^* satisfies (PGM-4). Hence $(X, G^*, \min)$ is a PGM-space.

(1) Let $\lambda \in (0,1)$. Define a mapping $f : X \to X$ by $fx = \lambda x$ for all $x \in X$. For any $t > 0$, we have

$$G^*_{fx,fy,fz}(t) = \frac{t}{t + \lambda(|x-y| + |y-z| + |z-x|)}$$

and

$$G^*_{x,y,z}(t/\lambda) = \frac{t/\lambda}{t/\lambda + (|x-y| + |y-z| + |z-x|)}.$$

Therefore, we conclude that f is a λ-contraction and f has a fixed point in X by Theorem 3.5. In fact, the fixed point is $x = 0$.

(2) Let $\lambda \in (0,1)$. Define a mapping $f : X \to X$ by $fx = 1$ for all $x \in X$. For any $t > 0$ and all $x,y,z \in X$, since

$$G^*_{fx,fy,fz}(t) = G^*_{1,1,1}(t) = 1$$

and

$$\frac{1}{3}\left[G^*_{x,fx,fx}(\lambda t) + G^*_{y,fy,fy}(\lambda t) + G^*_{z,fz,fz}(\lambda t)\right] \leq 1,$$

we conclude that

$$G^*_{fx,fy,fz}(t) \geq \frac{1}{3}\left[G^*_{x,fx,fx}(\lambda t) + G^*_{y,fy,fy}(\lambda t) + G^*_{z,fz,fz}(\lambda t)\right]$$

and hence f has a fixed point in X by Theorem 3.6. In fact, the fixed point is $x = 1$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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