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Iterative algorithms for quasi-variational inclusions and fixed point problems of pseudocontractions

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Abstract
In this paper, quasi-variational inclusions and fixed point problems of pseudocontractions are considered. An iterative algorithm is presented. A strong convergence theorem is demonstrated.
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1 Introduction

Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a single-valued nonlinear mapping and $B : H \to 2^H$ be a multi-valued mapping. The 'so called' quasi-variational inclusion problem is to find an $u \in 2^H$ such that

$$0 \in Au + Bu. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $(A + B)^{-1}(0)$. A number of problems arising in structural analysis, mechanics, and economics can be studied in the framework of this kind of variational inclusions; see for instance [1–4]. For related work, see [5–10]. The problem (1.1) includes many problems as special cases.

(1) If $B = \partial\phi : H \to 2^H$, where $\phi : H \to R \cup +\infty$ is a proper convex lower semi-continuous function and $\partial\phi$ is the subdifferential of ϕ, then the variational inclusion problem (1.1) is equivalent to finding $u \in H$ such that

$$\langle Au, y - u\rangle + \phi(y) - \phi(u) \geq 0, \quad \forall y \in H,$$

which is called the mixed quasi-variational inequality (see [11]).

(2) If $B = \partial\delta_C$, where C is a nonempty closed convex subset of H and $\delta_C : H \to [0, \infty]$ is the indicator function of C, *i.e.*,

$$\delta_C = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C, \end{cases}$$

then the variational inclusion problem (1.1) is equivalent to finding $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in C.$$

This problem is called the Hartman-Stampacchia variational inequality (see [12]).

Let $T : C \to C$ be a nonlinear mapping. The iterative scheme of Mann's type for approximating fixed points of T is the following: $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n,$$

for all $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $[0,1]$; see [13]. For two nonlinear mappings S and T, Takahashi and Tamura [14] considered the following iteration procedure: $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)S\big(\beta_n x_n + (1 - \beta_n)Tx_n\big),$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$. Algorithms for finding the fixed points of nonlinear mappings or for finding the zero points of maximal monotone operators have been studied by many authors. The reader can refer to [15–19]. Especially, Takahashi *et al.* [20] recently gave the following convergence result.

Theorem 1.1 *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let T be a nonexpansive mapping of C into itself, such that $F(T) \cap (A + B)^{-1}0 \neq \emptyset$. Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T\big(\alpha_n x + (1 - \alpha_n)J_{\lambda_n}^B(x_n - \lambda_n Ax_n)\big),$$

for all $n \geq 0$, where $\{\lambda_n\} \subset (0, 2\alpha)$, $\{\alpha_n\} \subset (0,1)$ and $\{\beta_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq b < 2\alpha, \qquad 0 < c \leq \beta_n \leq d < 1,$$
$$\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0, \qquad \lim_{n\to\infty}\alpha_n = 0 \quad \textit{and} \quad \sum_n \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point of $F(T) \cap (A + B)^{-1}0$.

Recently, Zhang *et al.* [21] introduced a new iterative scheme for finding a common element of the set of solutions to the inclusion problem and the set of fixed points of nonexpansive mappings in Hilbert spaces. Peng *et al.* [22] introduced another iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of a variational inclusion with set-valued maximal monotone mapping and inverse strongly monotone mappings, the set of solutions of an equilibrium problem, and the set of fixed points of a nonexpansive mapping.

Motivated and inspired by the works in this field, the purpose of this paper is to consider the quasi-variational inclusions and fixed point problems of pseudocontractions. An iterative algorithm is presented. A strong convergence theorem is demonstrated.

2 Notations and lemmas

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively. Let C be a nonempty closed convex subset of H. It is well known that in a real Hilbert space H, the following equality holds:

$$\left\| tx + (1-t)y \right\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $t \in [0,1]$.

Recall that a mapping $T : C \to C$ is called

- (D_1) *L-Lipschitzian* $\Longrightarrow$ there exists $L > 0$ such that $\|Tx - Ty\| \le L\|x-y\|$ for all $x, y \in C$; in the case of $L = 1$, T is said to be nonexpansive;
- (D_2) *Firmly nonexpansive* $\Longrightarrow$ $\|Tx - Ty\|^2 \le \|x-y\|^2 - \|(I-T)x - (I-T)y\|^2 \Longleftrightarrow \|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle$ for all $x, y \in C$;
- (D_3) *Pseudocontractive* $\Longrightarrow$ $\langle Tx - Ty, x - y\rangle \le \|x-y\|^2 \Longleftrightarrow \|Tx - Ty\|^2 \le \|x-y\|^2 + \|(I-T)x - (I-T)y\|^2$ for all $x, y \in C$;
- (D_4) *Strongly monotone* $\Longrightarrow$ there exists a positive constant $\tilde{\gamma}$ such that $\langle Tx - Ty, x - y\rangle \ge \tilde{\gamma}\|x-y\|$ for all $x, y \in C$;
- (D_5) *Inverse strongly monotone* $\Longrightarrow$ $\langle Tx - Ty, x - y\rangle \ge \alpha\|Tx - Ty\|^2$ for some $\alpha > 0$ and for all $x, y \in C$.

Let B be a mapping of H into 2^H. The effective domain of B is denoted by $\operatorname{dom}(B)$, that is, $\operatorname{dom}(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H iff

$$\langle x - y, u - v\rangle \ge 0$$

for all $x, y \in \operatorname{dom}(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal iff its graph is not strictly contained in the graph of any other monotone operator on H. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$.

For a maximal monotone operator B on H and $\lambda > 0$, we may define a single-valued operator $J_\lambda^B = (I + \lambda B)^{-1} : H \to \operatorname{dom}(B)$, which is called the resolvent of B for λ. It is known that the resolvent J_λ^B is firmly nonexpansive, *i.e.*,

$$\left\| J_\lambda^B x - J_\lambda^B y \right\|^2 \le \left\langle J_\lambda^B x - J_\lambda^B y, x - y \right\rangle$$

for all $x, y \in C$ and $B^{-1}0 = \operatorname{Fix}(J_\lambda^B)$ for all $\lambda > 0$.

Usually, the convergence of fixed point algorithms requires some additional smoothness properties of the mapping T such as demi-closedness.

Recall that a mapping T is said to be demiclosed if, for any sequence $\{x_n\}$ which weakly converges to $\tilde{x}$, and if the sequence $\{T(x_n)\}$ strongly converges to z, then $T(\tilde{x}) = z$. For the pseudocontractions, the following demiclosed principle is well known.

Lemma 2.1 ([23]) *Let H be a real Hilbert space, C a closed convex subset of H. Let $U : C \to C$ be a continuous pseudo-contractive mapping. Then*

(i) $\operatorname{Fix}(U)$ *is a closed convex subset of C,*
(ii) $(I - U)$ *is demiclosed at zero.*

Lemma 2.2 ([24]) *Let $\{r_n\}$ be a sequence of real numbers. Assume $\{r_n\}$ does not decrease at infinity, that is, there exists at least a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that $r_{n_k} \leq r_{n_k+1}$ for all $k \geq 0$. For every $n \geq N$, define an integer sequence $\{\tau(n)\}$ as*

$$\tau(n) = \max\{i \leq n : r_{n_i} < r_{n_i+1}\}.$$

Then $\tau(n) \to \infty$ as $n \to \infty$, and for all $n \geq N$

$$\max\{r_{\tau(n)}, r_n\} \leq r_{\tau(n)+1}.$$

Lemma 2.3 ([25]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \delta_n\gamma_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n\gamma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

In the sequel we shall use the following notations:
1. $\omega_w(u_n) = \{x : \exists u_{n_j} \to x \text{ weakly}\}$ denote the weak ω-limit set of $\{u_n\}$;
2. $u_n \rightharpoonup x$ stands for the weak convergence of $\{u_n\}$ to x;
3. $u_n \to x$ stands for the strong convergence of $\{u_n\}$ to x;
4. $\mathrm{Fix}(T)$ stands for the set of fixed points of T.

3 Main results

In this section, we consider a strong convergence theorem for quasi-variational inclusions and fixed point problems of pseudocontractive mappings in a Hilbert space.

Algorithm 3.1 *Let C be a nonempty closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda^B = (I+\lambda B)^{-1}$ be the resolvent of B for λ. Let $F : C \to H$ be an L_1-Lipschitzian and ς strongly monotone mapping and $f : C \to C$ be a ρ-contraction such that $\rho < \max\{1, \varsigma/2\}$. Let $T : C \to C$ be an $L_2(>1)$-Lipschitzian pseudocontraction. For $x_0 \in C$, define a sequence $\{x_n\}$ as follows*:

$$\begin{cases} z_n = J_\lambda^B(I-\lambda A)x_n, \\ y_n = \nu z_n + (1-\nu)T((1-\zeta)z_n + \zeta Tz_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)(\beta_n f(x_n) + (I-\beta_n F)y_n), \end{cases} \tag{3.1}$$

for all $n \in \mathbb{N}$, where λ, ν and ζ are three constants, $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0,1]$.

Now, we demonstrate the convergence analysis of the algorithm (3.1).

Theorem 3.2 *Suppose $\Gamma := \mathrm{Fix}(T) \cap (A+B)^{-1}(0) \neq \emptyset$. Assume the following conditions are satisfied*:

(C1) $\alpha_n \in [a,b] \subset (0,1)$;
(C2) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$;
(C3) $\lambda \in (0, 2\alpha)$ *and* $0 < 1 - \nu \leq \zeta < \frac{1}{\sqrt{1+L_2^2}+1}$.

Then the sequence $\{x_n\}$ *defined by* (3.1) *converges strongly to* $u = P_\Gamma(I - F + f)u$.

Proof Let $x^* \in \mathrm{Fix}(T) \cap (A+B)^{-1}(0)$. Then, we get $x^* = J_\lambda^B(I - \lambda A)x^* = Tx^*$. From (3.1), we have

$$\begin{aligned}
\|z_n - x^*\|^2 &= \|J_\lambda^B(I-\lambda A)x_n - J_\lambda^B(I-\lambda A)x^*\|^2 \\
&\leq \|x_n - x^* - \lambda(Ax_n - Ax^*)\|^2 \\
&= \|x_n - x^*\|^2 - 2\lambda\langle Ax_n - Ax^*, x_n - x^*\rangle + \lambda^2\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\lambda\alpha\|Ax_n - Ax^*\|^2 + \lambda^2\|Ax_n - Ax^*\|^2 \\
&= \|x_n - x^*\|^2 - \lambda(2\alpha - \lambda)\|Ax_n - Ax^*\|^2 \\
&\leq \|x_n - x^*\|^2. \qquad (3.2)
\end{aligned}$$

It follows that

$$\|z_n - x^*\| \leq \|x_n - x^*\|. \qquad (3.3)$$

Since $x^* \in \mathrm{Fix}(T)$, we have from (D_3) that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + \|Tx - x\|^2, \qquad (3.4)$$

for all $x \in C$.

Thus,

$$\begin{aligned}
\|T((1-\zeta)I + \zeta T)z_n - x^*\|^2 \leq{}& \|(1-\zeta)(z_n - x^*) + \zeta(Tz_n - x^*)\|^2 \\
&+ \|((1-\zeta)I + \zeta T)z_n - T((1-\zeta)I + \zeta T)z_n\|^2. \qquad (3.5)
\end{aligned}$$

By (3.4), (3.5), and (2.1), we obtain

$$\begin{aligned}
&\|T((1-\zeta)I + \zeta T)z_n - x^*\|^2 \\
&\quad\leq \|(1-\zeta)(z_n - x^*) + \zeta(Tz_n - x^*)\|^2 \\
&\qquad + \|((1-\zeta)I + \zeta T)z_n - T((1-\zeta)I + \zeta T)z_n\|^2 \\
&\quad= \|(1-\zeta)(z_n - T((1-\zeta)I + \zeta T)z_n) + \zeta(Tz_n - T((1-\zeta)I + \zeta T)z_n)\|^2 \\
&\qquad + \|(1-\zeta)(z_n - x^*) + \zeta(Tz_n - x^*)\|^2 \\
&\quad= (1-\zeta)\|z_n - T((1-\zeta)I + \zeta T)z_n\|^2 + \zeta\|Tz_n - T((1-\zeta)I + \zeta T)z_n\|^2 \\
&\qquad - \zeta(1-\zeta)\|z_n - Tz_n\|^2 + (1-\zeta)\|z_n - x^*\|^2 + \zeta\|Tz_n - x^*\|^2 - \zeta(1-\zeta)\|z_n - Tz_n\|^2 \\
&\quad\leq (1-\zeta)\|z_n - x^*\|^2 + \zeta(\|z_n - x^*\|^2 + \|z_n - Tz_n\|^2)
\end{aligned}$$

$$\begin{aligned}&-2\zeta(1-\zeta)\|z_n - Tz_n\|^2 + (1-\zeta)\big\|z_n - T\big((1-\zeta)I + \zeta T\big)z_n\big\|^2\\&+\zeta\big\|Tz_n - T\big((1-\zeta)I + \zeta T\big)z_n\big\|^2.\end{aligned}$$

Noting that T is L_2-Lipschitzian and $z_n - ((1-\zeta)I + \zeta T)z_n = \zeta(z_n - Tz_n)$, we have

$$\begin{aligned}&\big\|T\big((1-\zeta)I+\zeta T\big)z_n - x^*\big\|^2\\&\quad\le(1-\zeta)\big\|z_n - x^*\big\|^2 + \zeta\big(\big\|z_n - x^*\big\|^2 + \|z_n - Tz_n\|^2\big)\\&\qquad-2\zeta(1-\zeta)\|z_n - Tz_n\|^2 + (1-\zeta)\big\|z_n - T\big((1-\zeta)I+\zeta T\big)z_n\big\|^2 + \zeta^3L_2^2\|z_n - Tz_n\|^2\\&\quad=\big\|z_n - x^*\big\|^2 + (1-\zeta)\big\|z_n - T\big((1-\zeta)I+\zeta T\big)z_n\big\|^2\\&\qquad-\zeta\big(1-2\zeta-\zeta^2L_2^2\big)\|z_n - Tz_n\|^2.\end{aligned}\tag{3.6}$$

Since $\zeta < \frac{1}{\sqrt{1+L_2^2}+1}$, we have $1 - 2\zeta - \zeta^2L_2^2 > 0$. From (3.6), we can deduce

$$\big\|T\big((1-\zeta)I+\zeta T\big)z_n - x^*\big\|^2 \le \big\|z_n - x^*\big\|^2 + (1-\zeta)\big\|z_n - T\big((1-\zeta)I+\zeta T\big)z_n\big\|^2.\tag{3.7}$$

Hence,

$$\begin{aligned}\big\|y_n - x^*\big\|^2 &= \big\|\nu z_n + (1-\nu)T\big((1-\zeta)I+\zeta T\big)z_n - x^*\big\|^2\\&=\big\|\nu\big(z_n - x^*\big) + (1-\nu)\big(T\big((1-\zeta)I+\zeta T\big)z_n - x^*\big)\big\|^2\\&=\nu\big\|z_n - x^*\big\|^2 + (1-\nu)\big\|T\big((1-\zeta)I+\zeta T\big)z_n - x^*\big\|^2\\&\quad-\nu(1-\nu)\big\|T\big((1-\zeta)I+\zeta T\big)z_n - z_n\big\|^2\\&\le\nu\big\|z_n - x^*\big\|^2 + (1-\nu)\big[\big\|z_n - x^*\big\|^2 + (1-\zeta)\big\|z_n - T\big((1-\zeta)I+\zeta T\big)z_n\big\|^2\big]\\&\quad-\nu(1-\nu)\big\|T\big((1-\zeta)I+\zeta T\big)z_n - z_n\big\|^2\\&=\big\|z_n - x^*\big\|^2 + (1-\nu)(1-\zeta-\nu)\big\|T\big((1-\zeta)I+\zeta T\big)z_n - z_n\big\|^2.\end{aligned}\tag{3.8}$$

By (C3) and (3.8), we obtain

$$\big\|y_n - x^*\big\| \le \big\|z_n - x^*\big\|.\tag{3.9}$$

Let $u_n = \beta_n f(x_n) + (I - \beta_n F)y_n$ for all $n \ge 0$. Then, we have

$$\begin{aligned}\big\|u_n - x^*\big\| &= \big\|\beta_n f(x_n) + (I-\beta_n F)y_n - x^*\big\|\\&\le\beta_n\big\|f(x_n) - Fx^*\big\| + \big\|(I-\beta_n F)y_n - (I-\beta_n F)x^*\big\|\\&\le\beta_n\big\|f(x_n) - f\big(x^*\big)\big\| + \beta_n\big\|f\big(x^*\big) - Fx^*\big\|\\&\quad+\big\|(I-\beta_n F)y_n - (I-\beta_n F)x^*\big\|\\&\le\beta_n\rho\big\|x_n - x^*\big\| + \beta_n\big\|f\big(x^*\big) - Fx^*\big\|\\&\quad+\big\|(I-\beta_n F)y_n - (I-\beta_n F)x^*\big\|.\end{aligned}\tag{3.10}$$

Since F is L_1-Lipschitzian and ς strongly monotone, we have

$$\begin{aligned}
&\|(I-\beta_n F)y_n-(I-\beta_n F)x^*\|^2\\
&\quad=\|(y_n-x^*)-\beta_n(Fy_n-Fx^*)\|^2\\
&\quad=\|y_n-x^*\|^2-2\beta_n\langle Fy_n-Fx^*,y_n-x^*\rangle+\beta_n^2\|Fy_n-Fx^*\|^2\\
&\quad\le\|y_n-x^*\|^2-2\beta_n\varsigma\|y_n-x^*\|^2+\beta_n^2L_1^2\|y_n-x^*\|^2\\
&\quad=(1-2\beta_n\varsigma+\beta_n^2L_1^2)\|y_n-x^*\|^2.
\end{aligned}\tag{3.11}$$

Noting that $L_1\ge\varsigma$ and $\lim_{n\to\infty}\beta_n=0$, without loss of generality, we assume that $\beta_n<\frac{\varsigma}{L_1^2-\varsigma/4}$ for all $n\ge 0$. Thus, $1-2\beta_n\varsigma+\beta_n^2L_1^2\le(1-\beta_n\frac{\varsigma}{2})^2$. So,

$$\|(I-\beta_n F)y_n-(I-\beta_n F)x^*\|\le\left(1-\beta_n\frac{\varsigma}{2}\right)\|y_n-x^*\|.\tag{3.12}$$

We have from (3.9), (3.10), and (3.12)

$$\begin{aligned}
\|u_n-x^*\|&\le\beta_n\rho\|x_n-x^*\|+\beta_n\|f(x^*)-Fx^*\|+\left(1-\beta_n\frac{\varsigma}{2}\right)\|x_n-x^*\|\\
&=\left[1-\left(\frac{\varsigma}{2}-\rho\right)\beta_n\right]\|x_n-x^*\|+\beta_n\|f(x^*)-Fx^*\|.
\end{aligned}\tag{3.13}$$

From (3.1) and (3.13), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|&=\|\alpha_n(x_n-x^*)+(1-\alpha_n)(u_n-x^*)\|\\
&\le(1-\alpha_n)\left(\left[1-\left(\frac{\varsigma}{2}-\rho\right)\beta_n\right]\|x_n-x^*\|+\beta_n\|f(x^*)-Fx^*\|\right)\\
&\quad+\alpha_n\|x_n-x^*\|\\
&=\left[1-\left(\frac{\varsigma}{2}-\rho\right)(1-\alpha_n)\beta_n\right]\|x_n-x^*\|+(1-\alpha_n)\beta_n\|f(x^*)-Fx^*\|.
\end{aligned}\tag{3.14}$$

By the definition of x_n, we have

$$\begin{aligned}
x_{n+1}-x_n&=\alpha_nx_n+(1-\alpha_n)(\beta_nf(x_n)+(I-\beta_nF)y_n)-x_n\\
&=(1-\alpha_n)[\beta_nf(x_n)-\beta_nFy_n+y_n-x_n].
\end{aligned}\tag{3.15}$$

Hence,

$$\begin{aligned}
\langle x_{n+1}-x_n,x_n-x^*\rangle&=\langle(1-\alpha_n)[\beta_nf(x_n)-\beta_nFy_n+y_n-x_n],x_n-x^*\rangle\\
&=(1-\alpha_n)\beta_n\langle f(x_n),x_n-x^*\rangle-(1-\alpha_n)\beta_n\langle Fy_n,x_n-x^*\rangle\\
&\quad+(1-\alpha_n)\langle y_n-x_n,x_n-x^*\rangle.
\end{aligned}\tag{3.16}$$

Since $2\langle x_{n+1}-x_n, x_n-x^*\rangle = \|x_{n+1}-x^*\|^2 - \|x_n-x^*\|^2 - \|x_{n+1}-x_n\|^2$ and $2\langle y_n-x_n, x_n-x^*\rangle = \|y_n-x^*\|^2 - \|x_n-x^*\|^2 - \|y_n-x_n\|^2$, it follows from (3.16), (3.3), and (3.9) that

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 - \|x_n-x^*\|^2 - \|x_{n+1}-x_n\|^2 \\
&\quad = 2(1-\alpha_n)\beta_n\langle f(x_n), x_n-x^*\rangle - 2(1-\alpha_n)\beta_n\langle Fy_n, x_n-x^*\rangle \\
&\qquad + (1-\alpha_n)\big[\|y_n-x^*\|^2 - \|x_n-x^*\|^2 - \|y_n-x_n\|^2\big] \\
&\quad \le 2(1-\alpha_n)\beta_n\langle f(x_n), x_n-x^*\rangle - 2(1-\alpha_n)\beta_n\langle Fy_n, x_n-x^*\rangle \\
&\qquad - (1-\alpha_n)\|y_n-x_n\|^2. \qquad (3.17)
\end{aligned}$$

By (3.15), we obtain

$$\begin{aligned}
\|x_{n+1}-x_n\|^2 &\le (1-\alpha_n)^2\big[\beta_n\|f(x_n)-Fy_n\| + \|y_n-x_n\|\big]^2 \\
&= (1-\alpha_n)^2\big[\beta_n^2\|f(x_n)-Fy_n\|^2 + \|y_n-x_n\|^2 \\
&\quad + 2\beta_n\|f(x_n)-Fy_n\|\|y_n-x_n\|\big]. \qquad (3.18)
\end{aligned}$$

Combining (3.17) and (3.18) to deduce

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 - \|x_n-x^*\|^2 \\
&\quad \le 2(1-\alpha_n)\beta_n\langle f(x_n), x_n-x^*\rangle - 2(1-\alpha_n)\beta_n\langle Fy_n, x_n-x^*\rangle \\
&\qquad - (1-\alpha_n)\|y_n-x_n\|^2 + (1-\alpha_n)^2\big[\beta_n^2\|f(x_n)-Fy_n\|^2 \\
&\qquad + \|y_n-x_n\|^2 + 2\beta_n\|f(x_n)-Fy_n\|\|y_n-x_n\|\big] \\
&\quad \le 2(1-\alpha_n)\beta_n\langle f(x_n), x_n-x^*\rangle - 2(1-\alpha_n)\beta_n\langle Fy_n, x_n-x^*\rangle - (1-\alpha_n)\alpha_n\|y_n-x_n\|^2 \\
&\qquad + (1-\alpha_n)^2\big[\beta_n^2\|f(x_n)-Fy_n\|^2 + 2\beta_n\|f(x_n)-Fy_n\|\|y_n-x_n\|\big].
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\|x_{n+1}-x^*\|^2 - \|x_n-x^*\|^2 + (1-\alpha_n)\alpha_n\|y_n-x_n\|^2 \\
&\quad \le 2(1-\alpha_n)\beta_n\langle f(x_n), x_n-x^*\rangle - 2(1-\alpha_n)\beta_n\langle Fy_n, x_n-x^*\rangle \\
&\qquad + (1-\alpha_n)^2\big[\beta_n^2\|f(x_n)-Fy_n\|^2 + 2\beta_n\|f(x_n)-Fy_n\|\|y_n-x_n\|\big].
\end{aligned}$$

It follows that, hence, we obtain

$$\begin{aligned}
&(1-\alpha_n)\alpha_n\|y_n-x_n\|^2 \\
&\quad \le \|x_n-x^*\|^2 - \|x_{n+1}-x^*\|^2 + 2(1-\alpha_n)\beta_n\langle f(x_n), x_n-x^*\rangle \\
&\qquad - 2(1-\alpha_n)\beta_n\langle Fy_n, x_n-x^*\rangle \\
&\qquad + (1-\alpha_n)^2\big[\beta_n^2\|f(x_n)-Fy_n\|^2 + 2\beta_n\|f(x_n)-Fy_n\|\|y_n-x_n\|\big]. \qquad (3.19)
\end{aligned}$$

Next we divide our proof into two possible cases.

Case 1. There exists an integer number m such that $\|x_{n+1}-x^*\| \le \|x_n - x^*$ for all $n \ge m$. In this case, we have $\lim_{n\to\infty}\|x_n - x^*\|$ exists. Since $\alpha_n \in [a,b] \subset (0,1)$ and $\lim_{n\to\infty}\beta_n = 0$, by (3.19), we derive

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.20}$$

This together with (3.18) implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.21}$$

Note that

$$\begin{aligned}\|u_n - y_n\| &= \big\|\beta_n f(x_n) + (I - \beta_n F)y_n - y_n\big\| \\ &\le \beta_n \big\|f(x_n) - Fy_n\big\|.\end{aligned}$$

So,

$$\lim_{n\to\infty}\|u_n - y_n\| = 0. \tag{3.22}$$

By (3.20) and (3.22), we obtain

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.23}$$

From (3.2) and (3.9), we have

$$\big\|y_n - x^*\big\|^2 \le \big\|z_n - x^*\big\|^2 \le \big\|x_n - x^*\big\|^2 - \lambda(2\alpha - \lambda)\big\|Ax_n - Ax^*\big\|^2.$$

Hence,

$$\begin{aligned}\lambda(2\alpha - \lambda)\big\|Ax_n - Ax^*\big\|^2 &\le \big\|x_n - x^*\big\|^2 - \big\|y_n - x^*\big\|^2 \\ &\le \|x_n - y_n\|\big(\big\|x_n - x^*\big\| + \big\|y_n - x^*\big\|\big).\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\big\|Ax_n - Ax^*\big\| = 0. \tag{3.24}$$

Since J_λ^B is firmly nonexpansive and A is monotone, we have

$$\begin{aligned}\big\|z_n - x^*\big\|^2 &= \big\|J_\lambda^B(I-\lambda A)x_n - J_\lambda^B(I-\lambda A)x^*\big\|^2 \\ &\le \big\langle (I-\lambda A)x_n - (I-\lambda A)x^*, z_n - x^*\big\rangle \\ &= \big\langle z_n - x^*, x_n - x^*\big\rangle - \lambda\big\langle z_n - x^*, Ax_n - Ax^*\big\rangle \\ &= \frac{1}{2}\big(\big\|z_n - x^*\big\|^2 + \big\|x_n - x^*\big\|^2 - \|z_n - x_n\|^2\big) \\ &\quad - \lambda\big\langle x_n - x^*, Ax_n - Ax^*\big\rangle - \lambda\big\langle z_n - x_n, Ax_n - Ax^*\big\rangle\end{aligned}$$

$$\begin{aligned}&\leq \frac{1}{2}\left(\|z_n - x^*\|^2 + \|x_n - x^*\|^2 - \|z_n - x_n\|^2\right)\\&\quad + \lambda\|z_n - x_n\|\|Ax_n - Ax^*\|.\end{aligned}$$

It follows that

$$\|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2\lambda\|z_n - x_n\|\|Ax_n - Ax^*\|. \tag{3.25}$$

By (3.25) and (3.9), we deduce

$$\|y_n - x^*\|^2 \leq \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \|z_n - x_n\|^2 + 2\lambda\|z_n - x_n\|\|Ax_n - Ax^*\|.$$

Therefore,

$$\begin{aligned}\|z_n - x_n\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2 + 2\lambda\|z_n - x_n\|\|Ax_n - Ax^*\|\\&\leq \|x_n - y_n\|\left(\|x_n - x^*\| + \|y_n - x^*\|\right) + 2\lambda\|z_n - x_n\|\|Ax_n - Ax^*\|.\end{aligned} \tag{3.26}$$

Equations (3.20), (3.24), and (3.26) imply that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.27}$$

Notice that $F - f$ is $(\varsigma - \rho)$ strongly monotone. Thus, the variational inequality of finding $y \in \Gamma$ such that $\langle (F - f)y, x - y\rangle \geq 0$ for all $x \in \Gamma$ has a unique solution, denoted by x^*, that is, $x^* = P_\Gamma(I - V + F)(x^*)$. Next, we prove that

$$\limsup_{n\to\infty}\langle (f - F)x^*, u_n - x^*\rangle \leq 0.$$

Since u_n is bounded, without loss of generality, we assume that there exists a subsequence $\{z_{n_i}\}$ of $\{u_n\}$ such that $u_{n_i} \rightharpoonup \tilde{x}$ for some $\tilde{x} \in H$ and

$$\limsup_{n\to\infty}\langle (f - F)x^*, u_n - x^*\rangle = \limsup_{i\to\infty}\langle (f - F)x^*, u_{n_i} - x^*\rangle.$$

Thus, we have that $x_{n_i} \rightharpoonup \tilde{x}$ and

$$\lim_{i\to\infty}\|J_\lambda^B(I - \lambda A)x_{n_i} - x_{n_i}\| = 0.$$

Therefore, $\tilde{x} \in \mathrm{Fix}(J_\lambda^B(I - \lambda A)) = (A + B)^{-1}(0)$.

Next we show that $\tilde{x} \in \mathrm{Fix}(T)$. First, we show that $\mathrm{Fix}(T) = \mathrm{Fix}(T((1 - \zeta)I + \zeta T))$. As a matter of fact, $\mathrm{Fix}(T) \subset \mathrm{Fix}(T((1 - \zeta)I + \zeta T))$ is obvious. Next, we show that $\mathrm{Fix}(T((1 - \zeta)I + \zeta T)) \subset \mathrm{Fix}(T)$.

Take any $x^* \in \mathrm{Fix}(T((1 - \zeta)I + \zeta T))$. We have $T((1 - \zeta)I + \zeta T)x^* = x^*$. Set $S = (1 - \zeta)I + \zeta T$. We have $TSx^* = x^*$. Write $Sx^* = y^*$. Then, $Ty^* = x^*$. Now we show $x^* = y^*$. In fact,

$$\begin{aligned}\|x^* - y^*\| &= \|Ty^* - Sx^*\| = \|Ty^* - (1 - \zeta)x^* - \zeta Tx^*\|\\&= \zeta\|Ty^* - Tx^*\| \leq \zeta L_2\|y^* - x^*\|.\end{aligned}$$

Since, $\zeta < \frac{1}{\sqrt{1+L_2^2}+1} < \frac{1}{L_2}$, we deduce $y^* = x^* \in \mathrm{Fix}(S) = \mathrm{Fix}(T)$. Thus, $x^* \in \mathrm{Fix}(T)$. Hence, $\mathrm{Fix}(T((1-\zeta)I + \zeta T)) \subset \mathrm{Fix}(T)$. Therefore, $\mathrm{Fix}(T((1-\zeta)I + \zeta T)) = \mathrm{Fix}(T)$.

By (3.1), (3.20), and (3.27), we deduce

$$\lim_{n\to\infty} \big\| T\big((1-\zeta)I + \zeta T\big)x_n - x_n \big\| = 0. \tag{3.28}$$

Next we prove that $T((1-\zeta)I + \zeta T) - I$ is demiclosed at 0. Let the sequence $\{w_n\} \subset H_2$ satisfying $w_n \rightharpoonup x^\dagger$ and $w_n - T((1-\zeta)I + \zeta T)w_n \to 0$. Next, we will show that $x^\dagger \in \mathrm{Fix}(T((1-\zeta)I + \zeta T)) = \mathrm{Fix}(T)$.

Since T is L_2-Lipschizian, we have

$$\begin{aligned} \|w_n - Tw_n\| &\le \big\| w_n - T\big((1-\zeta)I + \zeta T\big)w_n \big\| + \big\| T\big((1-\zeta)I + \zeta T\big)w_n - Tw_n \big\| \\ &\le \big\| w_n - T\big((1-\zeta)I + \zeta T\big)w_n \big\| + \zeta L \|w_n - Tw_n\|. \end{aligned}$$

It follows that

$$\|w_n - Tw_n\| \le \frac{1}{1-\zeta L} \big\| w_n - T\big((1-\zeta)I + \zeta T\big)w_n \big\|.$$

Hence,

$$\lim_{n\to\infty} \|w_n - Tw_n\| = 0.$$

Since $T - I$ is demiclosed at 0 by Lemma 2.1, we immediately deduce $x^\dagger \in \mathrm{Fix}(T) = \mathrm{Fix}(T((1-\zeta)I + \zeta T))$. Therefore, $T((1-\zeta)I + \zeta T) - I$ is demiclosed at 0. By (3.28), we deduce $\tilde{x} \in \mathrm{Fix}(T)$. Hence, $\tilde{x} \in \Gamma$. So,

$$\begin{aligned} \limsup_{n\to\infty} \big\langle (f-F)x^*, u_n - x^* \big\rangle &= \limsup_{i\to\infty} \big\langle (f-F)x^*, u_{n_i} - x^* \big\rangle \\ &= \big\langle (f-F)x^*, \tilde{x} - x^* \big\rangle \\ &\le 0. \end{aligned} \tag{3.29}$$

Note that

$$\begin{aligned} \big\| u_n - x^* \big\|^2 &= \big\| \beta_n\big(f(x_n) - f\big(x^*\big)\big) + \beta_n\big(f\big(x^*\big) - Fx^*\big) + (I - \beta_n F)\big(y_n - x^*\big) \big\|^2 \\ &\le \big\| (I - \beta_n F)\big(y_n - x^*\big) \big\|^2 + 2\beta_n \big\langle f(x_n) - f\big(x^*\big), u_n - x^* \big\rangle \\ &\quad + 2\beta_n \big\langle f\big(x^*\big) - Fx^*, u_n - x^* \big\rangle \\ &\le \left(1 - \beta_n \frac{\varsigma}{2}\right)^2 \big\| x_n - x^* \big\|^2 + 2\beta_n \rho \big\| x_n - x^* \big\| \big\| u_n - x^* \big\| \\ &\quad + 2\beta_n \big\langle f\big(x^*\big) - Fx^*, u_n - x^* \big\rangle \\ &\le \left(1 - \beta_n \frac{\varsigma}{2}\right)^2 \big\| x_n - x^* \big\|^2 + 2\beta_n \rho \big\| x_n - x^* \big\|^2 + \frac{1}{2} \big\| u_n - x^* \big\|^2 \\ &\quad + 2\beta_n \big\langle f\big(x^*\big) - Fx^*, u_n - x^* \big\rangle. \end{aligned}$$

It follows that

$$\begin{aligned}\|u_n - x^*\|^2 &\le \left[1 - 2\left(\frac{\varsigma}{2} - \rho\right)\beta_n\right]\|x_n - x^*\|^2 + \beta_n^2\frac{\varsigma^2}{4}\|x_n - x^*\|^2 \\ &\quad + 4\beta_n\langle f(x^*) - Fx^*, u_n - x^*\rangle.\end{aligned}$$

So,

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= \|\alpha_n(x_n - x^*) + (1-\alpha_n)(u_n - x^*)\|^2 \\ &\le \alpha_n\|x_n - x^*\|^2 + (1-\alpha_n)\|u_n - x^*\|^2 \\ &\le \left[1 - 2\left(\frac{\varsigma}{2} - \rho\right)(1-\alpha_n)\beta_n\right]\|x_n - x^*\|^2 + (1-\alpha_n)\beta_n^2\frac{\varsigma^2}{4}\|x_n - x^*\|^2 \\ &\quad + 4(1-\alpha_n)\beta_n\langle f(x^*) - Fx^*, u_n - x^*\rangle \\ &= \left[1 - (\varsigma - 2\rho)(1-\alpha_n)\beta_n\right]\|x_n - x^*\|^2 \\ &\quad + (\varsigma - 2\rho)(1-\alpha_n)\beta_n\left\{\beta_n\frac{\varsigma^2}{4(\varsigma - 2\rho)}\|x_n - x^*\|^2\right. \\ &\quad \left. + \frac{4}{\varsigma - 2\rho}\langle f(x^*) - Fx^*, u_n - x^*\rangle\right\}.\end{aligned} \tag{3.30}$$

Applying Lemma 2.3 to (3.30) we deduce $x_n \to x^*$.

Case 2. Assume there exists an integer n_0 such that $\|x_{n_0} - x^*\| \le \|x_{n_0+1} - x^*\|$. In this case, we set $\omega_n = \{\|x_n - x^*\|\}$. Then, we have $\omega_{n_0} \le \omega_{n_0+1}$. Define an integer sequence $\{\tau_n\}$ for all $n \ge n_0$ as follows:

$$\tau(n) = \max\{l \in \mathbb{N} | n_0 \le l \le n, \omega_l \le \omega_{l+1}\}.$$

It is clear that $\tau(n)$ is a non-decreasing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = \infty$$

and

$$\omega_{\tau(n)} \le \omega_{\tau(n)+1},$$

for all $n \ge n_0$. From (3.19), we get

$$\begin{aligned}&(1-\alpha_{\tau(n)})\alpha_{\tau(n)}\|y_{\tau(n)} - x_{\tau(n)}\|^2 \\ &\quad \le \|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2 + 2(1-\alpha_{\tau(n)})\beta_{\tau(n)}\langle f(x_{\tau(n)}), x_{\tau(n)} - x^*\rangle \\ &\qquad - 2(1-\alpha_{\tau(n)})\beta_{\tau(n)}\langle Fy_{\tau(n)}, x_{\tau(n)} - x^*\rangle \\ &\qquad + (1-\alpha_{\tau(n)})^2\left[\beta_{\tau(n)}^2\|f(x_{\tau(n)}) - Fy_{\tau(n)}\|^2\right. \\ &\qquad \left. + 2\beta_{\tau(n)}\|f(x_{\tau(n)}) - Fy_{\tau(n)}\|\|y_{\tau(n)} - x_{\tau(n)}\|\right].\end{aligned} \tag{3.31}$$

It follows that

$$\lim_{n\to\infty} \|y_{\tau(n)} - x_{\tau(n)}\| = 0.$$

By a similar argument to that of (3.29) and (3.30), we can prove that

$$\limsup_{n\to\infty} \langle (f - F)x^*, u_{\tau(n)} - x^* \rangle \le 0, \tag{3.32}$$

and

$$\begin{aligned} \omega_{\tau(n)+1}^2 \le{}& \left[1 - 2\left(\frac{\varsigma}{2} - \rho\right)(1 - \alpha_{\tau(n)})\beta_{\tau(n)}\right]\omega_{\tau(n)}^2 \\ &+ (1 - \alpha_{\tau(n)})\beta_{\tau(n)}^2 \frac{\varsigma^2}{4}\omega_{\tau(n)}^2 \\ &+ 4(1 - \alpha_{\tau(n)})\beta_{\tau(n)} \langle f(x^*) - Fx^*, u_{\tau(n)} - x^* \rangle. \end{aligned} \tag{3.33}$$

Since $\omega_{\tau(n)} \le \omega_{\tau(n)+1}$, we have from (3.33)

$$\omega_{\tau(n)}^2 \le \frac{16}{4(\varsigma^2 - 2\rho) - \varsigma^2\beta_{\tau(n)}} \langle f(x^*) - Fx^*, u_\tau(n) - x^* \rangle. \tag{3.34}$$

Combining (3.33) and (3.34), we have

$$\limsup_{n\to\infty} \omega_{\tau(n)} \le 0,$$

and hence

$$\lim_{n\to\infty} \omega_{\tau(n)} = 0. \tag{3.35}$$

From (3.33), we also obtain

$$\limsup_{n\to\infty} \omega_{\tau(n)+1} \le \limsup_{n\to\infty} \omega_{\tau(n)}.$$

This together with (3.35) imply that

$$\lim_{n\to\infty} \omega_{\tau(n)+1} = 0.$$

Applying Lemma 2.2 to get

$$0 \le \omega_n \le \max\{\omega_{\tau(n)}, \omega_{\tau(n)+1}\}.$$

Therefore, $\omega_n \to 0$. That is, $x_n \to x^*$. This completes the proof. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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