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A class of nonlinear mixed ordered inclusion problems for ordered (α_A, λ)-$ANODM$ set-valued mappings with strong comparison mapping A

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Abstract
The purpose of this paper is to introduce and study a new class of nonlinear mixed ordered inclusion problems in ordered Banach spaces and to obtain an existence theorem and a comparability theorem of the resolvent operator. Further, by using fixed point theory and the resolvent operator, the authors constructed and studied an approximation algorithm for this kind of problems, and they show the relation between the first valued x_0 and the solution of the problems. The results obtained seem to be general in nature.

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1 Introduction

Well-known, generalized nonlinear inclusion (variational inequality and equation) have wide applications in many fields, including, for example, mathematics, physics, optimization and control, nonlinear programming, economic, and engineering sciences [1–6]. In 1972, the number of solutions of nonlinear equations had been introduced and studied by Amann [7], and in recent years, nonlinear mapping fixed point theory and applications have been intensively studied in ordered Banach space [8–10]. Therefore, it is very important and natural that generalized nonlinear ordered variational inequalities (ordered equations) are studied and discussed.

From 2008, the authors introduced and studied the approximation algorithm and the approximation solution theory for the generalized nonlinear ordered variational inclusion problems (inequalities, systems, and equations) in ordered Banach spaces; for example, in 2008, Li has introduced and studied the approximation algorithm and the approximation solution for a class of generalized nonlinear ordered variational inequality and ordered equation in ordered Banach spaces [11]. In 2009, by using the B-restricted-accretive method of mapping A with constants α_1, α_2, Li has introduced and studied an existence theorem and an approximation algorithm of solutions for a new class of general nonlinear ordered variational inequalities and equations in ordered Banach spaces [12]. In 2011, Li has introduced and studied a class of nonlinear inclusion problems for ordered *RME* set-valued mappings in order Hilbert spaces [13]; in 2012, Li has introduced and studied a class of nonlinear inclusion problems for ordered (α,λ)-*NODM* set-valued mappings, and then, applying the resolvent operator associated with the set-valued mappings, established an existence theorem on the solvability and a general algorithm applied to the approximation solvability of the nonlinear inclusion problem of this class of nonlinear inclusion problems in ordered Hilbert space [14], and have proved a sensitivity analysis of the solution for a new class of general nonlinear ordered parametric variational inequalities in 2012 [15]. Recently, Li *et al.* have studied the characterizations of ordered (α_A,λ)-weak-*ANODD* set-valued mappings, which was applied to solving approximate solution for a new class of general nonlinear mixed order quasi-variational inclusions involving the $\oplus$ operator, and a new class of generalized nonlinear mixed order variational inequalities systems with order Lipschitz continuous mappings in ordered Banach spaces [16, 17].

In this field, the obtained results seem to be general in nature. As regards new developments, it is exceedingly of interest to study the problems: for $w \in X$ and $\omega > 0$, find $x \in X$ such that $w \in f(x) + \omega M(x)$. A new class of nonlinear mixed ordered inclusion problems for ordered (α_A,λ)-*ANODM* set-valued mappings with strong comparison mapping A and characterizations of ordered $(\alpha_A,\frac{\lambda}{\omega})$-*ANODM* set-valued mappings are introduced in ordered Banach spaces. An existence theorem and a comparability theorem of the resolvent operator associated to a $(\alpha_A,\frac{\lambda}{\omega})$-*ANODM* set-valued mapping are established. By using fixed point theory and the resolvent operator associated for the $(\alpha_A,\frac{\lambda}{\omega})$-*ANODM* set-valued mapping, an existence theorem of solutions and an approximation algorithm for this kind of problems are studied, and the relation of between the first valued x_0 and the solution of the problems is discussed. The results obtained seem to be general in nature. For details, we refer the reader to [1–30] and the references therein.

Let X be a real ordered Banach space with norm $\|\cdot\|$, a zero θ, a normal cone $\mathbf{P}$, a normal constant N of $\mathbf{P}$ and a partial ordered relation $\leq$ defined by the cone $\mathbf{P}$ [11, 12]. Let $f: X \to X$ be a single-valued ordered compression mapping, and $M: X \to 2^X$ and

$$f(x) + M(x) = \{y | y = f(x) + u, \forall x \in X, u \in M(x)\} : X \to 2^X$$

be two set-valued mappings. We consider the following problem.

For $w \in X$, and any $\omega > 0$, find $x \in X$ such that

$$w \in f(x) + \omega M(x). \tag{1.1}$$

The problem (1.1) is called a nonlinear mixed ordered inclusion problems for the ordered *ANODM* set-valued mapping M in an ordered Banach space.

Remark 1.1 We have the following special cases of the problem (1.1):

(i) If $M(x) = F(g(x))$ be a single-valued mapping, $\omega = 1, f = 0$ and $w = \theta$, then the problem (2.1) in [11] can be obtained by the problem (1.1).
(ii) If $\omega = 1, f = 0$ and $w = \theta$, then the problem (1.1) changes to the problem (1.1) in [13] and [14].

Let us recall and discuss the following results and concepts for solving the problem (1.1).

2 Preliminaries

Let X be a real ordered Banach space with norm $\|\cdot\|$, a zero θ, a normal cone $\mathbf{P}$, normal constant N and a partial ordered relation $\leq$ defined by the cone $\mathbf{P}$. For arbitrary $x, y \in X$, $\operatorname{lub}\{x,y\}$ and $\operatorname{glb}\{x,y\}$ express the least upper bound of the set $\{x,y\}$ and the greatest lower bound of the set $\{x,y\}$ on the partial ordered relation $\leq$, respectively. Suppose $\operatorname{lub}\{x,y\}$ and $\operatorname{glb}\{x,y\}$ exist. Let us recall some concepts and results.

Definition 2.1 [11, 18] Let X be a real Banach space with norm $\|\cdot\|$, θ be a zero element in the X.

(i) A nonempty closed convex subsets $\mathbf{P}$ of X is said to be a cone, if
  (1) for any $x \in \mathbf{P}$ and any $\lambda > 0$, we have $\lambda x \in \mathbf{P}$,
  (2) $x \in \mathbf{P}$ and $-x \in \mathbf{P}$, then $x = \theta$;
(ii) $\mathbf{P}$ is said to be a normal cone if and only if there exists a constant $N > 0$, and a normal constant of $\mathbf{P}$ such that for $\theta \leq x \leq y$, we have $\|x\| \leq N\|y\|$;
(iii) for arbitrary $x, y \in X$, $x \leq y$ if and only if $x - y \in \mathbf{P}$;
(iv) for $x, y \in X$, x and y are said to be a comparison between each other, if and only if we have $x \leq y$ (or $y \leq x$) (denoted by $x \propto y$ for $x \leq y$ and $y \leq x$).

Lemma 2.2 [8] *If* $x \propto y$, *then* $\operatorname{lub}\{x,y\}$, *and* $\operatorname{glb}\{x,y\}$ *exist*, $x - y \propto y - x$, *and* $\theta \leq (x-y) \vee (y-x)$.

Lemma 2.3 *If for any natural number* n, $x \propto y_n$, *and* $y_n \to y^*$ $(n \to \infty)$, *then* $x \propto y^*$.

Proof If for any natural number n, $x \propto y_n$ and $y_n \to y^*$ $(n \to \infty)$, then $x - y_n \in \mathbf{P}$ or $y_n - x \in P$ for any natural number n. Since $\mathbf{P}$ is a nonempty closed convex subsets of X so that $x - y^* = \lim_{n\to\infty}(x - y_n) \in \mathbf{P}$ or $y^* - x = \lim_{n\to\infty}(y_n - x) \in \mathbf{P}$. Therefore, $x \propto y^*$. □

Lemma 2.4 [11, 12, 14, 15] *Let* X *be an ordered Banach space*, $\mathbf{P}$ *be a cone of* X, *and* $\leq$ *be a relation defined by the cone* $\mathbf{P}$ *in Definition* 2.1(iii). *For* $x, y, v, u \in X$, *then we have the following relations*:

(1) *the relation* $\leq$ *in* X *is a partial ordered relation in* X;
(2) $x \oplus y = y \oplus x$;
(3) $x \oplus x = \theta$;
(4) $\theta \leq x \oplus \theta$;
(5) *let* λ *be a real, then* $(\lambda x) \oplus (\lambda y) = |\lambda|(x \oplus y)$;
(6) *if* x, y, *and* w *can be comparative each other, then* $(x \oplus y) \leq x \oplus w + w \oplus y$;
(7) *let* $(x+y) \vee (u+v)$ *exist, and if* $x \propto u, v$ *and* $y \propto u, v$, *then* $(x+y) \oplus (u+v) \leq (x \oplus u + y \oplus v) \wedge (x \oplus v + y \oplus u)$;
(8) *if* x, y, z, w *can be compared with each other, then* $(x \wedge y) \oplus (z \wedge w) \leq ((x \oplus z) \vee (y \oplus w)) \wedge ((x \oplus w) \vee (y \oplus z))$;
(9) *if* $x \leq y$ *and* $u \leq v$, *then* $x + u \leq y + v$;
(10) *if* $x \propto \theta$, *then* $-x \oplus \theta \leq x \leq x \oplus \theta$;
(11) *if* $x \propto y$, *then* $(x \oplus \theta) \oplus (y \oplus \theta) \leq (x \oplus y) \oplus \theta = x \oplus y$;
(12) $(x \oplus \theta) - (y \oplus \theta) \leq (x - y) \oplus \theta$;
(13) *if* $\theta \leq x$ *and* $x \neq \theta$, *and* $\alpha > 0$, *then* $\theta \leq \alpha x$ *and* $\alpha x \neq \theta$.

Proof (1)-(8) come from Lemma 2.5 in [11] and Lemma 2.3 in [12], and (8)-(13) directly follow from (1)-(8). □

Definition 2.5 Let X be a real ordered Banach space, $A: X \to X$ be a single-valued mapping, and $M: X \to 2^X$ be a set-valued mapping. Then:

(1) a single-valued mapping A is said to be a γ-ordered non-extended mapping, if there exists a constant $\gamma > 0$ such that

$$\gamma(x \oplus y) \le A(x) \oplus A(y), \quad \forall x, y \in X;$$

(2) a single-valued mapping A is said to be a strong comparison mapping, if A is a comparison mapping, and $A(x) \propto A(y)$, then $x \propto y$ for any $x, y \in X$;

(3) a comparison mapping M is said to be an α_A-non-ordinary difference mapping with respect to A, if there exists a constant $\alpha_A > 0$ such that for each $x, y \in X$, $v_x \in M(x)$, and $v_y \in M(y)$,

$$(v_x \oplus v_y) \oplus \alpha_A\big(A(x) \oplus A(y)\big) = \theta;$$

(4) a comparison mapping M is said to be a λ-ordered monotone mapping with respect to B, if there exists a constant $\lambda > 0$ such that

$$\lambda(v_x - v_y) \ge x - y, \quad \forall x, y \in X, v_x \in M\big(B(x)\big), v_y \in M\big(B(y)\big);$$

(5) a comparison mapping M is said to be a (α_A, λ)-*ANODM* mapping, if M is a α_A-non-ordinary difference mapping with respect to A and a λ-ordered monotone mapping with respect to B, and $(A + \lambda M)(X) = X$ for $\alpha_A, \lambda > 0$.

Lemma 2.6 *Let X be a real ordered Banach space. If A is a γ-ordered non-extended mapping, and M is a λ-ordered monotone mapping and an α_A-non-ordinary difference mapping with respect to A, then for any $\omega > 0$, ωM is a $\frac{\lambda}{\omega}$-ordered monotone and an α_A-non-ordinary difference mapping with respect to A.*

Proof Let a comparison mapping M be a λ-ordered monotone mapping with respect to A, then it is obvious that ωM is a $\frac{\lambda}{\omega}$-ordered monotone mapping with respect to A. If M is an α_A-non-ordinary difference mapping with respect to A, then there exists a constant $\alpha_A > 0$ such that for each $x, y \in X$, and $v_x \in \omega M(x)$ and $v_y \in \omega M(y)$ ($v_x = \omega u_x$, $v_y = \omega u_y$, $u_x \in M(x)$, $u_y \in M(y)$) we have

$$(u_x \oplus u_y) \oplus \alpha_A\big(A(x) \oplus A(y)\big) = \theta$$

and

$$\omega\big((u_x \oplus u_y) \oplus \alpha_A\big(A(x) \oplus A(y)\big)\big) = \omega\theta = \theta.$$

By Lemma 2.4 and $\omega > 0$, we have

$$\big((\omega u_x \oplus \omega u_y) \oplus \alpha_A\big(A(x) \oplus A(y)\big)\big) = \theta.$$

Therefore,

$$(v_x \oplus v_y) \oplus \alpha_A\big(A(x) \oplus A(y)\big) = \theta.$$

It follows that ωM is a α_A-non-ordinary difference mapping with respect to A for any $\omega > 0$. □

Lemma 2.7 *Let X be a real ordered Banach space. If A is a γ-ordered non-extended mapping and a comparison mapping M is a (α_A, λ)-ANODM mapping, then ωM is a $(\alpha_A, \frac{\lambda}{\omega})$-ANODM mapping.*

Proof Let X be a real ordered Banach space, let A be a γ-ordered non-extended mapping and a comparison mapping M be a (α_A, λ)-*ANODM* mapping, then $(A + \lambda M)(X) = X$ for $\alpha_A, \lambda > 0$. It is follows that $(A + \frac{\lambda}{\omega}(\omega M))(X) = X$ for $\alpha_A, \frac{\lambda}{\omega} > 0$. Therefore, ωM is a $(\alpha_A, \frac{\lambda}{\omega})$-*ANODM* mapping by Lemma 2.6. □

Lemma 2.8 [14] *Let X be a real ordered Banach space. If A is a γ-ordered non-extended mapping and M is a α_A-non-ordinary difference mapping with respect to A, then an inverse mapping $J^A_{M,\lambda} = (A + \lambda M)^{-1} : X \to 2^X$ of $(A + \lambda M)$ is a single-valued mapping $(\alpha_A, \lambda > 0)$.*

Lemma 2.9 *Let X be a real ordered Banach space. If A is a γ-ordered non-extended mapping and M is a α_A-non-ordinary difference mapping with respect to A, then an inverse mapping $J^A_{\omega M,\frac{\lambda}{\omega}} = (A + \frac{\lambda}{\omega}(\omega M))^{-1} : X \to 2^X$ of $(A + \frac{\lambda}{\omega}(\omega M))$ is a single-valued mapping $(\alpha_A, \lambda > 0)$.*

Proof This directly follows from Lemma 2.6, Lemma 2.7, and Lemma 2.8. □

Lemma 2.10 [14] *Let X be a real ordered Banach space with norm $\|\cdot\|$, a zero θ, a normal cone $\mathbf{P}$, a normal constant N of $\mathbf{P}$ and a partial ordered relation $\leq$ defined by the cone $\mathbf{P}$, and the operator $\oplus$ be a XOR operator. If A is a strong comparison mapping, and $M : X \to 2^X$ is a λ-ordered monotone mapping with respect to $J^A_{M,\lambda}$, then the resolvent operator $J^A_{M,\lambda} : X \to X$ is a comparison mapping.*

Lemma 2.11 [14] *Let X be a real ordered Banach space with norm $\|\cdot\|$, a zero θ, a normal cone $\mathbf{P}$, a normal constant N of $\mathbf{P}$ and a partial ordered relation $\leq$ defined by the cone $\mathbf{P}$, and the operator $\oplus$ be a XOR operator. If A is a strong comparison mapping, and $M : X \to 2^X$ is a λ-ordered monotone mapping with respect to $J^A_{M,\lambda}$, then the resolvent operator $J^A_{\omega M,\frac{\lambda}{\omega}} : X \to X$ is a comparison mapping.*

Proof This directly follows from Lemma 2.6, Lemma 2.7, and Lemma 2.10. □

Lemma 2.12 [14] *Let X be a real ordered Banach space with norm $\|\cdot\|$, a zero θ, a normal cone $\mathbf{P}$, a normal constant N of $\mathbf{P}$ and a partial ordered relation $\leq$ defined by the cone $\mathbf{P}$. If A is a γ-ordered non-extended mapping, and $M : X \to 2^X$ is a (α_A, λ)-ANODM mapping, which is a α_A-non-ordinary difference mapping with respect to A and λ-ordered monotone mapping with respect to $J^A_{M,\lambda}$, then for the resolvent operator $J^A_{M,\lambda} : X \to X$, the following relation holds*:

$$J^A_{M,\lambda}(x) \oplus J^A_{M,\lambda}(y) \leq \frac{1}{\gamma(\alpha_A \lambda - 1)}(x \oplus y), \tag{2.1}$$

where $\alpha_A \lambda > 1$.

Lemma 2.13 *Let X be a real ordered Banach space with norm $\|\cdot\|$, a zero θ, a normal cone* **P**, *a normal constant N of* **P** *and a partial ordered relation $\leq$ defined by the cone* **P**. *If A is a γ-ordered non-extended mapping and $M : X \to 2^X$ is a (α_A, λ)-ANODM mapping, which is a α_A-non-ordinary difference mapping with respect to A and λ-ordered monotone mapping with respect to $J^A_{M,\lambda}$, then for the resolvent operator $J^A_{\omega M,\frac{\lambda}{\omega}} : X \to X$, the following relation holds*:

$$J^A_{\omega M,\frac{\lambda}{\omega}}(x) \oplus J^A_{\omega M,\frac{\lambda}{\omega}}(y) \leq \frac{\omega}{\gamma(\alpha_A\lambda - \omega)}(x \oplus y), \tag{2.2}$$

where $\alpha_A > \frac{\omega}{\lambda} > 0$.

Proof Let X be a real ordered Banach space, **P** be a normal cone with the normal constant N in the X, $\leq$ be a ordered relation defined by the cone **P**. For $x, y \in X$, let $u_x = J^A_{\omega M,\frac{\lambda}{\omega}}(x) \propto u_y = J^A_{\omega M,\frac{\lambda}{\omega}}(y)$ and $v_x = \frac{\omega}{\lambda}(x - A(u_x)) \in \omega M(u_x)$, $v_y = \frac{\omega}{\lambda}(y - A(u_y)) \in \omega M(u_y)$. Since $\omega M : X \to X$ is a $(\alpha_A, \frac{\lambda}{\omega})$-*ANODM* mapping with respect to A so that the following relations hold by (5) in Lemma 2.4 and the condition $(v_x \oplus v_y) \oplus \alpha_A(A(u_x) \oplus A(u_y)) = \theta$:

$$\begin{aligned}
&\frac{\omega}{\lambda}\big((x \oplus y) + \big(A(u_x) \oplus A(u_y)\big)\big) \\
&\quad \geq \omega v_x \oplus \omega v_y \\
&\quad = \alpha_A\big(A(u_x) \oplus A(u_y)\big).
\end{aligned}$$

It follows that $(\frac{\lambda}{\omega}\alpha_A - 1)(A(u_x) \oplus A(u_y)) \leq (x \oplus y)$ from the conditions $\alpha_A > \frac{\omega}{\lambda} > 0$ and $A(u_x) \oplus A(u_y) \geq \gamma(u_x \oplus u_y)$, and A is a γ-ordered non-extended mapping. The proof is completed. □

Remark 2.14 It is clear that Lemma 2.6, Theorem 3.2, and Theorem 3.3 in [14] are special cases of Lemma 2.6, Lemma 2.9, and Lemma 2.12, respectively, when $A = I$, the identity mapping in X.

3 Main results

In this section, we will show the algorithm of the approximation sequences for finding a solution of the problem (1.1), and we discuss the convergence and the relation between the first valued x_0 and the solution of the problem (1.1) in X, a real Banach space.

Theorem 3.1 *Let X be a real ordered Banach space with norm $\|\cdot\|$, a zero θ, a normal cone* **P**, *a normal constant N of* **P** *and a partial ordered relation $\leq$ defined by the cone* **P**, *and the operator $\oplus$ be a XOR operator. Let $A, f : X \to X$ be two single-valued ordered compression mappings and $A \propto f, f \propto \theta$. If A is a γ-ordered non-extended strong comparison mapping and $M : X \to 2^X$ is a α_A-non-ordinary difference mapping with respect to A, then the inclusion problem* (1.1) *has a solution x^* if and only if $x^* = J^A_{\omega M,\frac{\lambda}{\omega}}(A + \frac{\lambda}{\omega}(w - f))(x^*)$ in X.*

Proof This directly follows from the definition of the resolvent operator $J^A_{\omega M,\frac{\lambda}{\omega}}$ of $\omega M(x)$. □

Theorem 3.2 *Let X be a real ordered Banach space,* **P** *be a normal cone with the normal constant N in the X, $\leq$ be a partial ordered relation defined by the cone* **P**. *Let*

$A, f : X \to X$ be two single-valued β, ξ ordered compression mappings, respectively, A be a γ non-extended and strong compression mapping, and $M : X \to 2^X$ be a (α_A, λ)-ANODM mapping, which is a α_A-non-ordinary difference mapping with respect to A and λ-ordered monotone mapping with respect to $J^A_{M,\lambda}$. If $A \propto f$, $w \propto A, f, M, \alpha_A > \frac{\omega}{\lambda} > 0$, and

$$\beta\omega + \gamma\omega + \lambda\xi < \gamma\lambda\alpha_A \tag{3.1}$$

(*where* $\beta, \xi > 0$), *then the sequence* $\{x_n\}$ *converges strongly to* x^*, *the solution of the problem* (1.1), *which is generated by following algorithm.*

For any given $x_0 \in X$, let $x_1 = J^A_{\omega M, \frac{\lambda}{\omega}}(A + \frac{\lambda}{\omega}(w - f))(x_0)$, and for $n > 0$ and $0 < \varphi < 1$, set

$$x_{n+1} = (1-\varphi)x_n + \varphi J^A_{\omega M, \frac{\lambda}{\omega}}\left(A + \frac{\lambda}{\omega}(w - f)\right)(x_n).$$

For any $x_0 \in X$, we have

$$\begin{aligned}\|x^* - x_0\| &\leq \left(1 - N\left(1 - \frac{\gamma(\alpha_A\lambda - \omega)}{\varphi(\alpha_A\gamma\lambda - (\beta\omega + \gamma\omega + \lambda\xi))}\right)\right) \\ &\quad \times \left\|J^A_{\omega M, \frac{\lambda}{\omega}}\left(A + \frac{\lambda}{\omega}(w - f)\right)(x_0) - x_0\right\|.\end{aligned} \tag{3.2}$$

Proof Let X be a real ordered Banach space, let $\mathbf{P}$ be a normal cone with the normal constant N in the X, let $\leq$ be a partial ordered relation defined by the cone $\mathbf{P}$. For any $x_0 \in X$, we set $x_1 = (1-\varphi)x_0 + \varphi J^A_{\omega M, \frac{\lambda}{\omega}}(A + \frac{\lambda}{\omega}(w - f))(x_0)$. By using Lemma 2.7, Lemma 2.9, Lemma 2.12, $\frac{\lambda}{\omega}$-monotonicity of ωM, $(A + \frac{\lambda}{\omega}\omega M)(X) = X$, and the comparability of $J^A_{\omega M, \frac{\lambda}{\omega}}$, we know that $x_1 \propto x_0$. Further, we can obtain a sequence $\{x_n\}$, and $x_{n+1} \propto x_n$ (where $n = 0, 1, 2, \ldots$). Using Lemma 2.4, Lemma 2.7, Lemma 2.9, and Lemma 2.12, we have

$$\begin{aligned}\theta &\leq x_{n+1} \oplus x_n \\ &\leq \left((1-\varphi)x_n + \varphi J^A_{\omega M, \frac{\lambda}{\omega}}\left(A + \frac{\lambda}{\omega}(w - f)\right)(x_n)\right) \\ &\quad \oplus \left((1-\varphi)x_{n-1} + \varphi J^A_{\omega M, \frac{\lambda}{\omega}}\left(A + \frac{\lambda}{\omega}(w - f)\right)(x_{n-1})\right) \\ &\leq \varphi\frac{\omega}{\gamma(\alpha_A\lambda - \omega)}\left(\left(A + \frac{\lambda}{\omega}(w - f)\right)(x_n) \oplus \left(A + \frac{\lambda}{\omega}(w - f)\right)(x_{n-1})\right) \\ &\quad + (1-\varphi)(x_{n-1} \oplus x_n) \\ &\leq \varphi\frac{\omega}{\gamma(\alpha_A\lambda - \omega)}\left(A(x_n) \oplus A(x_{n-1}) + \frac{\lambda}{\omega}(w - f)(x_n) \oplus \frac{\lambda}{\omega}(w - f)(x_{n-1})\right) \\ &\quad + (1-\varphi)(x_{n-1} \oplus x_n) \\ &\leq \varphi\frac{1}{\gamma(\alpha_A\lambda - 1)}\big(A(x_n) \oplus A(x_{n-1}) + \lambda f(x_n) \oplus f(x_{n-1}) + \lambda(w \oplus w)\big) + (1-\varphi)(x_{n-1} \oplus x_n) \\ &\leq \varphi\frac{\omega}{\gamma(\alpha_A\lambda - \omega)}\left(\beta(x_n \oplus x_{n-1}) + \frac{\lambda}{\omega}\xi(x_n \oplus x_{n-1})\right) + (1-\varphi)(x_{n-1} \oplus x_n) \\ &\leq \left(1 - \varphi + \varphi\frac{\beta\omega + \lambda\xi}{\gamma(\alpha_A\lambda - \omega)}\right)(x_n \oplus x_{n-1}) \\ &\leq \left(1 - \varphi + \varphi\frac{\beta\omega + \lambda\xi}{\gamma(\alpha_A\lambda - \omega)}\right)^n(x_1 \oplus x_0);\end{aligned} \tag{3.3}$$

by Lemma 2.4 and Definition 2.1(ii), we obtain

$$\|x_{n+1}-x_n\| \le (1-\varphi+\varphi\delta)^n N\|x_1-x_0\|, \tag{3.4}$$

where $\delta = \frac{\beta\omega+\lambda\xi}{\gamma(\alpha_A\lambda-\omega)}$. Hence, for any $m > n > 0$, we have

$$\|x_m-x_n\| \le \sum_{i=n}^{m-1}\|x_{i+1}-x_i\| \le N\|x_1-x_0\|\sum_{i=n}^{m-1}(1-\varphi+\varphi\delta)^i.$$

It follows from the condition (3.1) that $0 < \delta < 1$, and $\|x_m - x_n\| \to 0$, as $n \to \infty$, and so $\{x_n\}$ is a Cauchy sequence in the complete space X. Let $x_n \to x^*$ as $n \to \infty$ ($x^* \in X$). By the conditions, we have

$$\begin{aligned} x^* &= \lim_{n\to\infty} x_{n+1} \\ &= \lim_{n\to\infty} J^A_{\omega M,\frac{\lambda}{\omega}}\left(A+\frac{\lambda}{\omega}(w-f)(x_n)\right) \\ &= J^A_{\omega M,\frac{\lambda}{\omega}}\left(A+\frac{\lambda}{\omega}(w-f)\right)(x^*). \end{aligned}$$

We know that x^* is a solution of the inclusion problem (1.1). It follows that $(J^A_{\varphi M,\frac{\lambda}{\omega}}(A + \frac{\lambda}{\omega}(w-f))(x_n)) \propto x^*$ ($n = 0,1,2,\ldots$) from Lemma 2.4 and (3.4). Also we have

$$\begin{aligned} \left\|x^*-x_0\right\| &= \lim_{n\to\infty}\|x_n-x_0\| \\ &\le \lim_{n\to\infty}\sum_{i=1}^{n}\|x_{i+1}-x_i\| \le \lim_{n\to\infty} N\sum_{i=2}^{n}(1-\varphi+\varphi\delta)^{n-1}\|x_1-x_0\| + \|x_1-x_0\| \\ &\le \left(\frac{1+(N-1)(1-\varphi+\varphi\delta)}{1-(1-\varphi+\varphi\delta)}\right)\left\|J^A_{\omega M,\frac{\lambda}{\omega}}\left(A+\frac{\lambda}{\omega}(w-f)\right)(x_0)-x_0\right\| \\ &\le \left(1-N\left(1-\frac{\gamma(\alpha_A\lambda-\omega)}{\varphi(\alpha_A\gamma\lambda-(\beta\omega+\gamma\omega+\lambda\xi))}\right)\right) \\ &\quad\times\left\|J^A_{\omega M,\frac{\lambda}{\omega}}\left(A+\frac{\lambda}{\omega}(w-f)\right)(x_0)-x_0\right\|. \end{aligned}$$

This completes the proof. □

Remark 3.3 Though the method of solving problem by the resolvent operator is the same as in [19, 20, 27], and [28] for a nonlinear inclusion problem, the character of the ordered (α_A,λ)-*ANODM* set-valued mapping is different from the one of the (A,η)-accretive mapping [19], (H,η)-monotone mapping [20], (G,η)-monotone mapping [27], and monotone mapping [28].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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