Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Approximating a common point of fixed points of a pseudocontractive mapping and zeros of sum of monotone mappings

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Abstract

Let C be a closed and convex subset of a real Hilbert space H. Let T be a Lipschitzian pseudocontractive mapping of C into itself, A be a γ-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. We introduce an iteration scheme for finding a minimum-norm point of $F(T) \cap (A+B)^{-1}(0)$. Application to a common element of the set of fixed points of a Lipschitzian pseudocontractive and solutions of variational inequality for α-inverse strongly monotone mappings is included. Our theorems improve and unify most of the results that have been proved in this direction for this important class of nonlinear mappings. To the best of our knowledge, approximating a common fixed point of pseudocontractive mappings with explicit scheme has not been possible and our result is even the first result that states the solution of a variational inequality in the set of fixed points of pseudocontractive mappings. Our scheme which is explicit is the best to use for the problem under consideration.

MSC: 47H05; 47H09; 47J25

Keywords: fixed points; monotone mappings; pseudocontractive mappings; strong convergence

1 Introduction

Let C be a closed convex subset of a real Hilbert space H. A mapping $T: C \to H$ is called *a contraction mapping* if there exists $L \in [0,1)$ such that $\|Tx - Ty\| \leq L\|x-y\|$ for all $x, y \in C$. If $L = 1$ then T is called *nonexpansive*. T is called *quasi-nonexpansive* if $\|Tx - Tp\| \leq \|x - p\|$ for all $x \in C$ and $p \in F(T)$, where $F(T) := \{x \in C : Tx = x\}$, the set of fixed points of T. A mapping T is called *γ-strictly pseudocontractive* [1] if and only if there exists $\gamma \in [0,1)$ such that

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \gamma \left\|(I-T)x - (I-T)y\right\|^2, \quad \text{for all } x, y \in C, \tag{1.1}$$

and T is called *pseudocontractive* if

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 + \left\|(I-T)x - (I-T)y\right\|^2, \quad \text{for all } x, y \in C, \tag{1.2}$$

where I is the identity mapping. We note that inequalities (1.1) and (1.2) can be equivalently written as

$$\langle x-y, Tx-Ty\rangle \leq \|x-y\|^2 - \lambda\big\|(I-T)x-(I-T)y\big\|^2, \tag{1.3}$$

for some $\lambda > 0$, and

$$\langle x-y, Tx-Ty\rangle \leq \|x-y\|^2, \quad \text{for all } x, y \in C, \tag{1.4}$$

respectively.

Clearly, the class of nonexpansive mappings is a subset of the class of γ-strictly pseudocontractive mappings and the class of γ-strictly pseudocontractive is contained in the class of pseudocontractive mappings. Moreover, this inclusion is strict due to the following example in [2].

Take $X = \mathbb{R}^2$, $B = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$, $B_1 = \{x \in B : \|x\| \leq \frac{1}{2}\}$, $B_2 = \{x \in B : \frac{1}{2} \leq \|x\| \leq 1\}$. If $x = (a,b) \in X$ we define $x\perp$ to be $(b,-a) \in X$. Define $T : B \to B$ by

$$Tx = \begin{cases} x + x\perp, & \text{if } x \in B_1, \\ \frac{x}{\|x\|} - x + x\perp, & \text{if } x \in B_2. \end{cases} \tag{1.5}$$

Then T is a Lipschitzian and pseudocontractive mapping but not a strictly pseudocontractive mapping.

Closely related to the class of pseudocontractive mappings is the class of monotone mappings. A mapping $A : C \to H$ is called *monotone* if

$$\langle x-y, Ax-Ay\rangle \geq 0, \quad \text{for all } x, y \in C, \tag{1.6}$$

and A is called *γ-inverse strongly monotone* if there exists a positive real number γ such that

$$\langle x-y, Ax-Ay\rangle \geq \gamma\|Ax-Ay\|^2, \quad \text{for all } x, y \in C. \tag{1.7}$$

If A is γ-inverse strongly monotone, then inequality (1.7) implies that A is Lipschitzian with constant $L := \frac{1}{\gamma}$, that is, $\|Ax-Ay\| \leq \frac{1}{\gamma}\|x-y\|$, for all $x, y \in C$.

We remark the T is γ-strictly pseudocontractive if and only if $A := (I-T)$ is γ-inverse strongly monotone and T is pseudocontractive if and only if $A := (I-T)$ is monotone. Clearly, the class of monotone mappings includes the class of γ-inverse strongly monotone mappings. We note that the inclusion is proper. This can be seen from the example in [2]. Take $A := (I-T)$, where T is as in (1.5). Then we see that A is monotone but not γ-inverse strongly monotone as T is not strictly pseudocontractive.

A mapping A is called *maximal monotone* if it is monotone and $\mathcal{R}(I+rA)$, the range of $(I+rA)$, is H for all $r > 0$. If A is maximal monotone, then to each $r > 0$ and $x \in H$, there corresponds a unique element $x_r \in D(A)$ satisfying

$$x \in x_r + rAx_r.$$

We denote the *resolvent* of A by $J_r x = x_r$. That is, $J_r = (I+rA)^{-1}$ for all $r > 0$. If A is monotone then $J_r := (I+rA)^{-1}$ is nonexpansive single valued mapping from $\mathcal{R}(I+rA)$ into $D(A)$ and $F(J_r) = N(A)$ (see [3]).

It is now well known (see *e.g.* [4]) that if A is monotone then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems. Consequently, considerable research efforts, especially within the past 20 years or so, have been devoted to iterative methods for approximating the zeros of monotone mapping A or fixed point of pseudocontractive mapping T (see, for example, [5–11]).

Let A be a nonlinear mapping on H. Consider the problem of finding

$$u \in C \text{ such that } 0 \in Au. \tag{1.8}$$

When A is a maximal monotone mapping, a well-known methods for solving (1.8) is the *proximal point algorithm*: $x_1 = x \in H$, and

$$x_{n+1} = J_{r_n} x_n, \quad n = 1, 2, 3, \dots,$$

where $J_{r_n} = (I + r_n A)^{-1}$ and $\{r_n\} \subset (0, \infty)$, then Rockafellar [12] (also see [13]) proved that the sequence $\{x_n\}$ *converges weakly* to an element of $A^{-1}(0)$.

In [14], Kamimura and Takahashi investigated the problem of finding a zero point of a maximal monotone mapping by considering the following iterative algorithm:

$$x_0 \in H, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) J_{\lambda_n} x_n, \quad n = 0, 1, \dots, \tag{1.9}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\lambda_n\}$ is a positive sequence, $A : H \to H$ is a maximal monotone, and $J_{\lambda_n} = (I + \lambda_n A)^{-1}$. They showed that the sequence $\{x_n\}$ generated by (1.9) converges *weakly* to some $z \in A^{-1}(0)$ in the framework of real Hilbert spaces, provided that the control sequences satisfy some restrictions.

Let C be a nonempty, closed and convex subset of H and $A : C \to H$ be a nonlinear mapping. The *variational inequality problem* which was introduced and studied by Stampacchia [15] is to:

$$\text{find } u \in C \text{ such that } \langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.10}$$

The set of solutions of the variational inequality problem is denoted by $\mathrm{VI}(C, A)$.

Variational inequality theory has emerged as an important tool in studying a wide class of numerous problems in physics, optimization, variational inequalities, minimax problems, and the Nash equilibrium problems in noncooperative games (see, for instance, [16–22]).

In [23], Takahashi and Toyoda investigated the problem of finding a common point of solutions of the variational inequality problem (1.10) for $A : C \to H$ a γ-inverse strongly monotone mapping and fixed points of a nonexpansive mapping $T : C \to C$ by considering the following iterative algorithm:

$$x_0 \in H, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T P_C (x_n - \lambda_n A x_n), \quad n = 0, 1, \dots, \tag{1.11}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, $\{\lambda_n\}$ is a positive sequence. They proved that the sequence $\{x_n\}$ generated by (1.11) converges *weakly* to some $z \in \mathrm{VI}(C, A) \cap F(T)$ provided that the control sequences satisfy some restrictions.

It is worth to mention that the methods studied above give weak convergence theorems in the framework of Hilbert spaces.

Regarding iterative method for a common point of fixed points of nonexpansive and zeros of sum of two monotone mappings, Takahashi *et al.* [24] proved the following theorem.

Theorem TT [24] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be a γ-inverse strongly monotone mapping of C into H and let B be a maximal monotone mapping on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B, for $\lambda > 0$, and let T be a nonexpansive mapping of C into itself such that $F(T) \cap (A + B)^{-1} \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T\big(\alpha_n x + (1 - \alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n)\big), \quad n = 1, 2, \ldots,$$

where $\{\lambda_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ satisfy certain conditions. Then $\{x_n\}$ converges strongly to a point of $F(T) \cap (A + B)^{-1}(0)$.

For other related results, we refer to [25–30].

A natural question arises: *can we obtain an iterative scheme which converges strongly to a common point of fixed points of the pseudocontractive mapping T and zeros of two monotone mappings*?

It is our purpose in this paper to introduce an iterative scheme which converges strongly to a common minimum-norm point of fixed points of a Lipschitzian pseudocontractive mapping and zeros of sum of two monotone mappings. Application to a common element of the set of fixed points of a Lipschitzian pseudocontractive mapping and solutions of variational inequality for γ-inverse strongly monotone mapping is included. The results obtained in this paper improve and extend the results of Kamimura and Takahashi [14], Takahashi and Toyoda [23], Takahashi *et al.* [24] and some other results in this direction.

2 Preliminaries

In what follows we shall make use of the following lemmas.

Lemma 2.1 [31] *Let C be a convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x - x_0 \rangle \leq 0, \quad \forall z \in C.$$

We also remark that in a real Hilbert space H, the following identity holds:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H. \tag{2.1}$$

Lemma 2.2 [32] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n \delta_n, \quad n \geq n_0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset R$ satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, *and* $\limsup_{n\to\infty} \delta_n \leq 0$. *Then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 [33] *Let H be a real Hilbert space, C a closed convex subset of H and $T : C \to C$ be a continuous pseudocontractive mapping, then*

(i) *$F(T)$ is closed convex subset of C;*

(ii) *$(I - T)$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, as $n \to \infty$, then $x = T(x)$.*

Lemma 2.4 [34] *Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and} \quad a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{j \leq k : a_j < a_{j+1}\}$.

Lemma 2.5 [35] *Let H be a real Hilbert space. Then for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 1, 2, \ldots, n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ the following equality holds:*

$$\|\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n\|^2 = \sum_{i=0}^{n} \alpha_i \|x_i\|^2 - \sum_{0 \leq i,j \leq n} \alpha_i \alpha_j \|x_i - x_j\|^2.$$

Lemma 2.6 [36] *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let $A : C \to E$ be γ-inverse strongly monotone mapping. Then, for $0 < \mu < 2\gamma$, the mapping $A_\mu x := (x - \mu Ax)$ is nonexpansive.*

Lemma 2.7 [37] *Let H be a Hilbert space. Let $A : D(A) \subseteq H \to 2^H$ and $B : D(B) \subseteq H \to 2^H$ be maximal monotone mappings. Suppose that $D(A) \cap \operatorname{int} D(B) \neq \emptyset$. Then $A + B$ is a maximal monotone mapping.*

3 Main result

Theorem 3.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant L. Let $A : C \to H$ be a γ-inverse strongly monotone mapping and B be a maximal monotone mapping on H such that the domain of B is subset of C. Assume that $\mathcal{F} = F(T) \cap (A+B)^{-1}(0)$ is nonempty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n Tx_n; \\ x_{n+1} = P_C[(1 - \alpha_n)(\theta_n x_n + \delta_n Ty_n + \gamma_n T_{\lambda_n} x_n)], \end{cases} \tag{3.1}$$

where $T_{\lambda_n}(x_n) := (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n$ and $\{\lambda_n\} \subset (a,b) \subset (a, 2\gamma)$, $\{\theta_n\}, \{\delta_n\}, \{\gamma_n\} \subset (c,d) \subset (0,1)$, $\{\alpha_n\} \subset (0,e) \subset (0,1)$, for some $a,b,c,d,e > 0$, satisfying the following conditions: (i) $\theta_n + \delta_n + \gamma_n = 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $\delta_n + \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$, $\forall n \geq 1$. Then $\{x_n\}$ converges strongly to the minimum-norm point x^* of $\mathcal{F}$.

Proof From Lemma 2.6 and the fact that J_{λ_n} is nonexpansive we see that T_{λ_n} is nonexpansive. Let $p \in \mathcal{F}$. Then from (3.1), (1.2), Lemma 2.5 and using the fact that $p = T_{\lambda_n}(p)$ we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \big\|P_C\big[(1-\alpha_n)(\theta_n x_n+\delta_n Ty_n+\gamma_n T_{\lambda_n}x_n)\big]-p\big\|^2\\
&\le \big\|(1-\alpha_n)(\theta_n x_n+\delta_n Ty_n+\gamma_n T_{\lambda_n}x_n)-p\big\|^2\\
&\le \alpha_n\|p\|^2+(1-\alpha_n)\big\|\theta_n(x_n-p)+\delta_n(Ty_n-p)+\gamma_n(T_{\lambda_n}x_n-p)\big\|^2\\
&\le \alpha_n\|p\|^2+(1-\alpha_n)\big[\theta_n\|x_n-p\|^2+\delta_n\|Ty_n-p\|^2\\
&\quad +\gamma_n\|T_{\lambda_n}x_n-p\|^2\big]-(1-\alpha_n)\delta_n\theta_n\|Ty_n-x_n\|^2\\
&\quad -(1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n-x_n\|^2\\
&\le \alpha_n\|p\|^2+(1-\alpha_n)(\theta_n+\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\delta_n\|Ty_n-p\|^2\\
&\quad -(1-\alpha_n)\delta_n\theta_n\|Ty_n-x_n\|^2-(1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n-x_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \alpha_n\|p\|^2+(1-\alpha_n)(\theta_n+\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\delta_n\big[\|y_n-p\|^2\\
&\quad +\|y_n-Ty_n\|^2\big]-(1-\alpha_n)\delta_n\theta_n\|Ty_n-x_n\|^2\\
&\quad -(1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n-x_n\|^2\\
&= \alpha_n\|p\|^2+(1-\alpha_n)(\theta_n+\gamma_n)\|x_n-p\|^2+(1-\alpha_n)\delta_n\|y_n-p\|^2\\
&\quad +(1-\alpha_n)\delta_n\|y_n-Ty_n\|^2-(1-\alpha_n)\delta_n\theta_n\|Ty_n-x_n\|^2\\
&\quad -(1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n-x_n\|^2.
\end{aligned}\tag{3.2}$$

In addition, from (3.1), Lemma 2.5, and (1.2) we get

$$\begin{aligned}
\|y_n-p\|^2 &= \big\|(1-\beta_n)(x_n-p)+\beta_n(Tx_n-p)\big\|^2\\
&= (1-\beta_n)\|x_n-p\|^2+\beta_n\|Tx_n-p\|^2\\
&\quad -\beta_n(1-\beta_n)\|x_n-Tx_n\|^2\\
&\le (1-\beta_n)\|x_n-p\|^2+\beta_n\big[\|x_n-p\|^2+\|x_n-Tx_n\|^2\big]\\
&\quad -\beta_n(1-\beta_n)\|x_n-Tx_n\|^2\\
&= \|x_n-p\|^2+\beta_n^2\|x_n-Tx_n\|^2
\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}
\|y_n-Ty_n\|^2 &= \big\|(1-\beta_n)(x_n-Ty_n)+\beta_n(Tx_n-Ty_n)\big\|^2\\
&= (1-\beta_n)\|x_n-Ty_n\|^2+\beta_n\|Tx_n-Ty_n\|^2\\
&\quad -\beta_n(1-\beta_n)\|x_n-Tx_n\|^2\\
&\le (1-\beta_n)\|x_n-Ty_n\|^2+\beta_n L^2\|x_n-y_n\|^2\\
&\quad -\beta_n(1-\beta_n)\|x_n-Tx_n\|^2\\
&= (1-\beta_n)\|x_n-Ty_n\|^2+\beta_n^3L^2\|x_n-Tx_n\|^2
\end{aligned}$$

$$\begin{aligned} &- \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 \\ &= (1-\beta_n)\|x_n - Ty_n\|^2 - \beta_n\bigl(1 - L^2\beta_n^2 - \beta_n\bigr)\|x_n - Tx_n\|^2. \end{aligned} \tag{3.4}$$

Substituting (3.3) and (3.4) into (3.2) we obtain

$$\begin{aligned} \|x_{n+1} - p\|^2 \le{}& \alpha_n\|p\|^2 + (1-\alpha_n)(\theta_n + \gamma_n)\|x_n - p\|^2 + (1-\alpha_n)\delta_n\bigl[\|x_n - p\|^2 \\ &+ \beta_n^2\|x_n - Tx_n\|^2\bigr] + (1-\alpha_n)\delta_n\bigl[(1-\beta_n)\|x_n - Ty_n\|^2 \\ &- \beta_n\bigl(1 - L^2\beta_n^2 - \beta_n\bigr)\|x_n - Tx_n\|^2\bigr] - (1-\alpha_n)\delta_n\theta_n\|Ty_n - x_n\|^2 \\ &- (1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n - x_n\|^2 \\ ={}& \alpha_n\|p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\delta_n\beta_n\bigl(1 - \bigl(L^2\beta_n^2 + 2\beta_n\bigr)\bigr) \\ &\times\|x_n - Tx_n\|^2 + (1-\alpha_n)\delta_n(1 - \theta_n - \beta_n)\|Ty_n - x_n\|^2 \\ &- (1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n - x_n\|^2, \end{aligned}$$

and hence

$$\begin{aligned} \|x_{n+1} - p\|^2 \le{}& \alpha_n\|p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\delta_n\beta_n \\ &\times\bigl(1 - \bigl(L^2\beta_n^2 + 2\beta_n\bigr)\bigr)\|x_n - Tx_n\|^2 \\ &+ (1-\alpha_n)\delta_n(\delta_n + \gamma_n - \beta_n)\|Ty_n - x_n\|^2 \\ &- (1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n - x_n\|^2. \end{aligned} \tag{3.5}$$

Now, from (iii) of the hypotheses we have

$$1 - 2\beta_n - L^2\beta_n^2 \ge 1 - 2\beta - L^2\beta^2 > 0 \tag{3.6}$$

and

$$(\delta_n + \gamma_n) - \beta_n \le 0, \quad \text{for all } n \ge 1. \tag{3.7}$$

Thus, inequality (3.5) implies that

$$\|x_{n+1} - p\|^2 \le \alpha_n\|p\|^2 + (1-\alpha_n)\|x_n - p\|^2. \tag{3.8}$$

Thus, by induction,

$$\|x_{n+1} - p\|^2 \le \max\bigl\{\|p\|^2, \|x_0 - p\|^2\bigr\}, \quad \forall n \ge 0,$$

which implies that $\{x_n\}$ and hence $\{y_n\}$ are bounded.

Let $w_n := (1-\alpha_n)(\theta_n x_n + \delta_n Ty_n + \gamma_n T_{\lambda_n}x_n)$. Then we see that $x_{n+1} = P_C w_n$. Let $x^* = P_{\mathcal{F}}(0)$. Then, using (3.1), (2.1) and following the methods used to get (3.5), we obtain

$$\begin{aligned} \bigl\|x_{n+1} - x^*\bigr\|^2 &= \bigl\|P_C\bigl[(1-\alpha_n)(\theta_n x_n + \delta_n Ty_n + \gamma_n T_{\lambda_n}x_n)\bigr] - x^*\bigr\|^2 \\ &\le \bigl\|\alpha_n\bigl(-x^*\bigr) + (1-\alpha_n)\bigl[\theta_n x_n + \delta_n Ty_n + \gamma_n T_{\lambda_n}x_n - x^*\bigr]\bigr\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n)\left\|\delta_n Ty_n + \theta_n x_n + \gamma_n T_{\lambda_n} x_n - x^*\right\|^2 \\
&\quad + 2\alpha_n\left\langle -x^*, w_n - x^*\right\rangle \\
&\leq (1-\alpha_n)\delta_n\left\|Ty_n - x^*\right\|^2 + (1-\alpha_n)\theta_n\left\|x_n - x^*\right\|^2 \\
&\quad + (1-\alpha_n)\gamma_n\left\|T_{\lambda_n}x_n - x^*\right\|^2 - (1-\alpha_n)\theta_n\delta_n\|Ty_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n - x_n\|^2 + 2\alpha_n\left\langle -x^*, w_n - x^*\right\rangle
\end{aligned}$$

and so

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &\leq (1-\alpha_n)\delta_n\left[\left\|y_n - x^*\right\|^2 + \|y_n - Ty_n\|^2\right] \\
&\quad + (1-\alpha_n)(\theta_n + \gamma_n)\left\|x_n - x^*\right\|^2 \\
&\quad - (1-\alpha_n)\theta_n\delta_n\|Ty_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n - x_n\|^2 + 2\alpha_n\left\langle -x^*, w_n - x^*\right\rangle \\
&\leq (1-\alpha_n)\delta_n\left[\left\|x_n - x^*\right\|^2 + \beta_n^2\|x_n - Tx_n\|^2\right] \\
&\quad + (1-\alpha_n)\delta_n\left[(1-\beta_n)\|x_n - Ty_n\|^2 - \beta_n\left(1 - L^2\beta_n^2 - \beta_n\right)\right. \\
&\quad \left.\times \|x_n - Tx_n\|^2\right] + (1-\alpha_n)(\theta_n + \gamma_n)\left\|x_n - x^*\right\|^2 \\
&\quad - (1-\alpha_n)\theta_n\delta_n\|Ty_n - x_n\|^2 - (1-\alpha_n)\theta_n\gamma_n\|T_{\lambda_n}x_n - x_n\|^2 \\
&\quad + 2\alpha_n\left\langle -x^*, w_n - x^*\right\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 &\leq (1-\alpha_n)\left\|x_n - x^*\right\|^2 - (1-\alpha_n)\delta_n\beta_n\left[1 - L^2\beta_n^2 - 2\beta_n\right] \\
&\quad \times \|x_n - Tx_n\|^2 + (1-\alpha_n)\delta_n(\delta_n + \gamma_n - \beta_n)\|x_n - Ty_n\|^2 \\
&\quad - (1-\alpha_n)\gamma_n\|x_n - T_{\lambda_n}x_n\|^2 + 2\alpha_n\left\langle -x^*, w_n - x^*\right\rangle &(3.9)\\
&\leq (1-\alpha_n)\left\|x_n - x^*\right\|^2 + 2\alpha_n\left\langle -x^*, w_n - x^*\right\rangle. &(3.10)
\end{aligned}$$

Now, we consider two cases.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}$ is decreasing for all $n \geq n_0$. Then we see that $\{\|x_n - x^*\|\}$ is convergent. Thus, from (3.9) and (3.6) we have

$$x_n - Tx_n \to 0, \qquad x_n - T_{\lambda_n}x_n \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

Moreover, from (3.1) and (3.11) we obtain

$$\|y_n - x_n\| = \beta_n\|x_n - Tx_n\| \to 0 \quad \text{as } n \to \infty, \tag{3.12}$$

and hence Lipschitz continuity of T, (3.12), (3.11) imply that

$$\begin{aligned}
\|Ty_n - x_n\| &\leq \|Ty_n - Tx_n\| + \|Tx_n - x_n\| \\
&\leq L\|y_n - x_n\| + \|Tx_n - x_n\| \to 0 \quad \text{as } n \to \infty. &(3.13)
\end{aligned}$$

In addition, from (3.13) and (3.11) we have

$$\begin{aligned}\|w_n - x_n\| &= \big\|(1-\alpha_n)(\theta_n x_n + \delta_n T y_n + \gamma_n T_{\lambda_n} x_n) - x_n\big\| \\ &\leq (1-\alpha_n)\delta_n\|T y_n - x_n\| + (1-\alpha_n)\gamma_n\|T_{\lambda_n} x_n - x_n\| \\ &\quad + \alpha_n\|x_n\| \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.14}$$

Furthermore, since $\{w_n\}$ is bounded subset of H which is reflexive, we can choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} \rightharpoonup w$ and $\limsup_{n\to\infty}\langle -x^*, w_n - x^*\rangle = \lim_{i\to\infty}\langle -x^*, w_{n_i} - x^*\rangle$. It follows from (3.14) that $x_{n_i} \rightharpoonup w$. Then, from (3.11) and Lemma 2.3, we have $w \in F(T)$.

Next, we show that $w \in (A+B)^{-1}(0)$. Let

$$z_n = J_{\lambda_n}(I - \lambda_n A)x_n. \tag{3.15}$$

Then from (3.11) we get $z_n - x_n \to 0$ as $n \to \infty$. In addition, for any $p \in \mathcal{F}$, we see that

$$\begin{aligned}\|z_n - p\|^2 &= \big\|J_{\lambda_n}(I-\lambda_n A)x_n - J_{\lambda_n}(I-\lambda_n A)p\big\|^2 \\ &\leq \|x_n - p\|^2 - 2\lambda_n\langle x_n - p, Ax_n - Ap\rangle + \lambda_n^2\|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - \lambda_n(2\gamma - \lambda_n)\|Ax_n - Ap\|^2. \end{aligned}$$

This implies that

$$\begin{aligned}\lambda_n(2\gamma - \lambda_n)\|Ax_n - Ap\|^2 &\leq \|x_n - p\|^2 - \|z_n - p\|^2 \\ &\leq \big(\|x_n - p\| + \|z_n - p\|\big)\|x_n - z_n\|, \end{aligned}$$

and hence we get

$$Ax_n - Ap \to 0 \quad \text{as } n \to \infty. \tag{3.16}$$

Now from (3.15) we obtain

$$x_{n_i} - \lambda_{n_i} A x_{n_i} \in (I + \lambda_{n_i} B) z_{n_i}.$$

That is,

$$\frac{x_{n_i} - z_{n_i}}{\lambda_{n_i}} - Ax_{n_i} \in Bz_{n_i}.$$

Since B is monotone, we get for any $(u,v) \in G(B)$, where $G(B)$ is the graph of B defined by $G(B) = \{(x,w) \in H \times H : x \in D(A), w \in Ax\}$,

$$\left\langle z_{n_i} - u, \frac{x_{n_i} - z_{n_i}}{\lambda_{n_i}} - Ax_{n_i} - v\right\rangle \geq 0. \tag{3.17}$$

On the other hand, since $\langle x_{n_i} - w, Ax_{n_i} - Aw\rangle \geq \gamma\|Ax_{n_i} - Aw\|^2$, $x_{n_i} \rightharpoonup w$ and $Ax_{n_i} \to Ap$, as $n \to \infty$ we have $Ax_{n_i} \to Aw$. Thus, letting $i \to \infty$, we obtain from (3.17)

$$\langle w - u, -Aw - v\rangle \geq 0.$$

Thus, maximality of B implies that $-Aw \in Bw$, that is, $0 \in (A+B)(w)$. Hence, we get $w \in (A+B)^{-1}(0)$.

Therefore, by Lemma 2.1, we immediately obtain

$$\limsup_{n\to\infty}\langle -x^*, w_n - x^*\rangle = \lim_{i\to\infty}\langle -x^*, w_{n_i} - x^*\rangle = \langle -x^*, w - x^*\rangle \leq 0. \tag{3.18}$$

Then it follows from (3.10), (3.18), and Lemma 2.2 that $\|x_n - x^*\| \to 0$ as $n \to \infty$. Consequently, $x_n \to x^* = P_{\mathcal{F}}(0)$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|x_{n_i} - x^*\| < \|x_{n_i+1} - x^*\|,$$

for all $i \in \mathbb{N}$. Then, by Lemma 2.4, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$\|x_{m_k} - x^*\| \leq \|x_{m_k+1} - x^*\| \quad \text{and} \quad \|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|, \tag{3.19}$$

for all $k \in \mathbb{N}$. Now, from (3.9) and (3.6) we get $x_{m_k} - Tx_{m_k} \to 0$, and $x_{m_k} - T_{\lambda_{m_k}} x_{m_k} \to 0$ as $k \to \infty$. Thus, like in Case 1, we obtain $w_{m_k} - x_{m_k} \to 0$ and

$$\limsup_{k\to\infty}\langle -x^*, w_{m_k} - x^*\rangle \leq 0. \tag{3.20}$$

Now, from (3.10) we have

$$\|x_{m_k+1} - x^*\|^2 \leq (1-\alpha_{m_k})\|x_{m_k} - x^*\|^2 + 2\alpha_{m_k}\langle -x^*, w_{m_k} - x^*\rangle, \tag{3.21}$$

and hence (3.19) and (3.21) imply that

$$\begin{aligned}\alpha_{m_k}\|x_{m_k} - x^*\|^2 &\leq \|x_{m_k} - x^*\|^2 - \|x_{m_k+1} - x^*\|^2 + 2\alpha_{m_k}\langle -x^*, w_{m_k} - x^*\rangle \\ &\leq -2\alpha_{m_k}\langle x^*, w_{m_k} - x^*\rangle.\end{aligned}$$

But using the fact that $\alpha_{m_k} > 0$ and (3.20) we obtain

$$\|x_{m_k} - x^*\| \to 0 \quad \text{as } k \to \infty.$$

This together with (3.21) implies that $\|x_{m_k+1} - x^*\| \to 0$ as $k \to \infty$. But $\|x_k - x^*\| \leq \|x_{m_k+1} - x^*\|$ for all $k \in \mathbb{N}$ and hence we obtain $x_k \to x^*$. Therefore, from the above two cases, we can conclude that $\{x_n\}$ converges strongly to the minimum-norm point of $\mathcal{F}$. The proof is complete. □

If, in Theorem 3.1, we assume that $A = 0$, then we get $T_{\lambda_n}(x_n) := (I + \lambda_n B)^{-1} x_n$ and hence we get the following corollary.

Corollary 3.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschtzian pseudocontractive mapping with Lipschitz constant L and*

$B : C \to 2^H$ be a maximal monotone mapping. Assume that $\mathcal{F} = F(T) \cap B^{-1}(0)$ is nonempty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in C$ by

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n; \\ x_{n+1} = P_C[(1-\alpha_n)(\theta_n x_n + \delta_n T y_n + \gamma_n T_{\lambda_n} x_n)], \end{cases} \tag{3.22}$$

where $T_{\lambda_n}(x_n) := (I + \lambda_n B)^{-1} x_n$ and $\{\lambda_n\} \subset (a,1)$, $\{\theta_n\}, \{\delta_n\}, \{\gamma_n\} \subset (c,d) \subset (0,1)$, $\{\alpha_n\} \subset (0,e) \subset (0,1)$, for some $a,c,d,e > 0$, satisfying the following conditions: (i) $\theta_n + \delta_n + \gamma_n = 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $\delta_n + \gamma_n \le \beta_n \le \beta < \frac{1}{\sqrt{1+L^2}+1}$, $\forall n \ge 1$. *Then $\{x_n\}$ converges strongly to the minimum-norm point x^* of $\mathcal{F}$.*

We also have the following theorem for two maximal monotone mappings.

Theorem 3.3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H such that* $\operatorname{int}(C) \neq \emptyset$. *Let $A, B : C \to H$ be maximal monotone mappings. Let $T : C \to C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant L such that $\mathcal{F} = F(T) \cap (A+B)^{-1}(0)$ is nonempty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n; \\ x_{n+1} = P_C[(1-\alpha_n)(\theta_n x_n + \delta_n T y_n + \gamma_n T_{\lambda_n} x_n)], \end{cases} \tag{3.23}$$

where $T_{\lambda_n}(x_n) := (I + \lambda_n (A+B))^{-1} x_n$ and $\{\lambda_n\} \subset (a,1)$, $\{\theta_n\}, \{\delta_n\}, \{\gamma_n\} \subset (c,d) \subset (0,1)$, $\{\alpha_n\} \subset (0,e) \subset (0,1)$, for some $a,c,d,e > 0$, satisfying the following conditions: (i) $\theta_n + \delta_n + \gamma_n = 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $\delta_n + \gamma_n \le \beta_n \le \beta < \frac{1}{\sqrt{1+L^2}+1}$, $\forall n \ge 1$. *Then $\{x_n\}$ converges strongly to the minimum-norm point x^* of $\mathcal{F}$.*

Proof From Lemma 2.7 we find that $A + B$ is a maximal monotone and hence by Corollary 3.2 we get the required assertion. □

If, in Theorem 3.3, we assume that $T = I$, the identity mapping on C, then we get the following corollary.

Corollary 3.4 *Let C be a nonempty, closed and convex subset of a real Hilbert space H such that* $\operatorname{int}(C) \neq \emptyset$. *Let $A, B : C \to H$ be maximal monotone mappings such that $\mathcal{F} = (A+B)^{-1}(0)$ is nonempty. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in C$ by*

$$x_{n+1} = P_C\big[(1-\alpha_n)\big((1-\gamma_n)x_n + \gamma_n T_{\lambda_n} x_n\big)\big],$$

where $T_{\lambda_n}(x_n) := (I + \lambda_n (A+B))^{-1} x_n$ and $\{\lambda_n\} \subset (a,1)$, $\{\gamma_n\} \subset (c,d) \subset (0,1)$, $\{\alpha_n\} \subset (0,e) \subset (0,1)$, for some $a,c,d,e > 0$, satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum \alpha_n = \infty$. *Then $\{x_n\}$ converges strongly to the minimum-norm point x^* of $\mathcal{F}$.*

4 Applications

We next study the problem of finding a solution of a variational inequality. Let C be a nonempty closed convex subset of a real Hilbert space H. The *normal cone* for C at a point $x \in C$, denoted by $N_C(x)$, is defined by

$$N_C(x) = \big\{x^* \in H : \langle y - x, x^* \rangle \le 0, \forall y \in C\big\}. \tag{4.1}$$

Let $f : H \to (-\infty, \infty]$ be a proper lower semicontinuous convex function. Define the subdifferential

$$\partial f(x) = \{z \in H : f(x) + \langle y - x, z\rangle \leq f(y), \forall y \in H\},$$

for all $x \in H$. Then from Rockafellar [38] we know that ∂f is maximal monotone mapping of H into itself. Let C be a nonempty closed convex subset of H and i_C be the indicator function of C, that is,

$$i_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases} \tag{4.2}$$

Then $i_C : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function on H and ∂i_C is a maximal monotone mapping. Let $J_\lambda x = (I + \lambda \partial i_C)^{-1} x$ for all $\lambda > 0$ and $x \in H$. From the fact that $\partial i_C x = N_C x$ and $x \in C$, we get

$$\begin{aligned} u \in J_\lambda x \quad &\Leftrightarrow \quad x \in u + \lambda \partial i_C u \quad \Leftrightarrow \quad x \in u + \lambda N_C u \\ &\Leftrightarrow \quad x - u \in \lambda N_C u \quad \Leftrightarrow \quad \langle x - u, y - u\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow \quad u = P_C x. \end{aligned}$$

Moreover,

$$\begin{aligned} x \in (A + \partial i_C)^{-1}(0) \quad &\Leftrightarrow \quad 0 \in (A + \partial i_C)x \quad \Leftrightarrow \quad -Ax \in \partial i_C x \\ &\Leftrightarrow \quad \langle -Ax, y - x\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow \quad x \in \mathrm{VI}(C, A), \end{aligned}$$

and hence $x \in (A + \partial i_C)^{-1}(0) \Leftrightarrow x \in \mathrm{VI}(C, A)$. Thus, the following corollary holds. Now, using Theorem 3.1, we obtain a strong convergence theorem for finding a common point of fixed points of Lipschtzian pseudocontractive mapping and solutions of the variational inequality problem for γ-inverse monotone mapping.

Theorem 4.1 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T : C \to C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant L and let $A : C \to H$ be a γ-inverse strongly monotone mapping such that $\mathcal{F} = F(T) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = (1 - \beta_n)x_n + \beta_n T x_n; \\ x_{n+1} = P_C[(1 - \alpha_n)(\theta_n x_n + \delta_n T y_n + \gamma_n P_C(x_n - \lambda_n A x_n))], \end{cases} \tag{4.3}$$

where $\{\lambda_n\} \subset (a, b) \subset (a, 2\gamma)$, $\{\theta_n\}, \{\delta_n\}, \{\gamma_n\} \subset (c, d) \subset (0, 1)$, $\{\alpha_n\} \subset (0, e) \subset (0, 1)$, for some $a, b, c, d, e > 0$, satisfying the following conditions: (i) $\theta_n + \delta_n + \gamma_n = 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $\delta_n + \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$, $\forall n \geq 1$. *Then $\{x_n\}$ converges strongly to the minimum-norm point x^* of $\mathcal{F}$.*

If, in Theorem 4.1, we take $T \equiv I$, the identity mapping on C we have the following corollary for a solution of variational inequality for a γ-inverse strongly monotone mapping.

Corollary 4.2 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A: C \to H$ be a γ-inverse strongly monotone mapping with $\mathrm{VI}(C,A) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in C$ by*

$$x_{n+1} = P_C\big[(1-\alpha_n)\big((1-\gamma_n)x_n + \gamma_n P_C(x_n - \lambda_n A x_n)\big)\big],$$

where $\{\lambda_n\} \subset (a,b) \subset (a, 2\gamma)$, $\{\gamma_n\} \subset (c,d) \subset (0,1)$, $\{\alpha_n\} \subset (0,e) \subset (0,1)$, for some $a, b, c, d, e > 0$, satisfying the following conditions: $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$. Then $\{x_n\}$ converges strongly to the minimum-norm point x^* of $\mathrm{VI}(C,A)$.

If, in Theorem 4.1, we take $A := (I - S)$, where S is a nonexpansive self mapping of C into itself, then we get the following corollary.

Corollary 4.3 *Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $T: C \to C$ be a Lipschitzian pseudocontractive mapping with Lipschitz constant L and let $S: C \to C$ be a nonexpansive mapping such that $\mathcal{F} = F(T) \cap F(S) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated from an arbitrary $x_0 \in C$ by*

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n T x_n; \\ x_{n+1} = P_C[(1-\alpha_n)(\theta_n x_n + \delta_n T y_n + \gamma_n((1-\lambda_n)x_n + \lambda_n S x_n))], \end{cases} \tag{4.4}$$

where $\{\lambda_n\} \subset (a,b) \subset (a, \frac{1}{2})$, $\{\theta_n\}, \{\delta_n\}, \{\gamma_n\} \subset (c,d) \subset (0,1)$, $\{\alpha_n\} \subset (0,e) \subset (0,1)$, for some $a, b, c, d, e > 0$, satisfying the following conditions: (i) $\theta_n + \delta_n + \gamma_n = 1$, (ii) $\lim_{n\to\infty} \alpha_n = 0$, $\sum \alpha_n = \infty$; (iii) $\delta_n + \gamma_n \leq \beta_n \leq \beta < \frac{1}{\sqrt{1+L^2}+1}$, $\forall n \geq 1$. *Then $\{x_n\}$ converges strongly to the minimum-norm point x^* of $F(T) \cap F(S)$.*

Proof Put $A := I - S$ in Theorem 4.1. Then we see that A is a $\frac{1}{4}$-inverse strongly monotone mapping. Furthermore, for $x \in C$ we have

$$P_C(x - \lambda A x) = P_C\big(x - \lambda(I - T)x\big) = (1-\lambda)x + \lambda T x$$

and

$$\begin{aligned} x^* \in \mathrm{VI}(C,A) \quad &\Leftrightarrow \quad x^* \in \mathrm{VI}(C, I - S) \\ &\Leftrightarrow \quad \big\langle Sx^* - x^*, y - x^* \big\rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow \quad P_C S x^* = x^* \quad \Leftrightarrow \quad Sx^* = x^*. \end{aligned} \tag{4.5}$$

Thus, we obtain $\mathrm{VI}(C,A) = F(S)$. Therefore, the conclusion holds by Theorem 4.1 □

Remark 4.4 Theorem 3.1 provides convergence sequence to a common point of fixed points of a Lipschitzian pseudocontractive mapping and zeros of two monotone mappings in Hilbert spaces.

Remark 4.5 Theorem 3.1 improves Theorem 3.1 of Takahashi *et al.* [24] in the sense that our convergence is to the common minimum-norm point of fixed points of a Lipschitzian pseudocontractive mapping and zeros of sum of two monotone mappings. Corollary 3.4

improves Theorem 1 of Kamimura and Takahashi [14] in the sense that our convergence is for the a zero of sum of two maximal monotone mappings. Theorem 4.1 extends Theorem 3.1 of Takahashi and Toyoda [23] in the sense that our convergence is to the common minimum-norm point of fixed points of a Lipschitzian pseudocontractive mapping and solutions of variational inequality for a γ-inverse strongly monotone mapping.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements
The authors are grateful to anonymous reviewers for their pertinent remarks and useful suggestions. This article was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The first author, therefore acknowledges with thanks DSR for financial support.

Received: 3 January 2014 Accepted: 14 March 2014 Published: 01 Apr 2014

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10.1186/1687-1812-2014-85
Cite this article as: Shahzad and Zegeye: **Approximating a common point of fixed points of a pseudocontractive mapping and zeros of sum of monotone mappings.** *Fixed Point Theory and Applications* 2014, 2014:85