Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Fixed point approximation of asymptotically nonexpansive mappings in hyperbolic spaces

Hafiz Fukhar-ud-din[1,2] and Amna Kalsoom[2*]

Dedicated to Professor Wataru Takahashi on his 70th birthday.

*Correspondence:
amna_iub@yahoo.com
[2]Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, 63100, Pakistan
Full list of author information is available at the end of the article

Abstract

Convergence theorems are established in a hyperbolic space for the modified Noor iterations with errors of asymptotically nonexpansive mappings. The obtained results extend and improve the several known results in Banach spaces and CAT(0) spaces simultaneously.

Keywords: asymptotically nonexpansive mapping; modified Noor iterative process; (UC) hyperbolic space; convergence

1 Introduction

Nonexpansive mappings are Lipschitzian with Lipschitz constant equal to 1. The class of nonexpansive mappings enjoys the fixed point property and even the approximate fixed point property in the general setting of metric spaces. The importance of this class lies in its powerful applications in initial value problems of the differential equations, game-theoretic model, image recovery and minimax problems. The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [1] as an important generalization of the class of nonexpansive mappings. Therefore, it is natural to extend powerful results for nonexpansive mappings to the class of asymptotically nonexpansive mappings. Iterative construction of fixed points of various nonlinear mappings emerged as the most powerful tool for solving such nonlinear problems. Approximation of fixed points of asymptotically nonexpansive mappings has been studied extensively by many authors; see for example [2–13] and the references cited therein.

In 1989, Glowinski and Le Tallec [14] used a three-step iterative process to find approximate solutions of elastoviscoplasticity problem, liquid crystal theory and eigenvalue computation. They observed that the three-step iterative process gives better numerical computations than two-step and one-step iterative processes. In 1998, Haubruge *et al.* [15] studied convergence analysis of a three-step iterative process of Glowinski and Le Tallec [14] and applied this process to obtain new splitting type iterations for solving variational inequalities, separable convex programming and minimization of a sum of convex functions. They also proved that the three-step iterative process leads to highly parallel iterations under certain conditions. Thus we conclude that the three-step iterative process plays an important and significant role in solving various numerical problems which arise in pure and applied sciences.

In 2000, Noor [5] introduced a three-step iterative process and studied the approximate solutions of variational inclusion in Hilbert spaces. In 2002, Xu and Noor [13] presented a three-step iterative process to approximate fixed points of asymptotically nonexpansive mappings in a Banach space. Cho *et al.* [2] extended Xu and Noor's iterative process to a three-step iterative process with errors in Banach spaces and used it to approximate fixed points of asymptotically nonexpansive mappings. In 2005, Suantai [9] proposed and analyzed the modified three-step Noor iterative process. This process was further studied for different kinds of mappings by Khan and Hussain [10] and Khan [11] for example. Nammanee *et al.* [12] extended this process to the one with errors as follows.

Let C be a nonempty convex subset of a Banach space X and let $T : C \to C$ be a given mapping. For given $x_1 \in C$, compute the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ by

$$\begin{aligned} z_n &= a_n T^n x_n + (1 - a_n - \gamma_n)x_n + \gamma_n u_n, \\ y_n &= b_n T^n z_n + c_n T^n x_n + (1 - b_n - c_n - \mu_n)x_n + \mu_n v_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n T^n z_n + (1 - \alpha_n - \beta_n - \lambda_n)x_n + \lambda_n w_n, \quad n \geq 1, \end{aligned} \tag{1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ are sequences in $[0,1]$; $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C.

By different choices of parameters $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n, \mu_n, \lambda_n$ to be zero, one can see that one-step iterations of Mann [3], two-step iterations of Ishikawa [4], three-step iterations of Xu and Noor [13], three-step iterations with errors of Cho *et al.* [2] and modified three-step iterations of Suantai [9] all are the special cases of iteration process (1).

Most of phenomena in nature are nonlinear. Therefore, mathematicians and scientists are always in pursuit of finding methods to solve nonlinear real world problems. So translating a linear version of known problems into its equivalent nonlinear version has a great importance.

Keeping in mind the occurrence of such phenomena, we translate modified three-step Noor iterations with errors in a nonlinear domain, namely, hyperbolic spaces and study their convergence analysis in a new setup.

A metric space (X,d) is hyperbolic [16] if there is a mapping $W : X^2 \times I \to X$ such that

$$\begin{aligned} &\text{(a)} \quad d\big(u, W(x,y,\alpha)\big) \leq \alpha d(u,x) + (1-\alpha)d(u,y), \\ &\text{(b)} \quad d\big(W(x,y,\alpha), W(x,y,\beta)\big) = |\alpha - \beta| d(x,y), \\ &\text{(c)} \quad W(x,y,\alpha) = W(y,x,1-\alpha), \\ &\text{(d)} \quad d\big(W(x,z,\alpha), W(y,w,\alpha)\big) \leq \alpha d(x,y) + (1-\alpha)d(z,w) \end{aligned} \tag{2}$$

for all $u,w,x,y,z \in X$ and $\alpha, \beta \in I = [0,1]$ (see also [17]); the space is convex [18] if only (a) is satisfied. A subset C of the hyperbolic space X is convex if $W(x,y,\alpha) \in C$ for all $x,y \in C$ and $\alpha \in I$. Normed spaces and their subsets are linear hyperbolic spaces while Hadamard manifolds [19], the Hilbert open unit ball equipped with the hyperbolic metric [20] and the CAT(0) spaces qualify for the criteria of nonlinear hyperbolic spaces [21–23].

Throughout the paper, a hyperbolic space (X, d, W) will simply be denoted by X. A hyperbolic space X is uniformly convex (UC) [24] if for any $u,x,y \in X$, $r > 0$ and $\varepsilon \in (0,2]$, there exists $\delta \in (0,1]$ such that $d(W(x,y,\frac{1}{2}),u) \leq (1-\delta)r < r$, whenever $d(x,u) \leq r, d(y,u) \leq r$ and $d(x,y) \geq r\varepsilon$.

A mapping $\eta : (0,\infty) \times (0,2] \to (0,1]$ such that $\eta(r,\varepsilon) = \delta$ for a given $r > 0$ and $\varepsilon \in (0,2]$ (as in the definition of UC) is known as a modulus of uniform convexity. We call η monotone if it decreases with respect to r (for a fixed ε).

Let C be a nonempty subset of a metric space X. A mapping $T : C \to C$ is asymptotically nonexpansive if there exists a sequence $\{k_n \geq 1\}$ with $\lim_{n\to\infty} k_n = 1$ such that

$$d\big(T^n x, T^n y\big) \leq k_n d(x,y) \quad \text{for } x,y \in C, n \geq 1;$$

it becomes nonexpansive if $k_n = 1$ for all $n \geq 1$. It was shown in [25] that an asymptotically nonexpansive mapping on a nonempty, bounded, closed and convex subset of a (UC) hyperbolic space has a fixed point.

We translate (1) in a hyperbolic space as follows.

Let C be a nonempty convex subset of a hyperbolic space X and $T : C \to C$ be an asymptotically nonexpansive mapping. Then, for arbitrarily chosen $x_1 \in C$, we construct the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in C as

$$\begin{aligned} z_n &= W\big(T^n x_n, W(x_n, u_n, \theta_{n_1}), a_n\big), \\ y_n &= W\left(T^n z_n, W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1-b_n}\right), b_n\right), \\ x_{n+1} &= W\left(T^n y_n, W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), \end{aligned} \tag{3}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$, $\{\lambda_n\}$ are sequences in $[0,1]$ and $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are bounded sequences in C and $\theta_{n_1} = 1 - \frac{\gamma_n}{1-a_n}$, $\theta_{n_2} = 1 - \frac{\mu_n}{1-b_n-c_n}$ and $\theta_{n_3} = 1 - \frac{\lambda_n}{1-\alpha_n-\beta_n}$.

Using Proposition 1.2(a) [26]: $W(x,y,0) = y$ for $x,y \in X$, the iteration process in (3) reduces to:

(i) modified Noor iterations (with $\gamma_n = \mu_n = \lambda_n = 0$):

$$\begin{aligned} z_n &= W\big(T^n x_n, x_n, a_n\big), \\ y_n &= W\left(T^n z_n, W\left(T^n x_n, x_n, \frac{c_n}{1-b_n}\right), b_n\right), \\ x_{n+1} &= W\left(T^n y_n, W\left(T^n z_n, x_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right); \end{aligned} \tag{4}$$

(ii) Noor iterations with errors (with $c_n = 0 = \beta_n$):

$$\begin{aligned} z_n &= W\left(T^n x_n, W\left(x_n, u_n, 1 - \frac{\gamma_n}{1-a_n}\right), a_n\right), \\ y_n &= W\left(T^n z_n, W\left(x_n, v_n, 1 - \frac{\mu_n}{1-b_n}\right), b_n\right), \\ x_{n+1} &= W\left(T^n y_n, W\left(x_n, w_n, 1 - \frac{\lambda_n}{1-\alpha_n}\right), \alpha_n\right); \end{aligned} \tag{5}$$

(iii) Noor iterations (with $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$):

$$\begin{aligned} z_n &= W\big(T^n x_n, x_n, a_n\big), \\ y_n &= W\big(T^n z_n, x_n, b_n\big), \\ x_{n+1} &= W\big(T^n y_n, x_n, \alpha_n\big); \end{aligned} \tag{6}$$

(iv) Ishikawa iterations (with $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n = 0$):

$$\begin{aligned} y_n &= W\big(T^n x_n, x_n, b_n\big), \\ x_{n+1} &= W\big(T^n y_n, x_n, \alpha_n\big); \end{aligned} \tag{7}$$

(v) Mann iterations (with $a_n = b_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n = 0$):

$$x_{n+1} = W\big(T^n x_n, x_n, \alpha_n\big). \tag{8}$$

The purpose of this paper is to establish convergence results of iteration process (3) for asymptotically nonexpansive mappings on a nonlinear domain ((UC) hyperbolic spaces) which includes both (UC) Banach spaces and CAT(0) spaces. Therefore, our results extend and improve the corresponding ones proved by Suantai [9], Xu and Noor [13] and others in a (UC) Banach space and are also valid in CAT(0) spaces, simultaneously.

In the sequel, we need the following lemmas.

Lemma 1.1 ([27]) *Let $\{a_n\}$, $\{\delta_n\}$ and $\{\theta_n\}$ be sequences of non-negative real numbers such that $\sum_{n=1}^{\infty} \theta_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$. If $a_{n+1} \le (1+\delta_n)a_n + \theta_n$, $n \ge 1$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 1.2 ([28]) *Let X be a (UC) hyperbolic space with monotone modulus of uniform convexity η. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[b,c]$ for some $b,c \in (0,1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n,x) \le r$, $\limsup_{n\to\infty} d(y_n,x) \le r$ and $\lim_{n\to\infty} d(W(x_n,y_n,\alpha_n),x) = r$ for some $r \ge 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$.*

2 Main results

The following lemma is crucial for proving the convergence results.

Lemma 2.1 *Let X be a (UC) hyperbolic space with monotone modulus of uniform convexity η, and let C be a nonempty, bounded, closed and convex subset of X. Let T be an asymptotically nonexpansive self-mapping on C with a sequence $\{k_n\} \subset [1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. For a given $x_1 \in C$, compute $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ as in (3) satisfying $0 < a \le \alpha_n, \beta_n, a_n, b_n \le b < 1$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$.*

Then we have the following conclusions:

(i) *If q is a fixed point of T, then $\lim_{n\to\infty} d(x_n,q)$ exists.*

(ii) *If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then*

$$\lim_{n\to\infty} d\left(T^n y_n, W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right)\right) = 0.$$

(iii) *If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$ and $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$, then*

$$\lim_{n\to\infty} d\left(T^n z_n, W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1-b_n}\right)\right) = 0.$$

(iv) *If $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$ and $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n + \gamma_n) < 1$, then*

$$\lim_{n\to\infty} d\big(T^n x_n, W(x_n, u_n, \theta_{n_1})\big) = 0.$$

Moreover, if $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$ *and* $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n + \gamma_n) < 1$, *then*

$$\lim_{n\to\infty} d(T^n x_n, x_n) = \lim_{n\to\infty} d(T^n z_n, x_n) = \lim_{n\to\infty} d(T^n y_n, x_n) = 0.$$

Proof (i) Applying (2)(a) with $u = q \in F(T)$ to the sequence $\{z_n\}$ in (3), we obtain

$$\begin{aligned} d(z_n, q) &= d\big(W\big(T^n x_n, W(x_n, u_n, \theta_{n_1}), a_n\big), q\big) \\ &\le a_n d\big(T^n x_n, q\big) + (1 - a_n) d\big(W(x_n, u_n, \theta_{n_1}), q\big) \\ &\le a_n d\big(T^n x_n, q\big) + (1 - a_n - \gamma_n) d(x_n, q) + \gamma_n d(u_n, q) \\ &\le a_n k_n d(x_n, q) + k_n(1 - a_n - \gamma_n) d(x_n, q) + \gamma_n d(u_n, q) \\ &\le k_n(1 - \gamma_n) d(x_n, q) + \gamma_n d(u_n, q). \end{aligned} \tag{9}$$

Again applying (2)(a) to the sequence $\{y_n\}$ in (3) and inserting (9), we have

$$\begin{aligned} d(y_n, q) &= d\left(W\left(T^n z_n, W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1 - b_n}\right), b_n\right), q\right) \\ &\le b_n d\big(T^n z_n, q\big) + (1 - b_n) d\left(W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1 - b_n}\right), q\right) \\ &\le b_n d\big(T^n z_n, q\big) + c_n d\big(T^n x_n, q\big) + (1 - b_n - c_n) d\big(W(x_n, v_n, \theta_{n_2}), q\big) \\ &\le b_n d\big(T^n z_n, q\big) + c_n d\big(T^n x_n, q\big) + (1 - b_n - c_n - \mu_n) d(x_n, q) + \mu_n d(v_n, q) \\ &\le b_n k_n\big[k_n(1 - \gamma_n) d(x_n, q) + \gamma_n d(u_n, q)\big] + c_n k_n d(x_n, q) \\ &\quad + (1 - b_n - c_n - \mu_n) d(x_n, q) + \mu_n d(v_n, q) \\ &\le b_n k_n^2(1 - \gamma_n) d(x_n, q) + b_n k_n \gamma_n d(u_n, q) + c_n k_n d(x_n, q) \\ &\quad + (1 - b_n - c_n - \mu_n) d(x_n, q) + \mu_n d(v_n, q) \\ &\le b_n k_n^2(1 - \gamma_n) d(x_n, q) + c_n k_n^2 d(x_n, q) + (1 - b_n - c_n - \mu_n) d(x_n, q) \\ &\quad + b_n k_n \gamma_n d(u_n, q) + \mu_n d(v_n, q) \\ &\le \big(b_n k_n^2 - b_n k_n^2 \gamma_n + c_n k_n^2\big) d(x_n, q) + k_n^2(1 - b_n - c_n - \mu_n) d(x_n, q) \\ &\quad + b_n k_n \gamma_n d(u_n, q) + \mu_n d(v_n, q) \\ &\le \big(b_n k_n^2 - b_n k_n^2 \gamma_n + c_n k_n^2 + k_n^2 - b_n k_n^2 - c_n k_n^2 - \mu_n k_n^2\big) d(x_n, q) \\ &\quad + b_n k_n \gamma_n d(u_n, q) + \mu_n d(v_n, q) \\ &\le \big(k_n^2 - b_n k_n^2 \gamma_n - \mu_n k_n^2\big) d(x_n, q) + b_n k_n \gamma_n d(u_n, q) + \mu_n d(v_n, q) \\ &\le k_n^2(1 - b_n \gamma_n - \mu_n) d(x_n, q) + b_n k_n \gamma_n d(u_n, q) + \mu_n d(v_n, q). \end{aligned}$$

That is,

$$\begin{aligned} d(y_n, q) \le{}& k_n^2(1 - b_n \gamma_n - \mu_n) d(x_n, q) \\ &+ b_n k_n \gamma_n d(u_n, q) + \mu_n d(v_n, q). \end{aligned} \tag{10}$$

Now it follows from (9) and (10) that

$$\begin{aligned}
d(x_{n+1},q) &= d\left(W\left(T^n y_n, W\left(T^n z_n, W(x_n,w_n,\theta_{n3}), \frac{\beta_n}{1-\alpha_n}\right),\alpha_n\right),q\right) \\
&\le \alpha_n d\left(T^n y_n,q\right) + (1-\alpha_n) d\left(W\left(T^n z_n, W(x_n,w_n,\theta_{n3}), \frac{\beta_n}{1-\alpha_n}\right),q\right) \\
&\le \alpha_n d\left(T^n y_n,q\right) + \beta_n d\left(T^n z_n,q\right) + (1-\alpha_n-\beta_n-\lambda_n) d(x_n,q) + \lambda_n d(w_n,q) \\
&\le \alpha_n k_n d(y_n,q) + \beta_n k_n d(z_n,q) + (1-\alpha_n-\beta_n-\lambda_n) d(x_n,q) + \lambda_n d(w_n,q) \\
&\le \alpha_n k_n \left[k_n^2(1-b_n\gamma_n-\mu_n) d(x_n,q) + b_n k_n \gamma_n d(u_n,q) + \mu_n d(v_n,q)\right] \\
&\quad + \beta_n k_n \left[k_n(1-\gamma_n) d(x_n,q) + \gamma_n d(u_n,q)\right] \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n) d(x_n,q) + \lambda_n d(w_n,q) \\
&\le \left(\alpha_n k_n^3 - \alpha_n k_n^3 b_n \gamma_n - \alpha_n k_n^3 \mu_n\right) d(x_n,q) + \alpha_n k_n^2 b_n \gamma_n d(u_n,q) \\
&\quad + \alpha_n k_n \mu_n d(v_n,q) + \left(\beta_n k_n^2 - \beta_n k_n^2 \gamma_n\right) d(x_n,q) + \beta_n k_n \gamma_n d(u_n,q) \\
&\quad + (1-\alpha_n-\beta_n-\lambda_n) d(x_n,q) + \lambda_n d(w_n,q) \\
&\le \left(\alpha_n k_n^3 - \alpha_n k_n^3 b_n \gamma_n - \alpha_n k_n^3 \mu_n\right) d(x_n,q) + \left(\beta_n k_n^3 - \beta_n k_n^2 \gamma_n\right) d(x_n,q) \\
&\quad + k_n^3 (1-\alpha_n-\beta_n-\lambda_n) d(x_n,q) + \alpha_n k_n^2 b_n \gamma_n d(u_n,q) \\
&\quad + \alpha_n k_n \mu_n d(v_n,q) + \beta_n k_n \gamma_n d(u_n,q) + \lambda_n d(w_n,q) \\
&\le \left[\alpha_n k_n^3 - \alpha_n k_n^3 b_n \gamma_n - \alpha_n k_n^3 \mu_n + \beta_n k_n^3\right. \\
&\quad \left. - \beta_n k_n^2 \gamma_n + k_n^3 - \alpha_n k_n^3 - \beta_n k_n^3 - \lambda_n k_n^3\right] d(x_n,q) \\
&\quad + \alpha_n k_n^2 b_n \gamma_n d(u_n,q) + \alpha_n k_n \mu_n d(v_n,q) + \beta_n k_n \gamma_n d(u_n,q) + \lambda_n d(w_n,q) \\
&\le \left[k_n^3 - \alpha_n k_n^3 b_n \gamma_n - \alpha_n k_n^3 \mu_n - \beta_n k_n^2 \gamma_n - \lambda_n k_n^3\right] d(x_n,q) \\
&\quad + \left(\alpha_n k_n^2 b_n \gamma_n + \beta_n k_n \gamma_n\right) d(u_n,q) + \alpha_n k_n \mu_n d(v_n,q) + \lambda_n d(w_n,q) \\
&\le k_n^3 d(x_n,q) + \left(k_n^2 + k_n\right) \gamma_n d(u_n,q) + k_n \mu_n d(v_n,q) + \lambda_n d(w_n,q).
\end{aligned}$$

Therefore, we have

$$d(x_{n+1},q) \le k_n^3 d(x_n,q) + \gamma_n A + \mu_n B + \lambda_n C,$$

where $A = \sup\{(k_n^2 + k_n) d(u_n,q) : n \ge 1\}$, $B = \sup\{k_n d(v_n,q) : n \ge 1\}$ and $C = \sup\{d(w_n,q) : n \ge 1\}$.

If we let $K = \max\{A,B,C\}$, then we have

$$d(x_{n+1},q) \le k_n^3 d(x_n,q) + K(\gamma_n + \mu_n + \lambda_n).$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$, it follows from Lemma 1.1 that $\lim_{n\to\infty} d(x_n,q)$ exists.

(ii) Since C is bounded, there exists $M > 0$ such that $\max\{d(x_n,u_n), d(x_n,v_n), d(x_n,w_n)\} \le M$.

If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then there exist $\sigma_1, \sigma_2 \in (0,1)$ such that $0 < \sigma_1 \le \alpha_n \le \alpha_n + \beta_n + \lambda_n \le \sigma_2 < 1$ for all $n \ge 1$. We have shown in part (i) that $\lim_{n\to\infty} d(x_n, q)$ exists, therefore $\lim_{n\to\infty} d(x_{n+1}, q) = c > 0$ (say).

That is,

$$\lim_{n\to\infty} W\left(T^n y_n, W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right) = c. \tag{11}$$

From (10), we have that

$$\limsup_{n\to\infty} d\bigl(T^n y_n, q\bigr) \le c. \tag{12}$$

Also

$$\begin{aligned}
&d\left(W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right), q\right) \\
&\quad\le \frac{\beta_n}{1-\alpha_n} d\bigl(T^n z_n, q\bigr) + \left(1 - \frac{\beta_n}{1-\alpha_n}\right) d\bigl(W(x_n, w_n, \theta_{n_3}), q\bigr) \\
&\quad\le \frac{\beta_n}{1-\alpha_n} d\bigl(T^n z_n, q\bigr) + \frac{1-\alpha_n-\beta_n-\lambda_n}{1-\alpha_n} d(x_n, q) + \frac{\lambda_n}{1-\alpha_n} d(w_n, q) \\
&\quad\le \frac{\beta_n}{1-\alpha_n}\bigl[k_n^2 d(x_n, q) + k_n \gamma_n M\bigr] + k_n^2\left(1 - \frac{\beta_n}{1-\alpha_n} - \frac{\lambda_n}{1-\alpha_n}\right) d(x_n, q) \\
&\qquad + \frac{\lambda_n}{1-\alpha_n} k_n^2 \bigl[d(x_n, q) + d(x_n, w_n)\bigr] \\
&\quad\le \frac{\beta_n}{1-\alpha_n} k_n^2 \gamma_n M + k_n^2 d(x_n, q) + \frac{\lambda_n}{1-\alpha_n} k_n^2 d(x_n, w_n) \\
&\quad\le \frac{\beta_n}{1-\alpha_n} k_n^2 \gamma_n M + k_n^2 d(x_n, q) + \frac{\lambda_n}{1-\alpha_n} k_n^2 d(x_n, w_n) \\
&\quad\le \frac{\beta_n}{1-\alpha_n} k_n^2 \gamma_n M + k_n^2 d(x_n, q) + \frac{1}{1-\alpha_n} k_n^2 \lambda_n M \\
&\quad\le \frac{b}{1-b} k_n^2 \gamma_n M + k_n^2 d(x_n, q) + \frac{1}{1-b} k_n^2 \lambda_n M
\end{aligned}$$

gives that

$$\limsup_{n\to\infty} d\left(W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right), q\right) \le c. \tag{13}$$

The hypothesis of Lemma 1.2 is satisfied in (11), (12) and (13), therefore we conclude

$$\lim_{n\to\infty} d\left(T^n y_n, W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right)\right) = 0. \tag{14}$$

(iii) If $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$, then there exist $\sigma_1, \sigma_2 \in (0,1)$ such that $0 < \sigma_1 \le \alpha_n \le \alpha_n + \beta_n + \lambda_n \le \sigma_2 < 1$ for all $n \ge 1$. Similarly, $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$ gives that there exist $\rho_1, \rho_2 \in (0,1)$ such that $0 < \rho_1 \le b_n \le b_n + c_n + \mu_n \le \rho_2 < 1$ for all $n \ge 1$.

Since

$$\begin{aligned} d(x_{n+1},q) &= d\left(W\left(T^n y_n, W\left(T^n z_n, W(x_n,w_n,\theta_{n3}), \frac{\beta_n}{1-\alpha_n}\right),\alpha_n\right),q\right) \\ &\le k_n d(y_n,q) + (1-a)d\left(W\left(T^n y_n, T^n z_n, W(x_n,w_n,\theta_{n3}), \frac{\beta_n}{1-\alpha_n}\right)\right), \end{aligned}$$

with the help of (14), we have

$$c \le \liminf_{n\to\infty} d(y_n,q) \le \limsup_{n\to\infty} d(y_n,q) \le c.$$

That is,

$$\lim_{n\to\infty} d(y_n,q) = c. \tag{15}$$

Obviously,

$$\limsup_{n\to\infty} d\big(T^n z_n,q\big) \le c \tag{16}$$

and

$$\begin{aligned} &d\left(W\left(T^n x_n, W(x_n,v_n,\theta_{n2}), \frac{c_n}{1-b_n}\right),q\right) \\ &\quad\le \frac{c_n}{1-b_n} d\big(T^n x_n,q\big) + \left(1-\frac{c_n}{1-b_n}\right) d\big(W(x_n,v_n,\theta_{n2}),q\big) \\ &\quad\le \frac{c_n}{1-b_n} k_n d(x_n,q) + \left(\frac{1-b_n-c_n-\mu_n}{1-b_n}\right) d(x_n,q) + \left(\frac{\mu_n}{1-b_n}\right) d(v_n,q) \\ &\quad\le \frac{c_n}{1-b_n} k_n d(x_n,q) + d(x_n,q) - \frac{c_n}{1-b_n} k_n d(x_n,q) - \frac{\mu_n}{1-b_n} d(x_n,q) \\ &\qquad + \left(\frac{\mu_n}{1-b_n}\right) d(x_n,q) + \left(\frac{\mu_n}{1-b_n}\right) d(x_n,v_n) \\ &\quad\le d(x_n,q) + \left(\frac{\mu_n}{1-b_n}\right) d(x_n,v_n) \le d(x_n,q) + \left(\frac{\mu_n}{1-b}\right) M \end{aligned}$$

gives that

$$\limsup_{n\to\infty} d\left(W\left(T^n x_n, W(x_n,v_n,\theta_{n2}), \frac{c_n}{1-b_n}\right),q\right) \le c. \tag{17}$$

Again the hypothesis of Lemma 1.2 is satisfied in (15), (16) and (17), therefore we get

$$\lim_{n\to\infty} d\left(T^n z_n, W\left(T^n x_n, W(x_n,v_n,\theta_{n2}), \frac{c_n}{1-b_n}\right)\right) = 0. \tag{18}$$

(iv) If $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$ and $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty}(a_n + \gamma_n) < 1$, then there exist $\rho_1,\rho_2,\tau_1,\tau_2 \in (0,1)$ such that $0 < \rho_1 \le b_n \le b_n + c_n + \mu_n \le \rho_2 < 1$ and $0 < \tau_1 \le a_n \le a_n + \gamma_n \le \tau_2 < 1$ for all $n \ge 1$.

Since

$$\begin{aligned} d(y_n, q) &\le d\left(W\left(T^n z_n, W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1-b_n}\right), b_n\right), q\right) \\ &\le (1-a) d\left(T^n z_n, W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1-b_n}\right)\right) + k_n d(z_n, q), \end{aligned}$$

with the help of (18), we have

$$c \le \liminf_{n\to\infty} d(z_n, q) \le \limsup_{n\to\infty} d(z_n, q) \le c.$$

That is,

$$\lim_{n\to\infty} d(z_n, q) = \lim_{n\to\infty} d\big(W\big(T^n x_n, W(x_n, u_n, \theta_{n_1}), a_n\big), q\big) = c. \tag{19}$$

Obviously,

$$\limsup_{n\to\infty} d\big(T^n x_n, q\big) \le c \tag{20}$$

and

$$\begin{aligned} d\big(W(x_n, u_n, \theta_{n_1}), q\big) &\le \left(\frac{1-a_n-\gamma_n}{1-a_n}\right) d(x_n, q) + \left(\frac{\gamma_n}{1-a_n}\right) d(x_n, u_n) \\ &\le \left(1 - \frac{\gamma_n}{1-a_n}\right) d(x_n, q) + \left(\frac{\gamma_n}{1-a_n}\right) d(x_n, u_n) \\ &\le d(x_n, q) - \frac{\gamma_n}{1-a_n} d(x_n, q) + \left(\frac{\gamma_n}{1-a_n}\right) d(x_n, u_n) \\ &\le d(x_n, q) + \left(\frac{\gamma_n}{1-b}\right) M \end{aligned}$$

provide that

$$\limsup_{n\to\infty} d\big(W(x_n, u_n, \theta_{n_1}), q\big) \le c. \tag{21}$$

Finally, appealing to Lemma 1.2 (using (19), (20), and (21)), we get that

$$\lim_{n\to\infty} d\big(T^n x_n, W(x_n, u_n, \theta_{n_1})\big) = 0. \tag{22}$$

Then

$$\begin{aligned} d\big(T^n x_n, x_n\big) &\le d\big(T^n x_n, W(x_n, u_n, \theta_{n_1})\big) + d\big(W(x_n, u_n, \theta_{n_1}), x_n\big) \\ &\le d\big(T^n x_n, W(x_n, u_n, \theta_{n_1})\big) + \left(\frac{\gamma_n}{1-a_n}\right) d(u_n, x_n) \\ &\le d\big(T^n x_n, W(x_n, u_n, \theta_{n_1})\big) + \left(\frac{\gamma_n}{1-b}\right) M \end{aligned}$$

together with (22) gives that

$$\lim_{n\to\infty} d\left(T^n x_n, x_n\right) = 0. \tag{23}$$

Next we show that $\lim_{n\to\infty} d(T^n z_n, x_n) = 0$ and $\lim_{n\to\infty} d(T^n y_n, x_n) = 0$.

The inequality

$$\begin{aligned}
d\left(T^n z_n, x_n\right) &\le d\left(T^n z_n, W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1-b_n}\right)\right) \\
&\quad + d\left(W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1-b_n}\right), x_n\right) \\
&\le d\left(T^n z_n, W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1-b_n}\right)\right) \\
&\quad + \frac{c_n}{1-b_n} d\left(T^n x_n, x_n\right) + \left(\frac{1-b_n-c_n}{1-b_n}\right) d\left(W(x_n, v_n, \theta_{n_2}), x_n\right) \\
&\le d\left(T^n z_n, W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1-b_n}\right)\right) \\
&\quad + \frac{c_n}{1-b_n} d\left(T^n x_n, x_n\right) + \frac{\mu_n}{1-b_n} d(x_n, v_n) \\
&\le d\left(T^n z_n, W\left(T^n x_n, W(x_n, v_n, \theta_{n_2}), \frac{c_n}{1-b_n}\right)\right) \\
&\quad + \frac{c_n}{1-b_n} d\left(T^n x_n, x_n\right) + \frac{\mu_n}{1-b_n} M
\end{aligned}$$

together with (23) gives that

$$\lim_{n\to\infty} d\left(T^n z_n, x_n\right) = 0.$$

Similarly,

$$\begin{aligned}
d\left(T^n y_n, x_n\right) &\le d\left(T^n y_n, W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right)\right) \\
&\quad + d\left(W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right), x_n\right) \\
&\le d\left(T^n y_n, W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right)\right) \\
&\quad + \frac{\beta_n}{1-\alpha_n} d\left(T^n z_n, x_n\right) + \left(\frac{\lambda_n}{1-\alpha_n}\right) d(x_n, w_n) \\
&\le d\left(T^n y_n, W\left(T^n z_n, W(x_n, w_n, \theta_{n_3}), \frac{\beta_n}{1-\alpha_n}\right)\right) \\
&\quad + \frac{b}{1-b} d\left(T^n z_n, x_n\right) + \left(\frac{\lambda_n}{1-b}\right) M
\end{aligned}$$

provides that

$$\lim_{n\to\infty} d\left(T^n y_n, x_n\right) = 0.$$

Hence

$$\lim_{n\to\infty} d\big(T^n x_n, x_n\big) = \lim_{n\to\infty} d\big(T^n y_n, x_n\big) = \lim_{n\to\infty} d\big(T^n z_n, x_n\big) = 0. \qquad \square$$

Theorem 2.2 *Let C be a nonempty bounded, closed and convex subset of a (UC) hyperbolic space with monotone modulus of uniform convexity η. Let T be a completely continuous and asymptotically nonexpansive self-mapping on C with $\{k_n \geq 1\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be control sequences in $[0,1]$ satisfying the following conditions*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,
(ii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,
(iii) $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \gamma_n) < 1$,
(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (3) *converge to the same fixed point of T.*

Proof By Lemma 2.1, we have

$$\lim_{n\to\infty} d\big(T^n x_n, x_n\big) = \lim_{n\to\infty} d\big(T^n y_n, x_n\big) = \lim_{n\to\infty} d\big(T^n z_n, x_n\big) = 0.$$

Since

$$\begin{aligned}
d(x_{n+1}, x_n) &= d\left(W\left(T^n y_n, W\left(T^n z_n, W(x_n, w_n, \theta_{n3}), \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), x_n\right) \\
&\leq \alpha_n d\big(T^n y_n, x_n\big) \\
&\quad + (1-\alpha_n) d\left(W\left(T^n z_n, W(x_n, w_n, \theta_{n3}), \frac{\beta_n}{1-\alpha_n}\right), x_n\right) \\
&\leq \alpha_n d\big(T^n y_n, x_n\big) + \beta_n d\big(T^n z_n, x_n\big) + \lambda_n d(x_n, w_n),
\end{aligned}$$

we have

$$\begin{aligned}
d\big(x_{n+1}, T^n x_{n+1}\big) &\leq d(x_{n+1}, x_n) + d\big(T^n x_{n+1}, T^n x_n,\big) + d\big(T^n x_n, x_n\big) \\
&\leq d(x_{n+1}, x_n) + k_n d(x_{n+1}, x_n) + d\big(T^n x_n, x_n\big) \\
&\leq (1 + k_n) d(x_{n+1}, x_n) + d\big(T^n x_n, x_n\big) \\
&\leq (1 + k_n)\alpha_n d\big(T^n y_n, x_n\big) + (1 + k_n)\beta_n d\big(T^n z_n, x_n\big) \\
&\quad + (1 + k_n)\lambda_n d(x_n, w_n) + d\big(T^n x_n, x_n\big).
\end{aligned}$$

This together with Lemma 2.1 implies that

$$\lim_{n\to\infty} d\big(x_{n+1}, T^n x_{n+1}\big) = 0.$$

Moreover, the estimate

$$\begin{aligned}
d(x_{n+1}, Tx_{n+1}) &\leq d\big(x_{n+1}, T^{n+1} x_{n+1}\big) + d\big(Tx_{n+1}, T^{n+1} x_{n+1}\big) \\
&\leq d\big(x_{n+1}, T^{n+1} x_{n+1}\big) + k_1 d\big(x_{n+1}, T^n x_{n+1}\big)
\end{aligned}$$

implies that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{24}$$

Since T is completely continuous and $\{x_n\} \subset C$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{Tx_{n_k}\}$ converges. Therefore from (24), $\{x_{n_k}\}$ converges. Let $\lim_{k\to\infty} x_{n_k} = q$. By the continuity of T and (24), we have that $Tq = q$, so q is a fixed point of T. By Lemma 2.1(i), $\lim_{n\to\infty} d(x_n, q)$ exists. But $\lim_{k\to\infty} d(x_{n_k}, q) = 0$. Thus $\lim_{n\to\infty} d(x_n, q) = 0$. Further the inequalities $d(y_n, x_n) \le b_n d(T^n z_n, x_n) + c_n d(T^n x_n, x_n) + \mu_n d(v_n, x_n)$ and $d(z_n, x_n) \le a_n d(T^n x_n, x_n) + \gamma_n d(u_n, x_n)$ give that $\lim_{n\to\infty} d(y_n, x_n) = 0$ and $\lim_{n\to\infty} d(z_n, x_n) = 0$, respectively.

That is,

$$\lim_{n\to\infty} y_n = q \quad \text{and} \quad \lim_{n\to\infty} z_n = q. \qquad \square$$

For $\gamma_n = \mu_n = \lambda_n = 0$, Theorem 2.2 reduces to the following.

Corollary 2.3 *Let C be a nonempty bounded, closed and convex subset of a (UC) hyperbolic space with monotone modulus of uniform convexity η. Let T be a completely continuous and asymptotically nonexpansive self-mapping on C with $\{k_n \ge 1\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ be in $[0,1]$ with $b_n + c_n, \alpha_n + \beta_n \in [0,1]$ for all $n \ge 1$ and*

(i) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty}(b_n + c_n) < 1$,

(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$.

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (4) converge to the same fixed point of T.

For $c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.2, we obtain the following result.

Corollary 2.4 *Let C be a nonempty bounded, closed and convex subset of a (UC) hyperbolic space with monotone modulus of uniform convexity η. Let T be a completely continuous and asymptotically nonexpansive self-mapping on C with $\{k_n \ge 1\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$ and $\{\alpha_n\}$ be in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$.

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in (6) converge to the same fixed point of T.

For $a_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$ in Theorem 2.2, we can obtain the Ishikawa-type convergence result.

Corollary 2.5 *Let C be a nonempty bounded, closed and convex subset of a (UC) hyperbolic space with monotone modulus of uniform convexity η. Let T be a completely continuous asymptotically nonexpansive self-mapping of C with $\{k_n \ge 1\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ and $\{b_n\}$ be real sequences in $[0,1]$ satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$, *and*

(ii) $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$.

Then $\{x_n\}$ and $\{y_n\}$ in (7) converge to the same fixed point of T.

For $a_n = b_n = c_n = \beta_n = \gamma_n = \mu_n = \lambda_n \equiv 0$, Theorem 2.2 reduces to the Mann-type convergence result.

Corollary 2.6 *Let C be a nonempty bounded, closed and convex subset of a uniformly convex hyperbolic space with monotone modulus of uniform convexity η. Let T be a completely continuous asymptotically nonexpansive self-map of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}$ be real sequences in* $[0,1]$ *satisfying* $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$. *Then $\{x_n\}$ in* (8) *converge to a fixed point of T.*

As a direct consequence of Theorem 2.2, we formulate the following result in CAT(0) spaces.

Corollary 2.7 *Let C be a nonempty bounded, closed and convex subset of a* CAT(0) *space. Let T be a completely continuous and asymptotically nonexpansive self-mapping on C with $\{k_n \geq 1\}$ satisfying $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\mu_n\}$ and $\{\lambda_n\}$ be control sequences in* $[0,1]$ *satisfying the following conditions*:

(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty}(\alpha_n + \beta_n + \lambda_n) < 1$,
(ii) $0 < \liminf_{n\to\infty} b_n \leq \limsup_{n\to\infty}(b_n + c_n + \mu_n) < 1$,
(iii) $0 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty}(a_n + \gamma_n) < 1$,
(iv) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$.

For a given $x_1 \in C$, compute $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ as

$$\begin{aligned}
z_n &= a_n T^n x_n \oplus (1 - \alpha_n)\left[\left(1 - \frac{\gamma_n}{1 - a_n}\right)x_n \oplus \frac{\gamma_n}{1 - a_n}u_n\right], \\
y_n &= b_n T^n z_n \oplus (1 - b_n)\left[\left(\frac{c_n}{1 - b_n}T^n x_n \oplus \left(1 - \frac{c_n}{1 - b_n}\right)\right.\right. \\
&\quad \times \left.\left.\left[\left(1 - \frac{\mu_n}{1 - b_n - c_n}\right)x_n \oplus \left(\frac{\mu_n}{1 - b_n - c_n}\right)v_n\right]\right)\right], \\
x_{n+1} &= \alpha_n T^n y_n \oplus (1 - \alpha_n)\left[\left(\frac{\beta_n}{1 - \alpha_n}T^n z_n \oplus \left(1 - \frac{\beta_n}{1 - \alpha_n}\right)\left[\theta_{n_3} x_n \oplus (1 - \theta_{n_3})w_n\right]\right)\right],
\end{aligned}$$

where $\lambda x \oplus (1 - \lambda)y$ is the geodesic path between x and y in X. Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge to the same fixed point of T.

Proof Any CAT(0) space is a (UC) hyperbolic space (take $W(x, y, \lambda) = \lambda x \oplus (1 - \lambda)y$), therefore conclusion follows from Theorem 2.2. □

Remark 2.8 (1) Our Theorem 2.2 and its corollaries extend and generalize corresponding theorems in a uniformly convex Banach space to a hyperbolic space. Some of these are given below:

(i) Theorem 2.2 itself is a nonlinear version of Theorem 2.3 in [12].
(ii) Corollary 2.3 extends and generalizes Theorem 2.3 in [9].
(iii) Corollary 2.4 extends and generalizes Theorem 2.1 in [13].
(iv) Corollary 2.5 extends and generalizes Theorem 3 in [29].
(v) Corollary 2.6 is a generalization and refinement of Theorem 2 in [29], Theorem 1.5 in [7] and Theorem 2.2 in [8].

(2) Our results also hold in a CAT(0) space.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, 31261, Saudi Arabia. [2]Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, 63100, Pakistan.

Acknowledgements
The first author owes a lot to Professor Wataru Takahashi from whom he received his doctorate at Tokyo Institute of Technology, Tokyo, Japan. He is extremely indebted to Professor Takahashi and wishes him a long healthy active life. The first author is also grateful to King Fahd University of Petroleum and Minerals for supporting research project IN121055. The second author Amna Kalsoom gratefully acknowledges Higher Education Commission (HEC) of Pakistan for financial support during this research.

Received: 31 October 2013 Accepted: 27 February 2014 Published: 17 Mar 2014

References
1. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171-174 (1972)
2. Cho, YJ, Zhou, HY, Guo, G: Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings. Comput. Math. Appl. **47**, 707-717 (2004)
3. Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-510 (1953)
4. Ishikawa, S: Fixed point by a new iteration. Proc. Am. Math. Soc. **44**, 147-150 (1974)
5. Noor, MA: New approximation schemes for general variational inequalities. J. Math. Anal. Appl. **251**, 217-229 (2000)
6. Noor, MA: Three-step iterative algorithms for multivalued quasi variational inclusions. J. Math. Anal. Appl. **255**, 589-604 (2001)
7. Schu, J: Iterative construction of fixed points of asymptotically nonexpansive mappings. J. Math. Anal. Appl. **158**, 407-413 (1991)
8. Schu, J: Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. Bull. Aust. Math. Soc. **43**, 153-159 (1991)
9. Suantai, S: Weak and strong convergence criteria of Noor iterations for asymptotically nonexpansive mappings. J. Math. Anal. Appl. **311**, 506-517 (2005)
10. Khan, SH, Hussain, N: Convergence theorems for nonself asymptotically nonexpansive mappings. Comput. Math. Appl. **55**, 2544-2553 (2008)
11. Khan, SH: Fixed points of quasi-contractive type operators in normed spaces by a three-step iteration process. In: Proceedings of the World Congress on Engineering, London, UK, 6-8 July 2011, vol. I (2011)
12. Nammanee, K, Noor, MA, Suantai, S: Convergence criteria of modified Noor iterations with errors for asymptotically nonexpansive mappings. J. Math. Anal. Appl. **314**, 320-334 (2006)
13. Xu, BL, Noor, MA: Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **267**, 444-453 (2002)
14. Glowinski, R, Le Tallec, P: Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics. SIAM, Philadelphia (1989)
15. Haubruge, S, Nguyen, VH, Strodiot, JJ: Convergence analysis and applications of Glowinski-Le Tallec splitting method for finding a zero of the sum of two maximal monotone operators. J. Optim. Theory Appl. **97**, 645-673 (1998)
16. Kohlenbach, U: Some logical metatheorems with applications in functional analysis. Trans. Am. Math. Soc. **357**, 89-128 (2005)
17. Abbas, M, Khamsi, MA, Khan, AR: Common fixed point and invariant approximation in hyperbolic ordered metric spaces. Fixed Point Theory Appl. **2011**, 25 (2011)
18. Takahashi, W: A convexity in metric spaces and nonexpansive mappings. Kodai Math. Semin. Rep. **22**, 142-149 (1970)
19. Busemann, H: Spaces with non-positive curvature. Acta Math. **80**, 259-310 (1948)
20. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Series of Monographs and Textbooks in Pure and Applied Mathematics, vol. 83. Dekker, New York (1984)
21. Kirk, WA: A fixed point theorem in CAT(0) spaces and *R*-trees. Fixed Point Theory Appl. **4**, 309-316 (2004)
22. Kirk, WA: Fixed Point Theory for Nonexpansive Mappings, vols. I, II. Lecture Notes in Mathematics, vol. 886, pp. 485-505. Springer, Berlin (1981)
23. Leustean, L: A quadratic rate of asymptotic regularity for CAT(0)-spaces. J. Math. Anal. Appl. **325**, 386-399 (2007)
24. Shimizu, T, Takahashi, W: Fixed points of multivalued mappings in certain convex metric spaces. Topol. Methods Nonlinear Anal. **8**, 197-203 (1996)
25. Miyake, H: A fixed point theorem for asymptotically nonexpansive mappings in metric spaces with convex structure. In: Proceedings of International Conference on Nonlinear Analysis and Convex Analysis, pp. 315-321 (2003)
26. Talman, LA: Fixed points for condensing multifunctions in metric spaces with convex structure. Kodai Math. Semin. Rep. **29**, 62-70 (1977)
27. Tan, KK, Xu, HK: Approximating fixed points of nonexpansive mapping by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301-308 (1993)
28. Khan, AR, Fukhar-ud-din, H, Khan, MAA: An implicit algorithm for two finite families of nonexpansive maps in hyperbolic spaces. Fixed Point Theory Appl. **2012**, 54 (2012)
29. Rhoades, BE: Fixed point iterations for certain nonlinear mappings. J. Math. Anal. Appl. **183**, 118-120 (1994)

10.1186/1687-1812-2014-64
Cite this article as: Fukhar-ud-din and Kalsoom: Fixed point approximation of asymptotically nonexpansive mappings in hyperbolic spaces. *Fixed Point Theory and Applications* 2014, 2014:64