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Common fixed point results for contractive mappings in complex valued metric spaces

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Abstract
In this paper, we obtain some common fixed point results for the mappings satisfying rational expressions on a closed ball in complex valued metric spaces. Our results improve several well-known conventional results.
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1 Introduction and preliminaries

Azam *et al.* [1] introduced new spaces called complex valued metric spaces and established the existence of fixed point theorems under the contraction condition. Subsequently, Rouzkard and Imdad [2] established some common fixed point theorems satisfying certain rational expressions in complex valued metric spaces which generalize, unify and complement the results of Azam *et al.* [1]. Sintunavarat and Kumam [3] obtained common fixed point results by replacing constant of contractive condition to control functions. Recently, Klin-eam and Suanoom [4] extend the concept of complex valued metric spaces and generalized the results of Azam *et al.* [1] and Rouzkard and Imdad [2]. For more on fixed point theory we refer the reader to [4–26].

The aim of this article is to extend and improve the conditions of contraction from the whole space to closed ball and establish the common fixed point theorems which are more general than the results of Klin-eam and Suanoom [4], Rouzkard and Imdad [2], and Azam *et al.* [1] on complex valued metric spaces.

Let $\mathbb{C}$ be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \quad \text{if and only if} \quad \mathrm{Re}(z_1) \leq \mathrm{Re}(z_2) \quad \text{and} \quad \mathrm{Im}(z_1) \leq \mathrm{Im}(z_2).$$

It follows that $z_1 \precsim z_2$ if and only if one of the following conditions is satisfied:

(i) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(ii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$,
(iii) $\mathrm{Re}(z_1) < \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) < \mathrm{Im}(z_2)$,
(iv) $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii), and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$\begin{aligned} &0 \precsim z_1 \precnsim z_2 \quad \Longrightarrow \quad |z_1| < |z_2|, \\ &z_1 \precsim z_2, \qquad z_2 \prec z_3 \quad \Longrightarrow \quad z_1 \prec z_3. \end{aligned}$$

Definition 1 Let X be a nonempty set. Suppose that the mapping $d : X \times X \to \mathbb{C}$ satisfies:

(1) $0 \precsim d(x,y)$ for all $x, y \in X$; and $d(x,y) = 0$ if and only if $x = y$;
(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(3) $d(x,y) \precsim d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(x,r) := \{y \in X : d(x,y) \prec r\} \subseteq A,$$

where $B(x,r)$ is an open ball. Then $\overline{B(x,r)} = \{y \in X : d(x,y) \preceq r\}$ is a closed ball.

A point $x \in X$ is called a limit point of A whenever for every $0 \prec r \in \mathbb{C}$, we have

$$B(x,r) \cap (A \setminus \{x\}) \neq \varnothing.$$

A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A. A subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B. The family

$$F := \{B(x,r) : x \in X, 0 \prec r\}$$

is a sub-basis for a Hausdorff topology τ on X.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec c$, for all $n > n_0$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x. We denote this by $\lim_n x_n = x$ or $x_n \to x$. If for every $c \in \mathbb{C}$ with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) \prec c$, for all $n, m > n_0$, then $\{x_n\}$ is called a Cauchy sequence. If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete complex valued metric space.

Example 2 Let $X = X_1 \cup X_2$ where

$$X_1 = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) = 0\}$$

and

$$X_2 = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0 \text{ and } \mathrm{Im}(z) \geq 0\}.$$

Define $d : X \times X \to \mathbb{C}$ as follows:

$$d(z_1,z_2) = \begin{cases} \frac{2}{3}|x_1 - x_2| + \frac{i}{2}|x_1 - x_2| & \text{if } z_1, z_2 \in X_1; \\ \frac{1}{2}|y_1 - y_2| + \frac{i}{3}|y_1 - y_2| & \text{if } z_1, z_2 \in X_2; \\ (\frac{2}{3}x_1 + \frac{1}{2}y_2) + i(\frac{1}{2}x_1 + \frac{1}{3}y_2) & \text{if } z_1 \in X_1, z_2 \in X_2; \\ (\frac{1}{2}y_1 + \frac{2}{3}x_2) + i(\frac{1}{3}y_1 + \frac{1}{2}x_2) & \text{if } z_1 \in X_2, z_2 \in X_1, \end{cases}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2 \in X$. Then (X,d) is a complete complex valued metric space.

Lemma 3 [1] *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n \to \infty$.*

Lemma 4 [1] *Let (X,d) be a complex valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n \to \infty$.*

Definition 5 [27] Two families of self-mappings $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are said to be pairwise commuting if:

(1) $T_iT_j = T_jT_i$ for all $i,j \in \{1,2,\dots,m\}$;
(2) $S_kS_l = S_lS_k$ for all $k,l \in \{1,2,\dots,n\}$;
(3) $T_iS_k = S_kT_i$ for all $i \in \{1,2,\dots,m\}$, $k \in \{1,2,\dots,n\}$.

2 Main result

In our main result, we discuss the existence of the common fixed point of the mappings satisfying a contractive condition on the closed ball. This result is very useful in the sense that it requires the contractiveness of the mappings only on a closed ball instead of the whole space.

Theorem 6 *Suppose that (X,d) is a complete complex valued metric space and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C, D and E be five nonnegative reals such that $A + B + C + 2D + 2E < 1$. Let $S,T : X \to X$ satisfy*

$$\begin{aligned} d(Sx,Ty) \precsim{} & Ad(x,y) + B\frac{d(x,Sx)d(y,Ty)}{1+d(x,y)} + C\frac{d(y,Sx)d(x,Ty)}{1+d(x,y)} \\ & + D\frac{d(x,Sx)d(x,Ty)}{1+d(x,y)} + E\frac{d(y,Sx)d(y,Ty)}{1+d(x,y)} \end{aligned} \tag{1}$$

for all $x,y \in \overline{B(x_0,r)}$. If

$$\bigl|d(x_0,Sx_0)\bigr| \le (1-\lambda)|r|, \tag{2}$$

where $\lambda = \max\{\frac{A+D}{1-B-D}, \frac{A+E}{1-B-E}\}$, then there exists a unique point $u \in \overline{B(x_0,r)}$ such that $u = Su = Tu$.

Proof Let x_0 be an arbitrary point in X and define

$$x_{2k+1} = Sx_{2k} \quad \text{and} \quad x_{2k+2} = Tx_{2k+1}, \quad \text{where } k = 0,1,2,\dots.$$

We will prove that $x_n \in \overline{B(x_0,r)}$ for all $n \in \mathbb{N}$ by mathematical induction.

Using inequality (2) and the fact that $\lambda = \max\{\frac{A+D}{1-B-D}, \frac{A+E}{1-B-E}\} < 1$, we have

$$\bigl|d(x_0,Sx_0)\bigr| \le |r|.$$

It implies that $x_1 \in \overline{B(x_0,r)}$. Let $x_2,\dots,x_j \in \overline{B(x_0,r)}$ for some $j \in N$. If $j = 2k+1$, where $k = 0,1,2,\dots,\frac{j-1}{2}$ or $j = 2k+2$ where $k = 0,1,2,\dots,\frac{j-2}{2}$, we obtain by using inequality (1)

$$\begin{aligned} d(x_{2k+1},x_{2k+2}) &= d(Sx_{2k},Tx_{2k+1}) \\ &\precsim Ad(x_{2k},x_{2k+1}) + B\frac{d(x_{2k+1},Tx_{2k+1})d(x_{2k},Sx_{2k})}{1+d(x_{2k},x_{2k+1})} \\ &\quad + C\frac{d(x_{2k},Tx_{2k+1})d(x_{2k+1},Sx_{2k})}{1+d(x_{2k},x_{2k+1})} \\ &\quad + D\frac{d(x_{2k},Tx_{2k+1})d(x_{2k},Sx_{2k})}{1+d(x_{2k},x_{2k+1})} \\ &\quad + E\frac{d(x_{2k+1},Tx_{2k+1})d(x_{2k+1},Sx_{2k})}{1+d(x_{2k},x_{2k+1})}. \end{aligned}$$

Now $x_{2k+1} = Sx_{2k}$ implies that $d(x_{2k+1},Sx_{2k}) = 0$, so we have

$$\begin{aligned} d(x_{2k+1},x_{2k+2}) &\precsim Ad(x_{2k},x_{2k+1}) + B\frac{d(x_{2k+1},x_{2k+2})d(x_{2k},x_{2k+1})}{1+d(x_{2k},x_{2k+1})} \\ &\quad + D\frac{d(x_{2k},x_{2k+2})d(x_{2k},x_{2k+1})}{1+d(x_{2k},x_{2k+1})}. \end{aligned}$$

This implies that

$$\begin{aligned} \left|d(x_{2k+1},x_{2k+2})\right| &\le A\left|d(x_{2k},x_{2k+1})\right| + B\frac{|d(x_{2k+1},x_{2k+2})||d(x_{2k},x_{2k+1})|}{|1+d(x_{2k},x_{2k+1})|} \\ &\quad + D\frac{|d(x_{2k},x_{2k+2})||d(x_{2k},x_{2k+1})|}{|1+d(x_{2k},x_{2k+1})|}. \end{aligned}$$

Since $|1+d(x_{2k},x_{2k+1})| > |d(x_{2k},x_{2k+1})|$, we have

$$\left|d(x_{2k+1},x_{2k+2})\right| \le A\left|d(x_{2k},x_{2k+1})\right| + B\left|d(x_{2k+1},x_{2k+2})\right| + D\left|d(x_{2k},x_{2k+2})\right|.$$

This implies by the triangular inequality that

$$\left|d(x_{2k+1},x_{2k+2})\right| \le \frac{A+D}{1-B-D}\left|d(x_{2k},x_{2k+1})\right|. \tag{3}$$

Similarly, we get

$$\left|d(x_{2k+2},x_{2k+3})\right| \le \frac{A+E}{1-B-E}\left|d(x_{2k+2},x_{2k+1})\right|. \tag{4}$$

Putting $\lambda = \max\{\frac{A+D}{1-B-D},\frac{A+E}{1-B-E}\}$, we obtain

$$\left|d(x_j,x_{j+1})\right| \le \lambda^j\left|d(x_0,x_1)\right| \quad \text{for all } j \in \mathbb{N}. \tag{5}$$

Now

$$\begin{aligned} \left|d(x_0,x_{j+1})\right| &\le \left|d(x_0,x_1)\right| + \cdots + \left|d(x_j,x_{j+1})\right| \\ &\le \left|d(x_0,x_1)\right| + \cdots + \lambda^j\left|d(x_0,x_1)\right| \end{aligned}$$

$$\begin{aligned}
&= \left|d(x_0,x_1)\right|\left[1+\cdots+\lambda^{j-1}+\lambda^j\right] \\
&\le (1-\lambda)|r|\frac{(1-\lambda^{j+1})}{1-\lambda} \\
&\le |r|
\end{aligned}$$

gives $x_{j+1} \in \overline{B(x_0,r)}$. Hence $x_n \in \overline{B(x_0,r)}$ for all $n \in \mathbb{N}$ and

$$\left|d(x_n,x_{n+1})\right| \le \lambda^n \left|d(x_0,x_1)\right|$$

for all $n \in \mathbb{N}$. Without loss of generality, we take $m > n$, then

$$\begin{aligned}
\left|d(x_n,x_m)\right| &\le \left|d(x_n,x_{n+1})\right| + \left|d(x_{n+1},x_{n+2})\right| + \cdots + \left|d(x_{m-1},x_m)\right| \\
&\le \left[\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}\right]\left|d(x_0,x_1)\right| \\
&\le \left[\frac{\lambda^n}{1-\lambda}\right]\left|d(x_0,x_1)\right| \to 0 \quad \text{as } m,n \to \infty.
\end{aligned}$$

This implies that the sequence $\{x_n\}$ is a Cauchy sequence in $\overline{B(x_0,r)}$. Therefore, there exists a point $u \in \overline{B(x_0,r)}$ with $\lim_{n\to\infty} x_n = u$.

We prove that $u = Su$. Let us consider

$$\begin{aligned}
\left|d(u,Su)\right| \le{}& \left|d(u,x_{2k+2})\right| + A\left|d(x_{2k+1},u)\right| + B\frac{|d(x_{2k+1},Tx_{2k+1})||d(u,Su)|}{|1+d(u,x_{2k+1})|} \\
&+ C\frac{|d(x_{2k+1},Su)||d(u,Tx_{2k+1})|}{|1+d(u,x_{2k+1})|} \\
&+ D\frac{|d(u,Tx_{2k+1})||d(u,Su)|}{|1+d(u,x_{2k+1})|} \\
&+ E\frac{|d(x_{2k+1},Tx_{2k+1})||d(x_{2k+1},Su)|}{|1+d(u,x_{2k+1})|}.
\end{aligned}$$

Notice that $\lim_{n\to\infty}|d(u,x_{2k+2})| = \lim_{n\to\infty}|d(x_{2k+1},u)| = |d(x_{2k+1},Su)| = 0$. Hence $|d(u,Su)| = 0$, that is, $u = Su$. It follows similarly that $u = Tu$. For uniqueness, assume that u^* in $\overline{B(x_0,r)}$ is a second common fixed point of S and T. Then

$$\begin{aligned}
\left|d(u,u^*)\right| \le{}& A\left|d(u,u^*)\right| + B\frac{|d(u,Su)||d(u^*,Tu^*)|}{|1+d(u,u^*)|} \\
&+ C\frac{|d(u^*,Su)||d(u,Tu^*)|}{|1+d(u,u^*)|} \\
&+ D\frac{|d(u,Su)||d(u,Tu^*)|}{|1+d(u,u^*)|} \\
&+ E\frac{|d(u^*,Su)||d(u^*,Tu^*)|}{|1+d(u,u^*)|}.
\end{aligned}$$

Since $|1+d(u,u^*)| > |d(u,u^*)|$, so we have

$$\left|d(u,u^*)\right| \le (A+C)\left|d(u,u^*)\right|.$$

This is contradiction because $A + C < 1$. Hence $u^* = u$. Therefore, u is a unique common fixed point of S and T. □

By setting $S = T$ in Theorem 6, we get the following corollary.

Corollary 7 *Suppose that (X, d) is a complete complex valued metric space and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C, D and E be five nonnegative reals such that $A + B + C + 2D + 2E < 1$. Let $T : X \to X$ satisfy*

$$\begin{aligned} d(Tx, Ty) \precsim{} & Ad(x,y) + B\frac{d(x, Tx)d(y, Ty)}{1 + d(x,y)} + C\frac{d(y, Tx)d(x, Ty)}{1 + d(x,y)} \\ & + D\frac{d(x, Tx)d(x, Ty)}{1 + d(x,y)} + E\frac{d(y, Tx)d(y, Ty)}{1 + d(x,y)} \end{aligned}$$

for all $x, y \in \overline{B(x_0, r)}$. If

$$\left|d(x_0, Tx_0)\right| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{\frac{A+D}{1-B-D}, \frac{A+E}{1-B-E}\}$, then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Tu$.

Remark 8 The conclusion of Theorem 6 remains true if the condition (2) is replaced by the condition $|d(x_0, Tx_0)| \leq (1 - \lambda)|r|$.

By choosing $E = 0$ in Theorem 6, we get the following corollary.

Corollary 9 *Suppose that (X, d) is a complete complex valued metric space and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C, D be four nonnegative reals such that $A + B + C + 2D < 1$. Let $S, T : X \to X$ satisfy*

$$d(Sx, Ty) \precsim Ad(x,y) + B\frac{d(x, Sx)d(y, Ty)}{1 + d(x,y)} + C\frac{d(y, Sx)d(x, Ty)}{1 + d(x,y)} + D\frac{d(x, Sx)d(x, Ty)}{1 + d(x,y)}$$

for all $x, y \in \overline{B(x_0, r)}$. If

$$\left|d(x_0, Sx_0)\right| \leq (1 - \lambda)|r|,$$

where $\lambda = \max\{\frac{A+D}{1-B-D}, \frac{A}{1-B}\}$, then there exists a unique point $u \in \overline{B(x_0, r)}$ such that $u = Su = Tu$.

By setting $S = T$ in Corollary 9, we get the following corollary.

Corollary 10 *Suppose that (X, d) is a complete complex valued metric space and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C, D be four nonnegative reals such that $A + B + C + 2D < 1$. Let $T : X \to X$ satisfy*

$$d(Tx, Ty) \precsim Ad(x,y) + B\frac{d(x, Tx)d(y, Ty)}{1 + d(x,y)} + C\frac{d(y, Tx)d(x, Ty)}{1 + d(x,y)} + D\frac{d(x, Tx)d(x, Ty)}{1 + d(x,y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$\left|d(x_0, Tx_0)\right| \le (1-\lambda)|r|,$$

where $\lambda = \max\{\frac{A+D}{1-B-D}, \frac{A}{1-B}\}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Tu$.

By choosing $D = 0$ in Theorem 6, we get the following corollary.

Corollary 11 *Suppose that* (X, d) *is a complete complex valued metric space and* $x_0 \in X$. *Let* $0 \prec r \in \mathbb{C}$ *and* A, B, C *and* E *be five nonnegative reals such that* $A + B + C + 2E < 1$. *Let* $S, T : X \to X$ *satisfy*

$$d(Sx, Ty) \precsim Ad(x, y) + B\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)} + E\frac{d(y, Sx)d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$\left|d(x_0, Sx_0)\right| \le (1-\lambda)|r|,$$

where $\lambda = \max\{\frac{A}{1-B}, \frac{A+E}{1-B-E}\}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Su = Tu$.

By setting $S = T$ in Corollary 11, we get the following corollary.

Corollary 12 *Suppose that* (X, d) *is a complete complex valued metric space and* $x_0 \in X$. *Let* $0 \prec r \in \mathbb{C}$ *and* A, B, C, *and* E *be five nonnegative reals such that* $A + B + C + 2E < 1$. *Let* $T : X \to X$ *satisfy*

$$d(Tx, Ty) \precsim Ad(x, y) + B\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Tx)d(x, Ty)}{1 + d(x, y)} + E\frac{d(y, Tx)d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$\left|d(x_0, Sx_0)\right| \le (1-\lambda)|r|,$$

where $\lambda = \max\{\frac{A}{1-B}, \frac{A+E}{1-B-E}\}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Tu$.

Remark 13 By equating A, B, C, D, and E to 0 in all possible combinations, one can derive a host of corollaries which include the Banach fixed point theorem for self-mappings on the closed ball in complex valued metric spaces.

By choosing $D = E = 0$ in Theorem 6, we get the extension of Theorem 2.1 of [2] to the closed ball as follows.

Corollary 14 *Suppose that* (X, d) *is a complete complex valued metric space and* $x_0 \in X$. *Let* $0 \prec r \in \mathbb{C}$ *and* A, B, C *be three nonnegative reals such that* $A + B + C < 1$. *Let* $S, T : X \to X$ *satisfy*

$$d(Sx, Ty) \precsim Ad(x, y) + B\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Sx)d(x, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$\left|d(x_0, Sx_0)\right| \le (1-\lambda)|r|,$$

where $\lambda = \frac{A}{1-B}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Su = Tu$.

By setting $S = T$ in Corollary 14, we get Corollary 2.3 of [16] on the closed ball as follows.

Corollary 15 *Suppose that* (X, d) *is a complete complex valued metric space and* $x_0 \in X$. *Let* $0 \prec r \in \mathbb{C}$ *and* A, B, C *be three nonnegative reals such that* $A + B + C < 1$. *Let* $T : X \to X$ *satisfy*

$$d(Tx, Ty) \precsim Ad(x, y) + B\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)} + C\frac{d(y, Tx)d(x, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$\left|d(x_0, Tx_0)\right| \le (1-\lambda)|r|,$$

where $\lambda = \frac{A}{1-B}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Tu$.

By choosing $C = D = E = 0$ in Theorem 6, we get the extension of Theorem 4 of [1] to the closed ball as follows.

Corollary 16 *Suppose that* (X, d) *is a complete complex valued metric space and* $x_0 \in X$. *Let* $0 \prec r \in \mathbb{C}$ *and* A, B *be nonnegative reals such that* $A + B < 1$. *Let* $S, T : X \to X$ *satisfy*

$$d(Sx, Ty) \precsim Ad(x, y) + B\frac{d(x, Sx)d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$\left|d(x_0, Sx_0)\right| \le (1-\lambda)|r|,$$

where $\lambda = \frac{A}{1-B}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Su = Tu$.

By setting $S = T$ in Corollary 16, we get Corollary 2.3 of [1] on the closed ball as follows.

Corollary 17 *Suppose that* (X, d) *is a complete complex valued metric space and* $x_0 \in X$. *Let* $0 \prec r \in \mathbb{C}$ *and* A, B *be nonnegative reals such that* $A + B < 1$. *Let* $T : X \to X$ *satisfy*

$$d(Tx, Ty) \precsim Ad(x, y) + B\frac{d(x, Tx)d(y, Ty)}{1 + d(x, y)}$$

for all $x, y \in \overline{B(x_0, r)}$. *If*

$$\left|d(x_0, Tx_0)\right| \le (1-\lambda)|r|,$$

where $\lambda = \frac{A}{1-B}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Tu$.

As an application of Theorem 6, we prove the following theorem for two finite families of mappings.

Theorem 18 *If $\{T_i\}_1^m$ and $\{S_i\}_1^n$ are two finite pairwise commuting finite families of self-mapping defined on a complete complex valued metric space (X,d) such that the mappings S and T (with $T = T_1T_2\cdots T_m$ and $S = S_1S_2\cdots S_n$) satisfy the contractive conditions* (1) *and* (2), *then the component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point.*

Proof From Theorem 6, we can say that the mappings T and S have a unique common fixed point u *i.e.* $Tu = Su = u$. Now our requirement is to show that u is a common fixed point of all the component mappings of both families. In view of pairwise commutativity of the families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ (for every $1 \leq k \leq m$), we can write $T_ku = T_kTu = TT_ku$ and $T_ku = T_kSu = ST_ku$ which show that T_ku (for every k) is also a common fixed point of T and S. By using the uniqueness of common fixed point, we can write $T_ku = u$ (for every k) which shows that u is a common fixed point of the family $\{T_i\}_1^m$. Using the same argument one can also show that (for every $1 \leq k \leq n$) $S_ku = u$. Thus the component maps of the two families $\{T_i\}_1^m$ and $\{S_i\}_1^n$ have a unique common fixed point. □

By setting $T_1 = T_2 = \cdots = T_m = F$ and $S_1 = S_2 = \cdots = S_n = G$ in Theorem 18, we get the following corollary.

Corollary 19 *Suppose that (X,d) is a complete complex valued metric space and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C, D and E be five nonnegative reals such that $A+B+C+2D+2E < 1$. Let $F,G : X \to X$ satisfy*

$$d\bigl(F^mx,G^ny\bigr) \precsim Ad(x,y) + B\frac{d(x,F^mx)d(y,G^ny)}{1+d(x,y)} + C\frac{d(y,F^mx)d(x,G^ny)}{1+d(x,y)} + D\frac{d(x,F^mx)d(x,G^ny)}{1+d(x,y)} + E\frac{d(y,F^mx)d(y,G^ny)}{1+d(x,y)}$$

for all $x,y \in \overline{B(x_0,r)}$ and

$$\bigl|d\bigl(x_0,G^nx_0\bigr)\bigr| \leq (1-\lambda)|r|,$$

where $\lambda = \max\{\frac{A+D}{1-B-D}, \frac{A+E}{1-B-E}\}$, then there exists a unique point $u \in \overline{B(x_0,r)}$ such that $u = Fu = Gu$.

By setting $m = n$ and $F = G = T$ in Corollary 19, we get the following corollary.

Corollary 20 *Suppose that (X,d) is a complete complex valued metric space and $x_0 \in X$. Let $0 \prec r \in \mathbb{C}$ and A, B, C, D and E be five nonnegative reals such that $A+B+C+2D+2E < 1$. Let $T : X \to X$ satisfy*

$$d\bigl(T^nx,T^ny\bigr) \precsim Ad(x,y) + B\frac{d(x,T^nx)d(y,T^ny)}{1+d(x,y)} + C\frac{d(y,T^nx)d(x,T^ny)}{1+d(x,y)} + D\frac{d(x,T^nx)d(x,T^ny)}{1+d(x,y)} + E\frac{d(y,T^nx)d(y,T^ny)}{1+d(x,y)}$$

for all $x,y \in \overline{B(x_0,r)}$ and

$$\bigl|d\bigl(x_0,T^nx_0\bigr)\bigr| \leq (1-\lambda)|r|,$$

where $\lambda = \max\{\frac{A+D}{1-B-D}, \frac{A+E}{1-B-E}\}$, *then there exists a unique point* $u \in \overline{B(x_0, r)}$ *such that* $u = Tu$.

Now we give an example satisfying our main result.

Example 21 Let $X_1 = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0 \text{ and } \mathrm{Im}(z) = 0\}$ and $X_2 = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0 \text{ and } \mathrm{Im}(z) \geq 0\}$ and let $X = X_1 \cup X_2$. Consider a metric $d : X \times X \to \mathbb{C}$ as follows:

$$d(z_1, z_2) = \begin{cases} \frac{2}{3}|x_1 - x_2| + \frac{i}{2}|x_1 - x_2| & \text{if } z_1, z_2 \in X_1; \\ \frac{1}{2}|y_1 - y_2| + \frac{i}{3}|y_1 - y_2| & \text{if } z_1, z_2 \in X_2; \\ \frac{2}{9}(x_1 + y_2) + \frac{i}{6}(x_1 + y_2) & \text{if } z_1 \in X_1, z_2 \in X_2; \\ \frac{i}{3}(x_2 + y_1) + \frac{2i}{9}(x_2 + y_1) & \text{if } z_1 \in X_2, z_2 \in X_1, \end{cases}$$

where $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in X$. Then (X, d) is a complex valued metric space. Take $z_0 = \frac{1}{2} + 0i$ and $r = \frac{1}{3} + \frac{1}{4}i$. Then

$$\overline{B(z_0, r)} = \{z \in X_1 : 0 \leq \mathrm{Re}(z) \leq 1\} \cup \{z \in X_2 : 0 \leq \mathrm{Im}(z) \leq 1\}.$$

Define $S, T : X \to X$ by

$$Sz = \begin{cases} 0 + \frac{x}{4}i & \text{if } z \in X_1 \text{ with } 0 \leq \mathrm{Re}(z) \leq 1, \mathrm{Im}(z) = 0; \\ \frac{5x}{6} + 0i & \text{if } z \in X_1 \text{ with } \mathrm{Re}(z) > 1, \mathrm{Im}(z) = 0; \\ \frac{y}{5} + 0i & \text{if } z \in X_2 \text{ with } 0 \leq \mathrm{Im}(z) \leq 1, \mathrm{Re}(z) = 0; \\ 0 + \frac{4y}{5}i & \text{if } z \in X_2 \text{ with } \mathrm{Im}(z) > 1, \mathrm{Re}(z) = 0; \end{cases}$$

$$Tz = \begin{cases} 0 + \frac{x}{6}i & \text{if } z \in X_1 \text{ with } 0 \leq \mathrm{Re}(z) \leq 1, \mathrm{Im}(z) = 0; \\ \frac{4x}{5} + 0i & \text{if } z \in X_1 \text{ with } \mathrm{Re}(z) > 1, \mathrm{Im}(z) = 0; \\ \frac{y}{7} + 0i & \text{if } z \in X_2 \text{ with } 0 \leq \mathrm{Im}(z) \leq 1, \mathrm{Re}(z) = 0; \\ 0 + \frac{5y}{6}i & \text{if } z \in X_2 \text{ with } \mathrm{Im}(z) > 1, \mathrm{Re}(z) = 0. \end{cases}$$

By a routine calculation, one can verify that the mappings S and T satisfy the conditions (1) and (2) of Theorem 6 with $A = \frac{1}{6}$, $B = \frac{1}{24}$, $C = \frac{1}{2}$, $D = \frac{1}{25}$ and $E = \frac{1}{26}$. Hence S and T are contractions on $\overline{B(z_0, r)}$ and $0 + 0i \in \overline{B(z_0, r)}$ is a unique common fixed point of mappings S and T.

It is interesting to notice that S and T are not contractions on the whole space X for $z_1 = z_2 = \frac{3}{2} + 0i \notin \overline{B(z_0, r)}$ as

$$\begin{aligned} d(Sz_1, Tz_2) &= \frac{1}{30} + \frac{1}{40}i \succ \frac{33{,}859}{3{,}744{,}000} + \frac{4{,}837}{156{,}000}i \\ &= Ad(z_1, z_2) + B\frac{d(z_1, Sz_1)d(z_2, Tz_2)}{1 + d(z_1, z_2)} + C\frac{d(z_2, Sz_1)d(z_1, Tz_2)}{1 + d(z_1, z_2)} \\ &\quad + D\frac{d(z_1, Sz_1)(z_1, Tz_2)}{1 + d(z_1, z_2)} + E\frac{d(z_2, Sz_1)d(z_2, Tz_2)}{1 + d(z_1, z_2)}. \end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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