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Strong and weak convergence theorems for common zeros of finite accretive mappings

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Dedicated to Professor Wataru Takahashi on the occasion of his 70th birthday

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Abstract
In this paper, we present a new iterative scheme to solve the problems of finding common zeros of finite m-accretive mappings in a real Hilbert space. Some strong and weak convergence theorems are established under different assumptions, which extends the corresponding works given by some authors.

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1 Introduction and preliminaries

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Then for $\forall x, y \in H$, and $\lambda \in [0,1]$,

$$\left\|\lambda x + (1-\lambda)y\right\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2. \tag{1.1}$$

We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x, and $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

Let C be a closed and convex subset of H. Then, for every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \le \|x - y\|$ for all $y \in C$. P_C is called the metric projection of H onto C. It is well known that $P_C : H \to C$ is characterized by the properties:

(i) $\langle x - P_C x, P_C x - y\rangle \ge 0$, for all $y \in C$ and $x \in H$;
(ii) For every $x, y \in H$, $\|P_C x - P_C y\|^2 \le \langle x - y, P_C x - P_C y\rangle$;
(iii) $\|P_C x - P_C y\| \le \|x - y\|$, for every $x, y \in H$;
(iv) $x_n \rightharpoonup x_0$ and $P_C x_n \to y_0$ imply that $P_C x_0 = y_0$.

A mapping $f : C \to C$ is called a contraction if there exists a constant $k \in (0,1)$ such that $\|f(x) - f(y)\| \le k\|x - y\|$, for $\forall x, y \in C$. We use $\sum_C$ to denote the collection of mappings f verifying the above inequality. That is, $\sum_C := \{f : C \to C | f$ is a contraction with constant $k\}$.

A mapping $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \le \|x - y\|$, for $\forall x, y \in C$. We use $F(T)$ to denote the fixed point set of T, that is, $F(T) := \{x \in C : Tx = x\}$.

A mapping $A : H \supset D(A) \to R(A) \subset H$ is called accretive if $\langle x - y, Ax - Ay\rangle \ge 0$, for $\forall x, y \in D(A)$ and it is called m-accretive if $R(I + \lambda A) = H$, for $\forall \lambda > 0$. An m-accretive mapping A

is demi-closed, that is, if $\{x_n\} \subset D(A)$ such that $x_n \rightharpoonup x$ and $Ax_n \to y$, then $x \in D(A)$ and $y = Ax$. Let $A^{-1}0$ denote the set of zeros of A, that is, $A^{-1}0 := \{x \in D(A) : Ax = 0\}$. We denote by J_r^A (for $r > 0$) the resolvent of A, that is, $J_r^A := (I + rA)^{-1}$. Then J_r^A is nonexpansive and $F(J_r^A) = A^{-1}0$.

Interest in accretive mappings, which is an important class of nonlinear operators, stems mainly from their firm connection with equations of evolution. It is well known that many physically significant problems can be modeled by initial value problems of the form

$$x'(t) + Ax(t) = 0, \qquad x(0) = x_0, \tag{1.2}$$

where A is an accretive mapping. Typical examples where such evolution equations occur can be found in the heat, wave or Schrödinger equations. If $x(t)$ is dependent on t, then (1.2) is reduced to

$$Au = 0, \tag{1.3}$$

whose solutions correspond to the equilibrium of the system (1.2). Consequently, within the past 40 years or so, considerable research efforts have been devoted to methods for finding approximate solutions of (1.3). An early fundamental result in the theory of accretive operators, due to Browder [1]. One classical method for studying the problem $0 \in Ax$, where A is an m-accretive mapping, is the following so-called proximal method (*cf.* [2]):

$$x_0 \in H, \quad x_{n+1} \approx J_{r_n} x_n, \quad n \geq 0, \tag{1.4}$$

where $J_{r_n} := (I + r_n A)^{-1}$. It was shown that the sequence generated by (1.4) converges weakly or strongly to a zero point of A under some conditions.

Recall that the following normal Mann iterative scheme to approximate the fixed point of a nonexpansive mapping $T : C \to C$ was introduced by Mann [3]:

$$x_0 \in C, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n, \quad n \geq 0. \tag{1.5}$$

It was proved that under some conditions, the sequence $\{x_n\}$ produced by (1.5) converges weakly to a point in $F(T)$.

Later, many mathematicians tried to combine the ideas of proximal method and Mann iterative method to approximate the zeros of m-accretive mappings; see, *e.g.* [4–11] and references therein.

Especially, in 2007, Qin and Su [4] presented the following iterative scheme:

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n x_n + (1 - \beta_n) J_{r_n} x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \end{cases} \tag{1.6}$$

where $J_{r_n} = (I + r_n A)^{-1}$. They showed that $\{x_n\}$ generated by the above scheme converges strongly to a zero of A.

Based on iterative schemes (1.4) and (1.5), Zegeye and Shahzad extended their discussion to the case of finite m-accretive mappings. They presented in [12] the following iterative scheme:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)S_r x_n, \quad n \geq 0, \tag{1.7}$$

where $S_r = a_0 I + a_1 J_{A_1} + a_2 J_{A_2} + \cdots + a_l J_{A_l}$ with $J_{A_i} = (I + A_i)^{-1}$ and $\sum_{i=0}^{l} a_i = 1$. If $\bigcap_{i=1}^{l} A_i^{-1}(0) \neq \emptyset$, they proved that $\{x_n\}$ generated by (1.7) converges strongly to the common zeros of A_i $(i = 1, 2, \ldots, l)$ under some conditions.

Later, their work was extended to the following one presented by Hu and Liu in [13]:

$$x_0 \in C, \quad x_{n+1} = \alpha_n u + \beta_n x_n + \vartheta_n S_{r_n} x_n, \quad n \geq 0, \tag{1.8}$$

where $S_{r_n} = a_0 I + a_1 J_{r_n}^{A_1} + a_2 J_{r_n}^{A_2} + \cdots + a_l J_{r_n}^{A_l}$ with $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$ and $\sum_{i=0}^{l} a_i = 1$. $\{\alpha_n\}, \{\beta_n\}, \{\vartheta_n\} \subset (0,1)$ and $\alpha_n + \beta_n + \vartheta_n = 1$. If $\bigcap_{i=1}^{l} A_i^{-1}(0) \neq \emptyset$, they proved that $\{x_n\}$ converges strongly to the common zeros of A_i $(i = 1, 2, \ldots, l)$ under some conditions.

In this paper, based on the work of (1.6), (1.7), and (1.8), we present the following iterative scheme:

$$\begin{cases} x_1 \in C, \\ y_n = \beta_n f(x_n) + (1-\beta_n) S_{r_n}^{A_m A_{m-1} \cdots A_1} x_n, \\ u_n = \vartheta_n f(y_n) + (1-\vartheta_n) W_{r_n} y_n, \\ x_{n+1} = \alpha_n f(u_n) + (1-\alpha_n) u_n, \end{cases} \tag{A}$$

where $S_{r_n}^{A_m A_{m-1} \cdots A_1} := J_{r_n}^{A_m} J_{r_n}^{A_{m-1}} \cdots J_{r_n}^{A_1}$, $W_{r_n} = a_0 I + a_1 J_{r_n}^{B_1} + a_2 J_{r_n}^{B_2} + \cdots + a_l J_{r_n}^{B_l}$, $J_{r_n}^{A_i} = (I + r_n A_i)^{-1}$ and $J_{r_n}^{B_j} = (I + r_n B_j)^{-1}$, for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, l$. $\sum_{k=0}^{l} a_k = 1$, $f : C \to C$ is a contraction, both $\{A_i\}_{i=1}^{m}$ and $\{B_j\}_{j=1}^{l}$ are finite families of m-accretive mappings. More details of iterative scheme (A) will be presented in Section 2. We shall prove a weak convergent theorem and a strong convergent theorem under different assumptions on $\{\alpha_n\}$, $\{\beta_n\}$, $\{\vartheta_n\}$, and $\{r_n\}$, respectively.

In order to prove our main results, we need the following lemmas.

By using the properties of the metric projection and m-accretive mappings, we can easily prove the following two lemmas.

Lemma 1.1 *For $\forall x \in H$ and $\forall y \in C$, $\|P_C x - y\|^2 + \|P_C x - x\|^2 \leq \|y - x\|^2$.*

Lemma 1.2 *For $\forall y \in A^{-1}0$, $\forall x \in H$ and $r > 0$,*

$$\left\|(I + rA)^{-1}x - y\right\|^2 + \left\|(I + rA)^{-1}x - x\right\|^2 \leq \|y - x\|^2.$$

Lemma 1.3 ([14]) *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq a_n + b_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} b_n < +\infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 1.4 ([15]) *Let H be a real Hilbert space and A be an m-accretive mapping. For $\lambda, \mu > 0$ and $x \in H$, we have*

$$J_\lambda^A x = J_\mu^A\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^A x\right),$$

where $J_\lambda^A = (I + \lambda A)^{-1}$ and $J_\mu^A = (I + \mu A)^{-1}$.

Lemma 1.5 ([16]) *Let H be a real Hilbert space and C be a closed convex subset of H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$, and $f \in \sum_C$. Then z_t, defined by*

$$z_t = tf(z_t) + (1 - t)Tz_t, \quad z_t \in C,$$

converges strongly to a point in $F(T)$. If one defines $Q : \sum_C \to F(T)$ by $Q(f) := \lim_{t\to 0} z_t$, $f \in \sum_C$, then $Q(f)$ solves the following variational inequality:

$$\langle (I - f)Q(f), Q(f) - p\rangle \leq 0, \quad \forall p \in F(T).$$

Lemma 1.6 ([17]) *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three sequences of nonnegative real numbers satisfying*

$$a_{n+1} \leq (1 - c_n)a_n + b_n c_n, \quad \forall n \geq 1,$$

where $\{c_n\} \subset (0,1)$ such that

(i) $c_n \to 0$ *and* $\sum_{n=1}^\infty c_n = +\infty$,

(ii) *either* $\limsup_{n\to\infty} b_n \leq 0$ *or* $\sum_{n=1}^\infty |b_n c_n| < +\infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.7 *In a Hilbert space H, we can easily get the following inequality*:

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in H.$$

2 Weak and strong convergence theorems

Lemma 2.1 *Let H be a real Hilbert space, C be a nonempty closed and convex subset of H and $A_i, B_j : C \to C$ ($i = 1,2,\ldots,m$; $j = 1,2,\ldots,l$) be finitely many m-accretive mappings such that $D := (\bigcap_{i=1}^m A_i^{-1}0) \cap (\bigcap_{j=1}^l B_j^{-1}0) \neq \emptyset$. Suppose $S_r^{A_m A_{m-1}\cdots A_1} := J_r^{A_m} J_r^{A_{m-1}} \cdots J_r^{A_1}$ and $W_r := a_0 I + a_1 J_r^{B_1} + a_2 J_r^{B_2} + \cdots + a_l J_r^{B_l}$, where $J_r^{A_i} = (I + rA_i)^{-1}$ ($i = 1,2,\ldots,m$), $J_r^{B_j} = (I + rB_j)^{-1}$ ($j = 1,2,\ldots,l$), $a_k \in (0,1)$, $k = 0,1,\ldots,l$, $\sum_{k=0}^l a_k = 1$, and $r > 0$. Then $S_r^{A_m A_{m-1}\cdots A_1} : C \to C$ and $W_r : C \to C$ are nonexpansive.*

Lemma 2.1 can easily be obtained in view of the facts that $(I + rA_i)^{-1}$ and $(I + rB_j)^{-1}$ are nonexpansive, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,l$.

Theorem 2.1 *Let H, C, D, and $A_i, B_j : C \to C$ ($i = 1,2,\ldots,m$; $j = 1,2,\ldots,l$) be the same as those in Lemma* 2.1. *Suppose that $D \neq \emptyset$. Let $\{x_n\}$ be generated by the iterative scheme* (A). *If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\vartheta_n\}$ are three sequences in $[0,1)$ such that $\sum_{n=1}^\infty \alpha_n < +\infty$, $\sum_{n=1}^\infty \beta_n < +\infty$,*

$\sum_{n=1}^{\infty} \vartheta_n < +\infty$, $\{r_n\} \subset (0,+\infty)$ with $\lim_{n\to\infty} r_n = +\infty$ and $f: C \to C$ is a contraction with contractive constant $k \in (0,1)$. Then $\{x_n\}$ converges weakly to a point $v_0 \in D$ satisfying

$$\lim_{n\to\infty} \|x_n - v_0\| = \min_{y\in D} \lim_{n\to\infty} \|x_n - y\|. \tag{2.1}$$

Proof We split our proof into five steps.

Step 1. $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are all bounded.

We can easily know that $\bigcap_{i=1}^{m} A_i^{-1}0 \subset F(S_{r_n}^{A_m\cdots A_1})$, and $\bigcap_{j=1}^{l} B_j^{-1}0 \subset F(W_{r_n})$. Then for $\forall p \in D$, from Lemma 2.1, we have

$$\|S_{r_n}^{A_m\cdots A_1}x_n - p\| = \|S_{r_n}^{A_m\cdots A_1}x_n - S_{r_n}^{A_m\cdots A_1}p\| \le \|x_n - p\|. \tag{2.2}$$

Based on (2.2), we know that

$$\begin{aligned}\|y_n - p\| &\le \beta_n\|f(x_n) - p\| + (1-\beta_n)\|S_{r_n}^{A_m\cdots A_1}x_n - p\| \\ &\le [1 - \beta_n(1-k)]\|x_n - p\| + \beta_n\|f(p) - p\|.\end{aligned} \tag{2.3}$$

Then (2.3) and Lemma 2.1 imply that

$$\begin{aligned}\|u_n - p\| &\le \vartheta_n\|f(y_n) - f(p)\| + \vartheta_n\|f(p) - p\| + (1-\vartheta_n)\|y_n - p\| \\ &\le [1 - \beta_n(1-k)][1 - \vartheta_n(1-k)]\|x_n - p\| \\ &\quad + [\vartheta_n + \beta_n - \vartheta_n\beta_n(1-k)]\|f(p) - p\|.\end{aligned} \tag{2.4}$$

Using (2.4), we know that

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n\|f(u_n) - f(p)\| + \alpha_n\|f(p) - p\| + (1-\alpha_n)\|u_n - p\| \\ &\le [1 - \beta_n(1-k)][1 - \alpha_n(1-k)][1 - \vartheta_n(1-k)]\|x_n - p\| \\ &\quad + \{[1 - \alpha_n(1-k)][\vartheta_n + \beta_n - \vartheta_n\beta_n(1-k)] + \alpha_n\}\|f(p) - p\| \\ &\le \|x_n - p\| + (\vartheta_n + \beta_n + \alpha_n)\|f(p) - p\|.\end{aligned} \tag{2.5}$$

Then Lemma 1.3 implies that $\lim_{n\to\infty} \|x_n - p\|$ exists, which ensures that $\{x_n\}$ is bounded. Combining with the fact that f is a contraction and noticing (2.2), (2.3), and (2.4), we can easily know that $\{f(x_n)\}$, $\{u_n\}$, $\{y_n\}$, $\{f(u_n)\}$, $\{f(y_n)\}$, $\{S_{r_n}^{A_i\cdots A_1}x_n\}$ ($i = 1,2,\ldots,m$), and $\{J_{r_n}^{B_j}x_n\}$ ($j = 1,2,\ldots,l$) are all bounded.

We may let $M_1 = \max\{\sup\{\|x_n\| : n \ge 1\}, \sup\{\|y_n\| : n \ge 1\}, \sup\{\|u_n\| : n \ge 1\}, \sup\{\|f(x_n)\| : n \ge 1\}, \sup\{\|f(y_n)\| : n \ge 1\}, \sup\{\|f(u_n)\| : n \ge 1\}, \sup\{\|S_{r_n}^{A_i\cdots A_1}x_n\| : n \ge 1, i = 1,2,\ldots,m\}, \sup\{\|J_{r_n}^{B_j}x_n\| : n \ge 1, j = 1,2,\ldots,l\}\}$.

Step 2. $\lim_{n\to\infty} \|P_D x_n - x_n\|$ exists.

In fact, it follows from the property of P_D that

$$\|P_D x_{n+1} - x_{n+1}\| \le \|P_D x_n - x_{n+1}\|. \tag{2.6}$$

In view of Lemma 1.1, we know that for $\forall v \in D$,

$$\|v - P_D x_n\|^2 \le \|v - x_n\|^2 - \|P_D x_n - x_n\|^2 \le \|x_n - v\|^2, \tag{2.7}$$

which implies that $\{P_D x_n\}$ is bounded since $\{x_n\}$ is bounded from step 1. Then $\{f(P_D x_n)\}$ is also bounded.

Let $M_2 = \max\{\sup\{\|P_D x_n\| : n \geq 1\}, \sup\{\|f(P_D x_n)\| : n \geq 1\}\}$.

Noticing (2.5) and (2.6), we have

$$\begin{aligned}\|x_{n+1} - P_D x_{n+1}\| &\leq \|x_n - P_D x_n\| + (\vartheta_n + \beta_n + \alpha_n)\big\|f(P_D x_n) - P_D x_n\big\| \\ &\leq \|x_n - P_D x_n\| + 2(\vartheta_n + \beta_n + \alpha_n)M_2.\end{aligned}$$

Therefore, in view of Lemma 1.3, $\lim_{n\to\infty} \|P_D x_n - x_n\|$ exists.

Step 3. $P_D x_n \to v_0$, where $v_0 \in D$ satisfies (2.1), as $n \to \infty$.

We first claim that there exists a unique element $v_0 \in D$ such that

$$\lim_{n\to\infty} \|x_n - v_0\| = \min_{y\in D} \lim_{n\to\infty} \|x_n - y\|.$$

In fact, if we let $h(y) = \lim_{n\to\infty} \|x_n - y\|$, $\forall y \in D$. Then we can easily find that $h(y) : D \to R^+$ is proper, strictly convex and lower-semi-continuous and $h(y) \to +\infty$ as $\|y\| \to +\infty$. This ensures that there exists a unique element $v_0 \in D$ such that $h(v_0) = \min_{y\in D} h(y)$.

From (2.7), we know that

$$\lim_{n\to\infty} \|v_0 - P_D x_n\|^2 \leq \lim_{n\to\infty} \big(\|v_0 - x_n\|^2 - \|P_D x_n - x_n\|^2\big) = h^2(v_0) - \lim_{n\to\infty} \|P_D x_n - x_n\|^2 \leq 0.$$

Therefore, $P_D x_n \to v_0$, as $n \to \infty$.

Step 4. $\omega(x_n) \subset D$, where $\omega(x_n)$ denotes the set consisting all of the weak limit points of $\{x_n\}$.

Since $\{x_n\}$ is bounded, then there exists a subsequence of $\{x_n\}$, for simplicity, we still denote it by $\{x_n\}$, such that $x_n \rightharpoonup x$, as $n \to \infty$.

Since $\|\cdot\|$ is convex, by using Lemma 1.2 and noticing (2.3), we have, for $\forall p \in D$,

$$\begin{aligned}\|y_n - p\|^2 &\leq \beta_n\big\|f(x_n) - p\big\|^2 + (1-\beta_n)\big\|S_{r_n}^{A_m\cdots A_1}x_n - p\big\|^2 \\ &\leq \beta_n\big\|f(x_n) - p\big\|^2 + (1-\beta_n)\big[\big\|S_{r_n}^{A_{m-1}\cdots A_1}x_n - p\big\|^2 \\ &\quad - \big\|S_{r_n}^{A_m\cdots A_1}x_n - S_{r_n}^{A_{m-1}\cdots A_1}x_n\big\|^2\big] \\ &\leq \beta_n k\|x_n - p\|^2 + (1-\beta_n)\big[\|x_n - p\|^2 - \big\|S_{r_n}^{A_m\cdots A_1}x_n - S_{r_n}^{A_{m-1}\cdots A_1}x_n\big\|^2\big] \\ &\quad + \beta_n\big\|f(p) - p\big\|^2 + 2\beta_n k\|x_n - p\|\big\|f(p) - p\big\| \\ &\leq \|x_n - p\|^2 - (1-\beta_n)\big\|S_{r_n}^{A_m\cdots A_1}x_n - S_{r_n}^{A_{m-1}\cdots A_1}x_n\big\|^2 \\ &\quad + \beta_n\big\|f(p) - p\big\|^2 + 2\beta_n k\|x_n - p\|\big\|f(p) - p\big\|.\end{aligned} \tag{2.8}$$

Then using (2.8), we have

$$\begin{aligned}&\|u_n - p\|^2 \\ &\quad \leq \vartheta_n\big\|f(y_n) - p\big\|^2 + (1-\vartheta_n)\|W_{r_n}y_n - W_{r_n}p\|^2 \\ &\quad \leq \big[1 - \vartheta_n(1-k)\big]\|y_n - p\|^2 + 2\vartheta_n k\|y_n - p\|\big\|f(p) - p\big\| + \vartheta_n\big\|f(p) - p\big\|^2\end{aligned}$$

$$\begin{aligned}&\le \|x_n-p\|^2-(1-\beta_n)\big\|S_{r_n}^{A_m\cdots A_1}x_n-S_{r_n}^{A_{m-1}\cdots A_1}x_n\big\|^2\\&\quad+(\vartheta_n+\beta_n)\big\|f(p)-p\big\|^2+2k\big(\beta_n\|x_n-p\|+\vartheta_n\|y_n-p\|\big)\big\|f(p)-p\big\|,\end{aligned}\tag{2.9}$$

which implies that

$$\begin{aligned}&\|x_{n+1}-p\|^2\\&\quad\le\big[1-\alpha_n(1-k)\big]\|u_n-p\|^2+2\alpha_nk\|u_n-p\|\big\|f(p)-p\big\|+\alpha_n\big\|f(p)-p\big\|^2\\&\quad\le\|x_n-p\|^2-(1-\beta_n)\big\|S_{r_n}^{A_m\cdots A_1}x_n-S_{r_n}^{A_{m-1}\cdots A_1}x_n\big\|^2\\&\qquad+(\alpha_n+\beta_n+\vartheta_n)\big\|f(p)-p\big\|^2\\&\qquad+2k\big(\alpha_n\|u_n-p\|+\beta_n\|x_n-p\|+\vartheta_n\|y_n-p\|\big)\big\|f(p)-p\big\|.\end{aligned}\tag{2.10}$$

Thus

$$\begin{aligned}&0\le(1-\beta_n)\big\|S_{r_n}^{A_m\cdots A_1}x_n-S_{r_n}^{A_{m-1}\cdots A_1}x_n\big\|^2\\&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n+\beta_n+\vartheta_n)\big\|f(p)-p\big\|^2\\&\qquad+2k\big(\alpha_n\|u_n-p\|+\beta_n\|x_n-p\|+\vartheta_n\|y_n-p\|\big)\big\|f(p)-p\big\|.\end{aligned}\tag{2.11}$$

Since from the proof of step 1, we know that $\lim_{n\to\infty}\|x_n-p\|$ exists, then $S_{r_n}^{A_m\cdots A_1}x_n-S_{r_n}^{A_{m-1}\cdots A_1}x_n\to 0$, as $n\to\infty$.

Going back to (2.8) again, we know that

$$\begin{aligned}\|y_n-p\|^2&\le\beta_n\big\|f(x_n)-p\big\|^2+(1-\beta_n)\big\|S_{r_n}^{A_{m-1}\cdots A_1}x_n-p\big\|^2\\&\le\beta_n\big\|f(x_n)-p\big\|^2\\&\quad+(1-\beta_n)\big[\big\|S_{r_n}^{A_{m-2}\cdots A_1}x_n-p\big\|^2-\big\|S_{r_n}^{A_{m-1}\cdots A_1}x_n-S_{r_n}^{A_{m-2}\cdots A_1}x_n\big\|^2\big]\\&\le\beta_nk\|x_n-p\|^2+(1-\beta_n)\big[\|x_n-p\|^2-\big\|S_{r_n}^{A_{m-1}\cdots A_1}x_n-S_{r_n}^{A_{m-2}\cdots A_1}x_n\big\|^2\big]\\&\quad+\beta_n\big\|f(p)-p\big\|^2+2\beta_nk\|x_n-p\|\big\|f(p)-p\big\|\\&\le\|x_n-p\|^2-(1-\beta_n)\big\|S_{r_n}^{A_{m-1}\cdots A_1}x_n-S_{r_n}^{A_{m-2}\cdots A_1}x_n\big\|^2\\&\quad+\beta_n\big\|f(p)-p\big\|^2+2\beta_nk\|x_n-p\|\big\|f(p)-p\big\|.\end{aligned}\tag{2.12}$$

Then using (2.12), repeating the processes of (2.9)-(2.11), we know that

$$S_{r_n}^{A_{m-1}\cdots A_1}x_n-S_{r_n}^{A_{m-2}\cdots A_1}x_n\to 0,\quad\text{as }n\to\infty.$$

By using the inductive method, we have the following results:

$$\begin{aligned}&S_{r_n}^{A_{m-2}\cdots A_1}x_n-S_{r_n}^{A_{m-3}\cdots A_1}x_n\to 0,\\&\cdots\\&(I+r_nA_1)^{-1}x_n-x_n\to 0,\end{aligned}$$

as $n \to \infty$. Therefore, $(I + r_nA_1)^{-1}x_n \rightharpoonup x, \ldots, S_{r_n}^{A_mA_{m-1}\cdots A_1}x_n = (I + r_nA_m)^{-1}\cdots(I + r_nA_1)^{-1}$ $x_n \rightharpoonup x$, as $n \to \infty$.

Let $v_{n,1} = (I+r_nA_1)^{-1}x_n$, then $A_1v_{n,1} = \frac{x_n-v_{n,1}}{r_n} \to 0$, since $r_n \to +\infty$ and both $\{x_n\}$ and $\{v_{n,1}\}$ are bounded. This ensures that $x \in A_1^{-1}0$.

Let $v_{n,2} = (I + r_nA_2)^{-1}(I + r_nA_1)^{-1}x_n = (I + r_nA_2)^{-1}v_{n,1}$, then $A_2v_{n,2} = \frac{v_{n,1}-v_{n,2}}{r_n} \to 0$, which implies that $x \in A_2^{-1}0$.

By induction, let $v_{n,m} = (I + r_nA_m)^{-1}\cdots(I + r_nA_1)^{-1}x_n = (I + r_nA_m)^{-1}v_{n,m-1}$, then $A_mv_{n,m} = \frac{v_{n,m-1}-v_{n,m}}{r_n} \to 0$, which implies that $x \in A_m^{-1}0$. Thus $x \in \bigcap_{i=1}^m A_i^{-1}0$.

Next, we shall show that $x \in \bigcap_{j=1}^l B_j^{-1}0$.

From step 1, we may assume that there exists $M_3 > 0$ such that $2\|x_n - p\|\|f(p) - p\| + \|f(p)-p\|^2 \le M_3$, $2\|y_n-p\|\|f(p)-p\| + \|f(p)-p\|^2 \le M_3$ and $2\|u_n-p\|\|f(p)-p\| + \|f(p)-p\|^2 \le M_3$.

Now, computing the following, $\forall p \in D$:

$$\begin{aligned} \|y_n - p\|^2 &\le \bigl[1 - \beta_n(1-k)\bigr]\|x_n - p\|^2 + \beta_n\bigl\|f(p) - p\bigr\|^2 \\ &\quad + 2\beta_nk\|x_n - p\|\bigl\|f(p) - p\bigr\| \\ &\le \bigl[1 - \beta_n(1-k)\bigr]\|x_n - p\|^2 + \beta_nM_3. \end{aligned} \tag{2.13}$$

By using Lemma 1.2,

$$\begin{aligned} \|u_n - p\|^2 &\le k\vartheta_n\|y_n - p\|^2 + 2\vartheta_nk\bigl\|f(p) - p\bigr\|\|y_n - p\| + \vartheta_n\bigl\|f(p) - p\bigr\|^2 \\ &\quad + (1-\vartheta_n)\left(a_0\|y_n - p\|^2 + \sum_{j=1}^l a_j\bigl\|(I + r_nB_j)^{-1}y_n - p\bigr\|^2\right) \\ &\le k\vartheta_n\|y_n - p\|^2 + 2\vartheta_nk\bigl\|f(p) - p\bigr\|\|y_n - p\| + \vartheta_n\bigl\|f(p) - p\bigr\|^2 \\ &\quad + (1-\vartheta_n)\left[a_0\|y_n - p\|^2 + \sum_{j=1}^l a_j\bigl(\|y_n - p\|^2 - \bigl\|(I + r_nB_j)^{-1}y_n - y_n\bigr\|^2\bigr)\right] \\ &= \bigl[1 - \vartheta_n(1-k)\bigr]\|y_n - p\|^2 + 2\vartheta_nk\|y_n - p\|\bigl\|f(p) - p\bigr\| + \vartheta_n\bigl\|f(p) - p\bigr\|^2 \\ &\quad - (1-\vartheta_n)\sum_{j=1}^l a_j\bigl\|(I + r_nB_j)^{-1}y_n - y_n\bigr\|^2 \\ &\le \|y_n - p\|^2 - (1-\vartheta_n)\sum_{j=1}^l a_j\bigl\|(I + r_nB_j)^{-1}y_n - y_n\bigr\|^2 + \vartheta_nM_3. \end{aligned} \tag{2.14}$$

Then (2.13) and (2.14) imply that

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &\quad \le \bigl[1 - \alpha_n(1-k)\bigr]\|u_n - p\|^2 + 2\alpha_nk\|u_n - p\|\bigl\|f(p) - p\bigr\| + \alpha_n\bigl\|f(p) - p\bigr\|^2 \\ &\quad \le \bigl[1 - \alpha_n(1-k)\bigr]\|u_n - p\|^2 + \alpha_nM_3 \\ &\quad \le \bigl[1 - \alpha_n(1-k)\bigr]\left[\|y_n - p\|^2 - (1-\vartheta_n)\sum_{j=1}^l a_j\bigl\|(I + r_nB_j)^{-1}y_n - y_n\bigr\|^2 + \vartheta_nM_3\right] + \alpha_nM_3 \end{aligned}$$

$$\begin{aligned}&\leq \big[1-\alpha_n(1-k)\big]\big[1-\beta_n(1-k)\big]\|x_n-p\|^2+\big[1-\alpha_n(1-k)\big]M_3(\beta_n+\vartheta_n)+\alpha_n M_3\\&\quad-\big[1-\alpha_n(1-k)\big](1-\vartheta_n)\sum_{j=1}^{l}a_j\big\|(I+r_nB_j)^{-1}y_n-y_n\big\|^2.\end{aligned}\tag{2.15}$$

From step 1, we know that $\lim_{n\to\infty}\|x_n-p\|$ exists, then (2.15) implies that

$$(I+r_nB_j)^{-1}y_n-y_n\to 0,\quad \text{as } n\to\infty,\ \text{for } j=1,2,\ldots,l.\tag{2.16}$$

From the iterative scheme (A), $\beta_n\to 0$, and the results of step 1, we know that

$$y_n-S_{r_n}^{A_mA_{m-1}\cdots A_1}x_n=\beta_n\big(f(x_n)-S_{r_n}^{A_mA_{m-1}\cdots A_1}x_n\big)\to 0,\quad \text{as } n\to\infty.$$

Then $y_n\rightharpoonup x$, since $S_{r_n}^{A_mA_{m-1}\cdots A_1}x_n\rightharpoonup x$, as $n\to\infty$.

Thus from (2.16), we have $(I+r_nB_j)^{-1}y_n\rightharpoonup x$, imitating the proof of $x\in\bigcap_{i=1}^m A_i^{-1}0$, we can see that $x\in\bigcap_{j=1}^l B_j^{-1}0$, and then $x\in D$.

Step 5. $x_n\rightharpoonup v_0=\lim_{n\to\infty}P_Dx_n$.

In fact, for $\forall y\in D$,

$$\langle P_Dx_n-y,P_Dx_n-x_n\rangle\leq 0.\tag{2.17}$$

From step 3, we know that $P_Dx_n\to v_0$, as $n\to\infty$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$ which is weakly convergent to x_0. Then $x_0\in D$ from step 4. Taking the limits on both sides of (2.17), we know that $\langle v_0-y,v_0-x_0\rangle\leq 0$.

Letting $y=x_0$, we have $x_0=v_0$.

Supposing $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ such that $x_{n_j}\rightharpoonup x_1$ as $j\to\infty$. Then repeating the above proof, we have $x_1=v_0$. Since all of the weakly convergent subsequences of $\{x_n\}$ converge to the same element v_0, then the whole sequence $\{x_n\}$ converges weakly to v_0.

This completes the proof. □

Remark 2.1 To prove the strong convergence result in Theorem 2.2, we need to prove the following two lemmas first and some new proof techniques can be seen.

Lemma 2.2 *Let $H,C,D,A_i,B_j:C\to C$ $(i=1,2,\ldots,m;j=1,2,\ldots,l)$, $S_r^{A_mA_{m-1}\cdots A_1}$ and W_r be the same as those in Lemma* 2.1. *Suppose that $D\neq\emptyset$. Then $F(S_r^{A_mA_{m-1}\cdots A_1})=\bigcap_{i=1}^m A_i^{-1}0$ and $F(W_r)=\bigcap_{j=1}^l B_j^{-1}0$, for $\forall r>0$.*

Proof It is easy to check $\bigcap_{i=1}^m A_i^{-1}0\subset F(S_r^{A_mA_{m-1}\cdots A_1})$ and $\bigcap_{j=1}^l B_j^{-1}0\subset F(W_r)$, for $\forall r>0$.

Next, we shall show that $F(W_r)\subset\bigcap_{j=1}^l B_j^{-1}0$.

For $\forall p\in F(W_r)$, $\forall q\in\bigcap_{j=1}^l B_j^{-1}0$. Since $\bigcap_{j=1}^l B_j^{-1}0\subset F(W_r)$, then $q=W_rq$. Thus

$$\begin{aligned}\|q-p\|&=\big\|a_0(q-p)+a_1\big(J_r^{B_1}q-J_r^{B_1}p\big)+\cdots+a_l\big(J_r^{B_l}q-J_r^{B_l}p\big)\big\|\\&\leq a_0\|q-p\|+a_1\big\|J_r^{B_1}q-J_r^{B_1}p\big\|+\cdots+a_l\big\|J_r^{B_l}q-J_r^{B_l}p\big\|\\&\leq\|q-p\|.\end{aligned}$$

Then $a_0(\|q-p\|-\|q-p\|)+a_1(\|q-p\|-\|J_r^{B_1}q-J_r^{B_1}p\|)+\cdots+a_l(\|q-p\|-\|J_r^{B_l}q-J_r^{B_l}p\|)=0$. Since $\|q-p\|-\|J_r^{B_j}q-J_r^{B_j}p\|\geq 0$, $j=1,2,\ldots,l$, then $\|q-p\|-\|J_r^{B_j}q-J_r^{B_j}p\|=0$, $j=1,2,\ldots,l$. That is,

$$\|q-p\|=\left\|J_r^{B_j}q-J_r^{B_j}p\right\|=\left\|q-J_r^{B_j}p\right\|,\quad j=1,2,\ldots,l. \tag{2.18}$$

By using Lemma 1.2 and (2.18), we know that $\|p-J_r^{B_j}p\|^2\leq\|q-p\|^2-\|q-J_r^{B_j}p\|^2=0$, $j=1,2,\ldots,l$. Thus $p=J_r^{B_j}p$, which implies that $p\in B_j^{-1}0$, $j=1,2,\ldots,l$. Then $F(W_r)\subset\bigcap_{j=1}^l B_j^{-1}0$, for $r>0$.

Finally, we shall show that $F(S_r^{A_mA_{m-1}\cdots A_1})\subset\bigcap_{i=1}^m A_i^{-1}0$.

For $\forall p\in F(S_r^{A_mA_{m-1}\cdots A_1})$, then $p=S_r^{A_mA_{m-1}\cdots A_1}p$. Let $q\in\bigcap_{i=1}^m A_i^{-1}0$, then $q=S_r^{A_mA_{m-1}\cdots A_1}q$, since $\bigcap_{i=1}^m A_i^{-1}0\subset F(S_r^{A_mA_{m-1}\cdots A_1})$. Therefore,

$$\begin{aligned}\|q-p\|&=\left\|S_r^{A_mA_{m-1}\cdots A_1}q-S_r^{A_mA_{m-1}\cdots A_1}p\right\|\\&\leq\left\|S_r^{A_{m-1}A_{m-2}\cdots A_1}q-S_r^{A_{m-1}A_{m-2}\cdots A_1}p\right\|\\&\leq\left\|S_r^{A_{m-2}A_{m-3}\cdots A_1}q-S_r^{A_{m-2}A_{m-3}\cdots A_1}p\right\|\\&\leq\cdots\\&\leq\left\|(I+rA_1)^{-1}q-(I+rA_1)^{-1}p\right\|\leq\|q-p\|.\end{aligned} \tag{2.19}$$

From (2.19), we know that

$$\left\|q-(I+rA_1)^{-1}p\right\|=\|q-p\|. \tag{2.20}$$

Noticing that (2.20) and (2.18) have the same form, then repeating the proof of $p=J_r^{B_j}p$, we know that $p=(I+rA_1)^{-1}p$ and then $p\in A_1^{-1}0$.

Since $p\in A_1^{-1}0$, using (2.19) again, we know that

$$\begin{aligned}\|q-p\|&=\left\|(I+rA_2)^{-1}(I+rA_1)^{-1}q-(I+rA_2)^{-1}(I+rA_1)^{-1}p\right\|\\&=\left\|q-(I+rA_2)^{-1}p\right\|.\end{aligned} \tag{2.21}$$

Repeating the above proof again, $p\in A_2^{-1}0$.

By induction, we have $p\in A_m^{-1}0$. Therefore, $F(S_r^{A_mA_{m-1}\cdots A_1})\subset\bigcap_{i=1}^m A_i^{-1}0$.

This completes the proof. □

Lemma 2.3 *Let $H,C,D,A_i,B_j:C\to C$ $(i=1,2,\ldots,m;\ j=1,2,\ldots,l)$, $S_r^{A_mA_{m-1}\cdots A_1}$ and W_r be the same as those in Lemma* 2.1. *Suppose that $D\neq\emptyset$. Then $W_rS_r^{A_mA_{m-1}\cdots A_1}:C\to C$ is nonexpansive and $F(W_rS_r^{A_mA_{m-1}\cdots A_1})=D$, for $\forall r>0$.*

Proof It is easy to check that $W_rS_r^{A_mA_{m-1}\cdots A_1}:C\to C$ is nonexpansive. We are left to show that $F(W_rS_r^{A_mA_{m-1}\cdots A_1})=D$.

$\forall p\in D$, then, from Lemma 2.2, $p=S_r^{A_mA_{m-1}\cdots A_1}p$ and $p=W_rp$. Thus $p=W_rS_r^{A_mA_{m-1}\cdots A_1}p$, which implies that $D\subset F(W_rS_r^{A_mA_{m-1}\cdots A_1})$.

On the other hand, let $p \in F(W_r S_r^{A_m A_{m-1} \cdots A_1})$, then $p = W_r S_r^{A_m A_{m-1} \cdots A_1} p$. Let $q \in D$, then $q = W_r S_r^{A_m A_{m-1} \cdots A_1} q$, since $D \subset F(W_r S_r^{A_m A_{m-1} \cdots A_1})$. Then Lemma 2.1 ensures that

$$\begin{aligned}\|p-q\| &\leq \left\|S_r^{A_m A_{m-1}\cdots A_1} p - S_r^{A_m A_{m-1}\cdots A_1} q\right\| \\ &\leq \left\|S_r^{A_{m-1}\cdots A_1} p - S_r^{A_{m-1}\cdots A_1} q\right\| \\ &\leq \cdots \leq \left\|J_r^{A_1} p - J_r^{A_1} q\right\| \leq \|p-q\|,\end{aligned}$$

which implies that

$$\left\|J_r^{A_1} p - q\right\| = \left\|S_r^{A_2 A_1} p - q\right\| = \cdots = \left\|S_r^{A_m A_{m-1}\cdots A_1} p - q\right\| = \|p-q\|.$$

Using the same method as that in Lemma 2.2, $p \in \bigcap_{i=1}^m A_i^{-1}0$. Thus $p = S_r^{A_m A_{m-1} \cdots A_1} p$. Since $p = W_r S_r^{A_m A_{m-1} \cdots A_1} p$, then $p = W_r p$, which implies that $p \in \bigcap_{j=1}^l B_j^{-1}0$ from Lemma 2.2. Therefore, $F(W_r S_r^{A_m A_{m-1} \cdots A_1}) \subset D$.

This completes the proof. □

Theorem 2.2 *Suppose H, D, C, $\{A_i\}_{i=1}^m$, $\{B_j\}_{j=1}^l$ and f are the same as those in Theorem* 2.1. *Let $\{x_n\}$ be generated by the iterative scheme* (A). *If $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\vartheta_n\}$ are three sequences in* $(0,1)$ *and $\{r_n\} \subset (0,+\infty)$ satisfy the following conditions*:

(i) $\sum_{n=1}^\infty |\alpha_{n+1} - \alpha_n| < +\infty$, *and* $\alpha_n \to 0$, *as* $n \to \infty$;
(ii) $\sum_{n=1}^\infty \beta_n = +\infty$, $\sum_{n=1}^\infty |\beta_{n+1} - \beta_n| < +\infty$, *and* $\beta_n \to 0$, *as* $n \to \infty$;
(iii) $\sum_{n=1}^\infty |\vartheta_{n+1} - \vartheta_n| < +\infty$, *and* $\vartheta_n \to 0$, *as* $n \to \infty$;
(iv) $\sum_{n=1}^\infty |r_{n+1} - r_n| < +\infty$, *and* $r_n \to r^* \geq \varepsilon > 0$, *as* $n \to \infty$.

Then $\{x_n\}$ converges strongly to a point $p_0 \in D$, which is the unique solution of the following variational inequality:

$$\langle f(p_0) - p_0, p_0 - q\rangle \geq 0, \quad \forall q \in D. \tag{2.22}$$

Proof We shall split the proof into five steps:

Step 1. $\{x_n\}$ is bounded.

$$\begin{aligned}&\forall p \in D, \quad \|y_n - p\| \leq \left[1 - \beta_n(1-k)\right]\|x_n - p\| + \beta_n\left\|f(p) - p\right\|, \\ &\|u_n - p\| \leq \left[1 - \vartheta_n(1-k)\right]\|y_n - p\| + \vartheta_n\left\|f(p) - p\right\|.\end{aligned}$$

Letting $\delta_n = \alpha_n + \beta_n + \vartheta_n - (\alpha_n\beta_n + \alpha_n\vartheta_n + \beta_n\vartheta_n)(1-k) + \alpha_n\beta_n\vartheta_n(1-k)^2$. Then

$$\begin{aligned}\|x_{n+1} - p\| &\leq \left[1 - \alpha_n(1-k)\right]\|u_n - p\| + \alpha_n\left\|f(p) - p\right\| \\ &\leq \left[1 - \alpha_n(1-k)\right]\left[1 - \beta_n(1-k)\right]\left[1 - \vartheta_n(1-k)\right]\|x_n - p\| \\ &\quad + \left\{\left[1 - \alpha_n(1-k)\right]\vartheta_n + \alpha_n + \left[1 - \alpha_n(1-k)\right]\left[1 - \vartheta_n(1-k)\right]\beta_n\right\}\left\|f(p) - p\right\| \\ &= \left[1 - \delta_n(1-k)\right]\|x_n - p\| + \delta_n\left\|f(p) - p\right\| \\ &\leq \max\left\{\|x_n - p\|, \frac{1}{1-k}\left\|f(p) - p\right\|\right\}, \quad n \geq 1.\end{aligned}$$

By induction, $\|x_n - p\| \leq \max\{\|x_1 - p\|, \frac{1}{1-k}\|f(p) - p\|\}$, $n \geq 1$. Thus $\{x_n\}$ is bounded.

Step 2. $\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0$ and $\lim_{n\to\infty}\|x_n-u_n\| = 0$.
In fact,

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad\le |\beta_n-\beta_{n-1}|\left\|f(x_n)-S_{r_n}^{A_m\cdots A_1}x_n\right\| + \beta_{n-1}\left\|f(x_n)-f(x_{n-1})\right\| \\
&\qquad + (1-\beta_{n-1})\left\|S_{r_n}^{A_m\cdots A_1}x_n - S_{r_{n-1}}^{A_m\cdots A_1}x_{n-1}\right\| \\
&\quad\le 2M_1|\beta_n-\beta_{n-1}| + \beta_{n-1}k\|x_n-x_{n-1}\| \\
&\qquad + (1-\beta_{n-1})\left\|S_{r_n}^{A_m\cdots A_1}x_n - S_{r_{n-1}}^{A_m\cdots A_1}x_{n-1}\right\|.
\end{aligned}\tag{2.23}$$

Next we discuss $\|S_{r_n}^{A_m\cdots A_1}x_n - S_{r_{n-1}}^{A_m\cdots A_1}x_{n-1}\|$.
If $r_{n-1}\le r_n$, then in view of Lemma 1.4,

$$\begin{aligned}
&\left\|J_{r_n}^{A_1}x_n - J_{r_{n-1}}^{A_1}x_{n-1}\right\| \\
&\quad= \left\|J_{r_{n-1}}^{A_1}\left(\frac{r_{n-1}}{r_n}x_n + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}^{A_1}x_n\right) - J_{r_{n-1}}^{A_1}x_{n-1}\right\| \\
&\quad\le \left\|\frac{r_{n-1}}{r_n}x_n + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}^{A_1}x_n - x_{n-1}\right\| \\
&\quad\le \frac{r_{n-1}}{r_n}\|x_n-x_{n-1}\| + \left(1-\frac{r_{n-1}}{r_n}\right)\left\|J_{r_n}^{A_1}x_n - x_{n-1}\right\| \\
&\quad\le \|x_n-x_{n-1}\| + \frac{r_n-r_{n-1}}{\varepsilon}\left\|J_{r_n}^{A_1}x_n - x_{n-1}\right\|.
\end{aligned}\tag{2.24}$$

For $\forall p\in D$, let $M_4 = M_1 + \|p\|$, then

$$\begin{aligned}
&\left\|J_{r_n}^{A_1}x_n - x_{n-1}\right\| \\
&\quad\le \left\|(I+r_nA_1)^{-1}x_n - p\right\| + \|p-x_{n-1}\| \\
&\quad\le \|x_n-p\| + \|p-x_{n-1}\| \le 2M_4.
\end{aligned}\tag{2.25}$$

From (2.24) and (2.25), we know that

$$\left\|J_{r_n}^{A_1}x_n - J_{r_{n-1}}^{A_1}x_{n-1}\right\| \le \|x_n-x_{n-1}\| + 2M_4\frac{r_n-r_{n-1}}{\varepsilon}.\tag{2.26}$$

Notice that $S_{r_n}^{A_2A_1}x_n = J_{r_n}^{A_2}J_{r_n}^{A_1}x_n$ and $S_{r_{n-1}}^{A_2A_1}x_{n-1} = J_{r_{n-1}}^{A_2}J_{r_{n-1}}^{A_1}x_{n-1}$; similar to (2.26), we have

$$\left\|S_{r_n}^{A_2A_1}x_n - S_{r_{n-1}}^{A_2A_1}x_{n-1}\right\| \le \left\|J_{r_n}^{A_1}x_n - J_{r_{n-1}}^{A_1}x_{n-1}\right\| + 2M_4\frac{r_n-r_{n-1}}{\varepsilon}.\tag{2.27}$$

Following from (2.26) and (2.27), we have

$$\left\|S_{r_n}^{A_2A_1}x_n - S_{r_{n-1}}^{A_2A_1}x_{n-1}\right\| \le \|x_n-x_{n-1}\| + 2\times 2M_4\frac{r_n-r_{n-1}}{\varepsilon}.$$

Then by induction, we can get the following result:

$$\left\|S_{r_n}^{A_m\cdots A_1}x_n - S_{r_{n-1}}^{A_m\cdots A_1}x_{n-1}\right\| \le \|x_n-x_{n-1}\| + 2\times mM_4\frac{r_n-r_{n-1}}{\varepsilon}.\tag{2.28}$$

Putting (2.28) into (2.23), and letting $M_5 = \max\{\frac{2\times mM_4}{\varepsilon}, 2M_1\}$,

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \leq \bigl[1 - \beta_n(1-k)\bigr]\|x_n - x_{n-1}\| + \frac{2\times mM_4}{\varepsilon}(r_n - r_{n-1}) + 2M_1|\beta_n - \beta_{n-1}| \\
&\quad \leq \bigl[1 - \beta_n(1-k)\bigr]\|x_n - x_{n-1}\| + M_5\bigl[(r_n - r_{n-1}) + |\beta_n - \beta_{n-1}|\bigr].
\end{aligned} \tag{2.29}$$

If $r_n \leq r_{n-1}$, then imitating the above proof, we have

$$\|y_n - y_{n-1}\| \leq \bigl[1 - \beta_n(1-k)\bigr]\|x_n - x_{n-1}\| + M_5\bigl[(r_{n-1} - r_n) + |\beta_n - \beta_{n-1}|\bigr]. \tag{2.30}$$

Combining (2.29) and (2.30),

$$\|y_n - y_{n-1}\| \leq \bigl[1 - \beta_n(1-k)\bigr]\|x_n - x_{n-1}\| + M_5\bigl(|r_{n-1} - r_n| + |\beta_n - \beta_{n-1}|\bigr). \tag{2.31}$$

Similar to the discussion of (2.24), we have

$$\begin{aligned}
&\|W_{r_n}y_n - W_{r_{n-1}}y_{n-1}\| \\
&\quad \leq a_0\|y_n - y_{n-1}\| + \sum_{j=1}^{l} a_j \bigl\|J_{r_n}^{B_j}y_n - J_{r_{n-1}}^{B_j}y_{n-1}\bigr\| \\
&\quad \leq a_0\|y_n - y_{n-1}\| + \sum_{j=1}^{l} a_j \biggl(\|y_n - y_{n-1}\| + \frac{|r_n - r_{n-1}|}{\varepsilon}\bigl\|J_{r_n}^{B_j}y_n - y_{n-1}\bigr\|\biggr) \\
&\quad \leq \|y_n - y_{n-1}\| + 2M_1\frac{|r_n - r_{n-1}|}{\varepsilon}.
\end{aligned} \tag{2.32}$$

Using (2.32), then

$$\begin{aligned}
&\|u_n - u_{n-1}\| \\
&\quad \leq \vartheta_n k\|y_n - y_{n-1}\| + |\vartheta_n - \vartheta_{n-1}|\bigl(\bigl\|f(y_{n-1})\bigr\| + \|W_{r_{n-1}}y_{n-1}\|\bigr) \\
&\qquad + (1-\vartheta_n)\|W_{r_n}y_n - W_{r_{n-1}}y_{n-1}\| \\
&\quad \leq \bigl[1 - \vartheta_n(1-k)\bigr]\|y_n - y_{n-1}\| + 2M_1|\vartheta_n - \vartheta_{n-1}| + \frac{2M_1}{\varepsilon}|r_n - r_{n-1}|.
\end{aligned} \tag{2.33}$$

Based on (2.31) and (2.33), and letting $M_6 = M_5 + \frac{2M_1}{\varepsilon}$, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \leq \alpha_n\bigl\|f(u_n) - f(u_{n-1})\bigr\| + |\alpha_n - \alpha_{n-1}|\bigl\|f(u_{n-1})\bigr\| + (1-\alpha_n)\|u_n - u_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}|\|u_{n-1}\| \\
&\quad \leq \bigl[1 - \alpha_n(1-k)\bigr]\|u_n - u_{n-1}\| + 2M_1|\alpha_n - \alpha_{n-1}| \\
&\quad \leq \bigl[1 - \alpha_n(1-k)\bigr]\bigl[1 - \vartheta_n(1-k)\bigr]\|y_n - y_{n-1}\| + 2M_1\bigl(|\vartheta_n - \vartheta_{n-1}| + |\alpha_n - \alpha_{n-1}|\bigr) \\
&\qquad + \frac{2M_1}{\varepsilon}|r_n - r_{n-1}|
\end{aligned}$$

$$\begin{aligned}
&\leq \bigl[1-\beta_n(1-k)\bigr]\|x_n-x_{n-1}\| + M_6\bigl(|r_n-r_{n-1}| + |\beta_n-\beta_{n-1}| + |\alpha_n-\alpha_{n-1}| \\
&\quad + |\vartheta_n-\vartheta_{n-1}|\bigr).
\end{aligned}$$

In view of Lemma 1.6, we know that $\|x_{n+1}-x_n\| \to 0$, as $n \to \infty$. Combining with the fact that $\|x_{n+1}-u_n\| = \alpha_n\|f(u_n)-u_n\| \to 0$, we can easily know that $\|x_n-u_n\| \leq \|x_{n+1}-x_n\| + \|x_{n+1}-u_n\| \to 0$, as $n \to \infty$.

Step 3. $\|W_r u_n - u_n\| \to 0$, and $\|S_r^{A_m A_{m-1}\cdots A_1} u_n - u_n\| \to 0$, as $n \to \infty$. In view of Lemma 1.4 again, we know that

$$\begin{aligned}
&\bigl\|S_{r_n}^{A_1}x_n - S_r^{A_1}x_n\bigr\| \\
&\quad = \left\|J_r^{A_1}\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}^{A_1}x_n\right) - J_r^{A_1}x_n\right\| \\
&\quad \leq \left|1-\frac{r}{r_n}\right| \bigl\|J_{r_n}^{A_1}x_n - x_n\bigr\| \leq 2M_1\left|1-\frac{r}{r_n}\right|,
\end{aligned}$$

and then

$$\begin{aligned}
&\bigl\|S_{r_n}^{A_2A_1}x_n - S_r^{A_2A_1}x_n\bigr\| \\
&\quad \leq \frac{r}{r_n}\bigl\|J_{r_n}^{A_1}x_n - J_r^{A_1}x_n\bigr\| + \left|1-\frac{r}{r_n}\right| \bigl\|S_{r_n}^{A_2A_1}x_n - J_r^{A_1}x_n\bigr\| \leq 2M_1\left|1-\frac{r}{r_n}\right|\left(\frac{r}{r_n}+1\right).
\end{aligned}$$

By induction,

$$\bigl\|S_{r_n}^{A_m\cdots A_1}x_n - S_r^{A_m\cdots A_1}x_n\bigr\| \leq 2M_1\left|1-\frac{r}{r_n}\right|\left[\left(\frac{r}{r_n}\right)^{m-1} + \cdots + \frac{r}{r_n} + 1\right] \to 0, \tag{2.34}$$

as $n \to \infty$, since $r_n \to r^*$.

$\forall p \in D$, continuing the computation of (2.15), we have

$$\begin{aligned}
&0 \leq \bigl[1-\alpha_n(1-k)\bigr](1-\vartheta_n)\sum_{j=1}^{l} a_j\bigl\|(I+r_nB_j)^{-1}y_n - y_n\bigr\|^2 \\
&\quad \leq \|x_n-p\|^2 - \|x_{n+1}-p\|^2 + M_3(\alpha_n+\beta_n+\vartheta_n).
\end{aligned}$$

From step 2, we know that $\|x_n-x_{n+1}\| \to 0$, then $\|(I+r_nB_j)^{-1}y_n - y_n\| \to 0$, $j = 1, 2, \ldots, l$, which implies that

$$W_{r_n}y_n - y_n \to 0, \quad \text{as } n \to \infty. \tag{2.35}$$

Noticing that $\|u_n - W_{r_n}y_n\| = \vartheta_n\|f(y_n) - W_{r_n}y_n\| \to 0$, and $\|y_n - S_{r_n}^{A_m\cdots A_1}x_n\| = \beta_n\|f(x_n) - S_{r_n}^{A_m\cdots A_1}x_n\| \to 0$, as $n \to \infty$.

Combining with the facts of (2.34), (2.35), and step 2, we know that

$$\begin{aligned}
&\bigl\|u_n - S_r^{A_m\cdots A_1}u_n\bigr\| \\
&\quad \leq \|u_n - W_{r_n}y_n\| + \|W_{r_n}y_n - y_n\| + \bigl\|y_n - S_{r_n}^{A_m\cdots A_1}x_n\bigr\| \\
&\qquad + \bigl\|S_{r_n}^{A_m\cdots A_1}x_n - S_r^{A_m\cdots A_1}x_n\bigr\| + \bigl\|S_r^{A_m\cdots A_1}x_n - S_r^{A_m\cdots A_1}u_n\bigr\| \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

Using Lemma 1.4 again, then

$$\|W_{r_n}y_n - W_r y_n\| \le \sum_{j=1}^{l} a_j \|J_{r_n}^{B_j} y_n - J_r^{B_j} y_n\| \le 2M_1(1-a_0)\left|1 - \frac{r}{r_n}\right| \to 0.$$

Since $\|W_r u_n - W_r y_n\| \le \|u_n - y_n\| \le \vartheta_n \|f(y_n) - y_n\| + (1-\vartheta_n)\|W_{r_n}y_n - y_n\| \to 0$, then $\|u_n - W_r u_n\| \le \|u_n - W_{r_n}y_n\| + \|W_{r_n}y_n - W_r y_n\| + \|W_r y_n - W_r u_n\| \to 0$, as $n \to \infty$.

Step 4. $\limsup_{n\to\infty}\langle f(p_0) - p_0, u_n - p_0\rangle \le 0$, $\limsup_{n\to\infty}\langle f(p_0) - p_0, x_{n+1} - p_0\rangle \le 0$, $\limsup_{n\to\infty}\langle f(p_0) - p_0, y_n - p_0\rangle \le 0$, where p_0 satisfies (2.22).

Using Lemmas 1.5 and 2.3, we know that if we let $z_t = tf(z_t) + (1-t)W_r S_r^{A_m A_{m-1}\cdots A_1} z_t$, $r > 0$ and $t \in (0,1)$, then $z_t \to p_0 \in F(W_r S_r^{A_m A_{m-1}\cdots A_1}) = D$, as $t \to 0^+$. And, p_0 satisfies (2.22).

From step 3, we may choose $t_n \in (0,1)$ such that $t_n \to 0$, $\frac{\|S_r^{A_m\cdots A_1}u_n - u_n\|}{t_n} \to 0$, and $\frac{\|W_r u_n - u_n\|}{t_n} \to 0$, as $n \to \infty$.

Using Lemma 1.7,

$$\begin{aligned}
&\|z_{t_n} - u_n\|^2 \\
&\quad \le (1-t_n)^2 \big\|W_r S_r^{A_m\cdots A_1} z_{t_n} - u_n\big\|^2 + 2t_n\big\langle f(z_{t_n}) - u_n, z_{t_n} - u_n\big\rangle \\
&\quad \le (1-t_n)^2\big[\|z_{t_n} - u_n\| + \big\|u_n - S_r^{A_m\cdots A_1}u_n\big\| + \|u_n - W_r u_n\|\big]^2 \\
&\qquad + 2t_n\big\langle f(z_{t_n}) - z_{t_n}, z_{t_n} - u_n\big\rangle + 2t_n\|z_{t_n} - u_n\|^2.
\end{aligned}$$

Then

$$\begin{aligned}
&\big\langle f(z_{t_n}) - z_{t_n}, u_n - z_{t_n}\big\rangle \\
&\quad \le \frac{t_n}{2}\|z_{t_n} - u_n\|^2 + \frac{(1-t_n)^2}{t_n}\|z_{t_n} - u_n\|\big(\big\|S_r^{A_m\cdots A_1}u_n - u_n\big\| + \|u_n - W_r u_n\|\big) \\
&\qquad + \frac{(1-t_n)^2}{2t_n}\big(\big\|S_r^{A_m\cdots A_1}u_n - u_n\big\| + \|W_r u_n - u_n\|\big)^2.
\end{aligned} \tag{2.36}$$

Since $\{S_r^{A_m\cdots A_1}u_n\}$, $\{W_r u_n\}$, $\{x_n\}$, $\{u_n\}$ and $\{z_{t_n}\}$ are all bounded, and $\frac{\|S_r^{A_m\cdots A_1}u_n - u_n\|}{t_n} \to 0$, and $\frac{\|W_r u_n - u_n\|}{t_n} \to 0$, from (2.36), $\limsup_{n\to\infty}\langle f(z_{t_n}) - z_{t_n}, u_n - z_{t_n}\rangle \le 0$.

Recalling that $z_{t_n} \to p_0$, then $\langle z_{t_n} - p_0, u_n - z_{t_n}\rangle \to 0$. Thus $\limsup_{n\to\infty}\langle f(z_{t_n}) - p_0, u_n - z_{t_n}\rangle \le 0$. Since $\langle f(z_{t_n}) - p_0, u_n - p_0\rangle = \langle f(z_{t_n}) - p_0, u_n - z_{t_n}\rangle + \langle f(z_{t_n}) - p_0, z_{t_n} - p_0\rangle$, then $\limsup_{n\to\infty}\langle f(p_0) - p_0, u_n - p_0\rangle \le 0$. Then from step 2, $\limsup_{n\to\infty}\langle f(p_0) - p_0, x_{n+1} - p_0\rangle \le 0$.

Noticing that

$$\begin{aligned}
&\big\langle f(p_0) - p_0, y_n - p_0\big\rangle \\
&\quad = \big\langle f(p_0) - p_0, y_n - W_{r_n}y_n\big\rangle + \big\langle f(p_0) - p_0, W_{r_n}y_n - u_n\big\rangle \\
&\qquad + \big\langle f(p_0) - p_0, u_n - x_{n+1}\big\rangle + \big\langle f(p_0) - p_0, x_{n+1} - p_0\big\rangle,
\end{aligned}$$

and using (2.35), iterative scheme (A) and the result of step 2, we have $\limsup_{n\to\infty}\langle f(p_0) - p_0, y_n - p_0\rangle \le 0$.

Step 5. $x_n \to p_0$, which satisfies (2.22), as $n \to \infty$.

Using Lemma 1.7, we know that

$$\|y_n - p_0\|^2 \le \big[1 - \beta_n(1-k)\big]\|x_n - p_0\|^2 + 2\beta_n\big\langle f(p_0) - p_0, y_n - p_0\big\rangle. \tag{2.37}$$

We have

$$\|u_n - p_0\|^2 \le \big[1 - \vartheta_n(1-k)\big]\|y_n - p_0\|^2 + 2\vartheta_n\big\langle f(p_0) - p_0, u_n - p_0\big\rangle. \tag{2.38}$$

Letting $M_7 = \max\{(M_1 + \|p_0\|)^2, 2(M_1 + \|p_0\|)(\|f(p_0)\| + \|p_0\|)\}$ and using (2.37) and (2.38), we have

$$\begin{aligned}
&\|x_{n+1} - p_0\|^2 \\
&\quad\le \big[1 - \alpha_n(1-k)\big]\|u_n - p_0\|^2 + 2\alpha_n\big\langle f(p_0) - p_0, x_{n+1} - p_0\big\rangle \\
&\quad\le \big[1 - \alpha_n(1-k)\big]\big[1 - \beta_n(1-k)\big]\big[1 - \vartheta_n(1-k)\big]\|x_n - p_0\|^2 \\
&\qquad + 2\big[1 - \alpha_n(1-k)\big]\big[1 - \vartheta_n(1-k)\big]\beta_n\big\langle f(p_0) - p_0, y_n - p_0\big\rangle \\
&\qquad + 2\big[1 - \alpha_n(1-k)\big]\vartheta_n\big\langle f(p_0) - p_0, u_n - p_0\big\rangle + 2\alpha_n\big\langle f(p_0) - p_0, x_{n+1} - p_0\big\rangle \\
&\quad\le \big[1 - (1-k)(\alpha_n + \beta_n + \vartheta_n)\big]\|x_n - p_0\|^2 + M_7(1-k)^2(\alpha_n\beta_n + \beta_n\vartheta_n + \alpha_n\vartheta_n) \\
&\qquad + 2\alpha_n\vartheta_n(1-k)\big\langle p_0 - f(p_0), u_n - p_0\big\rangle + 2(\alpha_n\beta_n + \beta_n\vartheta_n)(1-k)\big\langle p_0 - f(p_0), y_n - p_0\big\rangle \\
&\qquad + 2\alpha_n\beta_n\vartheta_n(1-k)^2\big\langle f(p_0) - p_0, y_n - p_0\big\rangle \\
&\qquad + 2\alpha_n\big\langle f(p_0) - p_0, x_{n+1} - p_0\big\rangle + 2\beta_n\big\langle f(p_0) - p_0, y_n - p_0\big\rangle + 2\vartheta_n\big\langle f(p_0) - p_0, u_n - p_0\big\rangle \\
&\quad\le \big[1 - (1-k)(\alpha_n + \beta_n + \vartheta_n)\big]\|x_n - p_0\|^2 + M_7(1-k)^2(\alpha_n\beta_n + \beta_n\vartheta_n + \alpha_n\vartheta_n) \\
&\qquad + M_7(1-k)(\alpha_n\beta_n + \beta_n\vartheta_n + \alpha_n\vartheta_n) + 2M_7\alpha_n\beta_n\vartheta_n(1-k)^2 \\
&\qquad + 2\alpha_n\big\langle f(p_0) - p_0, x_{n+1} - p_0\big\rangle \\
&\qquad + 2\beta_n\big\langle f(p_0) - p_0, y_n - p_0\big\rangle + 2\vartheta_n\big\langle f(p_0) - p_0, u_n - p_0\big\rangle.
\end{aligned} \tag{2.39}$$

Let $c_n = (\alpha_n + \beta_n + \vartheta_n)(1-k)$, then $c_n \to 0$ and $\sum_{n=1}^{\infty} c_n = +\infty$.

Let $b_n = M_7[\frac{(2-k)(\alpha_n\beta_n + \beta_n\vartheta_n + \alpha_n\vartheta_n)}{\alpha_n + \beta_n + \vartheta_n} + \frac{2(1-k)\alpha_n\beta_n\vartheta_n}{\alpha_n + \beta_n + \vartheta_n}] + \frac{2\alpha_n}{(\alpha_n + \beta_n + \vartheta_n)(1-k)}\langle f(p_0) - p_0, x_{n+1} - p_0\rangle + \frac{2\vartheta_n}{(\alpha_n + \beta_n + \vartheta_n)(1-k)}\langle f(p_0) - p_0, u_n - p_0\rangle + \frac{2\beta_n}{(\alpha_n + \beta_n + \vartheta_n)(1-k)}\langle f(p_0) - p_0, y_n - p_0\rangle$.

Notice that $\lim_{n\to\infty} \frac{\alpha_n\beta_n + \beta_n\vartheta_n + \alpha_n\vartheta_n}{\alpha_n + \beta_n + \vartheta_n} = 0$, $\lim_{n\to\infty} \frac{\alpha_n\beta_n\vartheta_n}{\alpha_n + \beta_n + \vartheta_n} = 0$ and from the results in step 4, we have $\limsup_{n\to+\infty} b_n \le 0$.

Using Lemma 1.6, $x_n \to p_0$, which satisfies (2.22), as $n \to \infty$.

This completes the proof. □

Remark 2.2 The iterative construction in this paper generalizes and extends some corresponding ones in [2, 4, 12, 13], *etc.*, in Hilbert spaces.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors read and approved the final manuscript.

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