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A new explicit iterative algorithm for solving a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings

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Abstract
In this paper, we introduce a new explicit iterative algorithm for finding a solution for a class of variational inequalities over the common fixed points set of a finite family of nonexpansive mappings in Hilbert spaces. Under suitable assumptions, we prove that the sequence generated by the iterative algorithm converges strongly to the unique solution of the variational inequality. Our result improves and extends the corresponding results announced by many others. At the end of the paper, we extend our result to the more broad family of λ-strictly pseudo-contractive mappings.

Keywords: nonexpansive mapping; strong convergence; variational inequalities; common fixed points

1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Throughout this paper, we always assume that T is a nonexpansive operator on H. The fixed point set of T is denoted by $\mathrm{Fix}(T)$, *i.e.*, $\mathrm{Fix}(T) = \{x \in H : Tx = x\}$. The typical problem is to minimize a quadratic function on a real Hilbert space H:

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, u\rangle, \tag{1.1}$$

where C is a nonempty closed convex subset of H, u is a given point in H and A is a strongly positive bounded linear operator on H.

In 2003, Xu [1] introduced the following iterative scheme:

$$x_{n+1} = \alpha_n u + (I - \alpha_n A)Tx_n, \tag{1.2}$$

where u is some point of H and $\{\alpha_n\}$ is a sequence in $(0,1)$. He proved that the sequence $\{x_n\}$ converges strongly to the unique solution of the minimization problem (1.1) with $C = \mathrm{Fix}(T)$.

In 2006, Marino and Xu [2] considered the viscosity method on the iterative scheme (1.2), and they gave the following general iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n, \tag{1.3}$$

where f is a contraction on H. They proved the above sequence $\{x_n\}$ converges strongly to the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T),$$

which is the optimality condition for the minimization problem

$$\min_{x \in \mathrm{Fix}(T)} \frac{1}{2} \langle Ax, x \rangle - h(x),$$

where h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$ for $x \in H$).

In 2001, Yamada [3] considered the following hybrid iterative method:

$$x_{n+1} = Tx_n - \mu \lambda_n F(Tx_n), \tag{1.4}$$

where F is L-Lipschitzian continuous and η-strongly monotone operator with $L > 0$, $\eta > 0$ and $0 < \mu < 2\eta/L^2$. Under some appropriate conditions, the sequence $\{x_n\}$ generated by (1.4) converges strongly to the unique solution of the variational inequality

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathrm{Fix}(T).$$

Combining (1.3) and (1.4), Tian [4] considered the following general viscosity type iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \mu \alpha_n F)Tx_n. \tag{1.5}$$

Improving and extending the corresponding results given by Marino *et al.*, he proved that the sequence $\{x_n\}$ generated by (1.5) converges strongly to the unique solution $x^* \in \mathrm{Fix}(T)$ of the variational inequality

$$\langle (\gamma f - \mu F)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T).$$

In [5], Tian generalized the iterative method (1.5) replacing the contraction operator f with a Lipschitzian continuous operator V to solve the following variational inequality:

$$\langle (\gamma V - \mu F)\tilde{x}, x - \tilde{x} \rangle \leq 0, \quad \forall x \in \mathrm{Fix}(T). \tag{1.6}$$

On the other hand, let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-mappings of H. Assume $\bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$. In [1], Xu also defined the following sequence $\{x_n\}$:

$$x_{n+1} = \alpha_n u + (I - \alpha_n A)T_{n+1}x_n, \quad n \geq 0, \tag{1.7}$$

where $T_n = T_{n \bmod N}$ and the mod function takes values in $\{1, 2, \ldots, N\}$. He found that the sequence $\{x_n\}$ generated by (1.7) converges strongly to the unique solution of the minimization problem (1.1) with $C = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ under suitable conditions on $\{\alpha_n\}$ and the following additional condition on $\{T_n\}$:

$$F(T_N \cdots T_2 T_1) = F(T_1 T_N \cdots T_3 T_2) = \cdots = F(T_{N-1} \cdots T_1 T_N). \tag{1.8}$$

In fact, there are many nonexpansive mappings which do not satisfy (1.8).

In 1999, Atsushiba and Takahashi [6] defined the W_n-mappings generated by $T_1, T_2, \ldots, T_N$ and $\{\gamma_{n,1}\}, \{\gamma_{n,2}\}, \ldots, \{\gamma_{n,N}\} \subset [0,1]$ as follows:

$$\begin{aligned}
&U_{n,0} = I,\\
&U_{n,1} = \gamma_{n,1} T_1 U_{n,0} + (1-\gamma_{n,1})I,\\
&U_{n,2} = \gamma_{n,2} T_2 U_{n,1} + (1-\gamma_{n,2})I,\\
&\vdots\\
&U_{n,N-1} = \gamma_{n,N-1} T_{N-1} U_{n,N-2} + (1-\gamma_{n,N-1})I,\\
&W_n = U_{n,N} = \gamma_{n,N} T_N U_{n,N-1} + (1-\gamma_{n,N})I.
\end{aligned}$$

From [6, Lemma 3.1], we know that $F(W_n) = \bigcap_{i=1}^N F(T_i)$.

In 2006, Yao [7] introduced the following iterative method:

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta x_n + \bigl((1-\beta)I - \alpha_n A\bigr) W_n x_n. \tag{1.9}$$

Without the condition (1.8), he proved that the sequence $\{x_n\}$ generated by (1.9) converges strongly to the unique solution of the following variational inequality:

$$\bigl\langle (A-\gamma f)x^*, x^* - x\bigr\rangle \le 0, \quad \forall x \in \bigcap_{i=1}^N \mathrm{Fix}(T_i), \tag{1.10}$$

which is the optimality condition for the minimization problem

$$\min_{x\in C} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.11}$$

where $C = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$ and h is a potential function for γf (*i.e.*, $h'(x) = \gamma f(x)$).

Shang *et al.* [8] introduced the following scheme:

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n) W_n x_n,\\ x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n. \end{cases} \tag{1.12}$$

Under certain appropriate conditions, without (1.8), they proved that $\{x_n\}$ defined by (1.12) converges strongly to the unique solution of (1.10) which is also the optimality condition for (1.11).

Recently, combining the Krasnoselskii-Mann type algorithm and the steepest-descent method, Buong and Duong [9] introduced a new explicit iterative algorithm:

$$x_{k+1} = \bigl(1-\beta_k^0\bigr)x_k + \beta_k^0 T_0^k T_N^k \cdots T_1^k x_k, \tag{1.13}$$

where $T_i^k = (1-\beta_k^i)I + \beta_k^i T_i$ for $i = 1,2,\ldots,N$, $T_0^k = I - \lambda_k \mu F$, and F is an L-Lipschitz continuous and η-strongly monotone mapping. Under some appropriate assumptions, they proved that the sequence $\{x_k\}$ converges strongly to the unique solution of the following variational inequality:

$$\langle F(x^*), x - x^* \rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i). \tag{1.14}$$

Very recently, Zhou and Wang [10] proposed a simpler iterative algorithm than the iterative algorithm (1.13) given by Buong and Duong:

$$x_{k+1} = (I - \lambda_k \mu F) T_N^k \cdots T_1^k x_k. \tag{1.15}$$

They proved that the sequence $\{x_k\}$ defined by (1.15) converges strongly to the unique solution of the variational inequality (1.14) in a faster rate of convergence.

Motivated and inspired by the results of Zhou *et al.*, in this paper, we consider a new iterative algorithm to solve the class of variational inequalities (1.6). The iterative algorithm improves and extends the results of Yao *et al.*, and the corresponding results announced by many others. At the end of this paper, we extend our iterative algorithm to the more broad family of λ-strictly pseudo-contractive mappings.

2 Preliminaries

Throughout this paper, we write $x_n \rightharpoonup x$ and $x_n \to x$ to indicate that $\{x_n\}$ converges weakly to x and converges strongly to x, respectively.

An operator $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. It is well known that $\mathrm{Fix}(T)$ is closed and convex. A is called strongly positive if there exists a constant $\gamma > 0$ such that $\langle Ax, x \rangle \geq \gamma \|x\|^2$ for all $x \in H$. The operator F is called η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle x - y, Fx - Fy \rangle \geq \eta \|x - y\|^2$$

for all $x, y \in H$.

In order to prove our results, we collect some necessary conceptions and lemmas in this section.

Definition 2.1 A mapping $T : H \to H$ is said to be an averaged mapping if there exists some number $\alpha \in (0, 1)$ such that

$$T = (1 - \alpha) I + \alpha S, \tag{2.1}$$

where $I : H \to H$ is the identity mapping and $S : H \to H$ is nonexpansive. More precisely, when (2.1) holds, we say that T is α-averaged.

Lemma 2.1 ([11]) (i) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings* $\{T_i\}_{i=1}^N$ *is averaged, then so is the composite* $T_1 \cdots T_N$. *In particular, if* T_1 *is* α_1*-averaged and* T_2 *is* α_2*-averaged, where* $\alpha_1, \alpha_2 \in (0, 1)$, *then both* $T_1 T_2$ *and* $T_2 T_1$ *are* α*-averaged, where* $\alpha = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2$.

(ii) *If the mappings* $\{T_i\}_{i=1}^N$ *are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 \cdots T_N).$$

In particular, if $N = 2$, *we have* $\mathrm{Fix}(T_1) \cap \mathrm{Fix}(T_2) = \mathrm{Fix}(T_1 T_2) = \mathrm{Fix}(T_2 T_1)$.

Lemma 2.2 ([12]) *Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if the following inequality holds*:

$$\langle x - y, z - y \rangle \leq 0$$

for every $z \in C$.

Lemma 2.3 ([5]) *Assume V is a contraction on a Hilbert space H with coefficient $\alpha > 0$, and $F : H \to H$ is an L-Lipschitzian continuous and η-strongly monotone operator with $L > 0$, $\eta > 0$. Then, for $0 < \gamma < \frac{\mu\eta}{\alpha}$, $\mu F - \gamma V$ is strongly monotone with coefficient $\mu\eta - \gamma\alpha$.*

Lemma 2.4 ([13]) *Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to $x \in C$ and $\{(I - T)x_n\}$ converges strongly to $y \in C$, then $(I - T)x = y$. In particular, if $y = 0$, then $x \in \mathrm{Fix}(T)$.*

Lemma 2.5 ([14]) *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence in $[0,1]$ which satisfies the following condition*:

$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1.$$

Suppose $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$ for all integers $n \geq 0$ and

$$\limsup_{n\to\infty} \bigl(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|\bigr) \leq 0.$$

Then $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 2.6 ([1]) *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7 ([15]) *Assume S is a λ-strictly pseudo-contractive mapping on a Hilbert space H. Define a mapping T by $Tx = \alpha x + (1 - \alpha)Sx$ for all $x \in H$ and $\alpha \in [\lambda, 1)$. Then T is a nonexpansive mapping such that $\mathrm{Fix}(T) = \mathrm{Fix}(S)$.*

3 Main results

Now we state and prove our main results in this paper.

Theorem 3.1 *Let $\{T_i\}_{i=1}^N$ be N nonexpansive mappings of a real Hilbert space H such that $C = \bigcap_{i=1}^N \mathrm{Fix}(T_i) \neq \emptyset$, F be an L-Lipschitzian continuous and η-strongly monotone operator on H with $L > 0$ and $\eta > 0$, V be an α-Lipschitzian on H with $\alpha > 0$. Suppose $x_1 \in H$ and $0 < \mu < \frac{2\eta}{L^2}$. Define a sequence $\{x_k\}$ as follows*:

$$x_{k+1} = \alpha_k \gamma V(x_k) + (I - \mu\alpha_k F)T_N^k T_{N-1}^k \cdots T_1^k x_k, \quad k \geq 0, \tag{3.1}$$

where $0 < \gamma < \frac{\tau}{\alpha}$ *with* $\tau = \mu(\eta - \frac{1}{2}\mu L^2)$ *and* $T_i^k = (1-\beta_k^i)I + \beta_k^i T_i$ *for* $i = 1,2,\ldots,N$. *Suppose* $\alpha_k \in (0,1)$ *and* $\beta_k^i \in (\xi,\zeta)$ *for some* $\xi,\zeta \in (0,1)$. *If the following conditions are satisfied*:

(i) $\lim_{k\to\infty} \alpha_k = 0$;

(ii) $\sum_{k=1}^{\infty} \alpha_k = \infty$;

(iii) $\lim_{k\to\infty} |\beta_{k+1}^i - \beta_k^i| = 0$ *for* $i = 1,2,\ldots,N$.

Then the sequence $\{x_k\}$ *converges strongly to the unique solution* x^* *of the variational inequality*:

$$\langle(\mu F - \gamma V)x^*, x - x^*\rangle \geq 0, \quad \forall x \in \bigcap_{i=1}^{N} \operatorname{Fix}(T_i). \tag{3.2}$$

Equivalently, we have $P_C(I - \mu F + \gamma V)x^* = x^*$.

Proof Since our methods easily deduce the general case, we prove Theorem 3.1 for $N = 2$.

First, we show $\{x_k\}$ is bounded. In fact, for some point $p \in C$, by (3.1) we have

$$\begin{aligned}
\|x_{k+1} - p\| &= \|\alpha_k\gamma Vx_k + (I - \mu\alpha_k F)T_2^k T_1^k x_k - p\| \\
&= \|(I - \mu\alpha_k F)T_2^k T_1^k x_k - (I - \mu\alpha_k F)p + \alpha_k(\gamma Vx_k - \mu Fp)\| \\
&\leq (1 - \alpha_k\tau)\|T_2^k T_1^k x_k - T_2^k T_1^k p\| + \alpha_k\big(\|\gamma Vx_k - \gamma Vp\| + \|\gamma Vp - \mu Fp\|\big) \\
&\leq (1 - \alpha_k\tau)\|x_k - p\| + \alpha_k\gamma\alpha\|x_k - p\| + \alpha_k\|\gamma Vp - \mu Fp\| \\
&= \big(1 - \alpha_k(\tau - \gamma\alpha)\big)\|x_k - p\| + \alpha_k(\tau - \gamma\alpha)\frac{\|\gamma Vp - \mu Fp\|}{\tau - \gamma\alpha} \\
&\leq \max\left\{\|x_k - p\|, \frac{1}{\tau - \gamma\alpha}\|\gamma Vp - \mu Fp\|\right\} \\
&\leq \cdots \leq \max\left\{\|x_0 - p\|, \frac{1}{\tau - \gamma\alpha}\|\gamma Vp - \mu Fp\|\right\}.
\end{aligned}$$

Therefore, $\{x_k\}$ is bounded. Hence we also see that $\{T_2^k T_1^k x_k\}$, $\{FT_2^k T_1^k x_k\}$, and $\{Vx_k\}$ are all bounded. From (3.1), it follows that

$$\lim_{k\to\infty} \|x_{k+1} - T_2^k T_1^k x_k\| = 0. \tag{3.3}$$

We next show that $\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0$. Noting that T_1^k and T_2^k are β_k^1-averaged and β_k^2-averaged, respectively, by Lemma 2.1, we find that $T_2^k T_1^k$ is t_k-averaged for every k, where $t_k = \beta_k^1 + \beta_k^2 - \beta_k^1\beta_k^2$. Set $\xi^* = 2\xi - \xi^2$ and $\zeta^* = 2\zeta - \zeta^2$. It is easy to deduce that $0 < \xi^* \leq t_k \leq \zeta^* < 1$ for all k and

$$\lim_{k\to\infty} \|t_{k+1} - t_k\| = 0. \tag{3.4}$$

Since for every k, $T_2^k T_1^k$ is t_k-averaged, we can find a family of nonexpansive mappings $\{S_k\}_{k\geq 0}$ on H such that

$$T_2^k T_1^k = (1 - t_k)I + t_k S_k, \quad k \geq 0. \tag{3.5}$$

Substituting (3.4) into (3.1) yields

$$\begin{aligned} x_{k+1} &= \alpha_k \gamma V x_k + (I - \mu\alpha_k F)\bigl[(1-t_k)x_k + t_k S_k x_k\bigr] \\ &= (1-t_k)x_k + t_k\biggl[S_k x_k + \frac{\alpha_k}{t_k}\bigl(\gamma V x_k - \mu F T_2^k T_1^k x_k\bigr)\biggr]. \end{aligned}$$

Define a sequence $\{z_k\}$ by $z_k = S_k x_k + \frac{\alpha_k}{t_k}(\gamma V x_k - \mu F T_2^k T_1^k x_k)$, so

$$x_{k+1} = (1-t_k)x_k + t_k z_k. \tag{3.6}$$

Now, we claim that

$$\limsup_{k\to\infty}\bigl(\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|\bigr) \le 0.$$

To this end, we observe that

$$\begin{aligned} \|z_{k+1} - z_k\| &\le \|S_{k+1}x_{k+1} - S_k x_k\| + \frac{\alpha_{k+1}}{t_{k+1}}\bigl\|\gamma V x_{k+1} - \mu F T_2^{k+1} T_1^{k+1} x_{k+1}\bigr\| \\ &\quad + \frac{\alpha_k}{t_k}\bigl\|\gamma V x_k - \mu F T_2^k T_1^k x_k\bigr\| \\ &\le \|S_{k+1}x_{k+1} - S_{k+1}x_k\| + \|S_{k+1}x_k - S_k x_k\| \\ &\quad + \frac{\alpha_{k+1}}{t_{k+1}}\bigl(\|\gamma V x_{k+1}\| + \bigl\|\mu F T_2^{k+1} T_1^{k+1} x_{k+1}\bigr\|\bigr) \\ &\quad + \frac{\alpha_k}{t_k}\bigl(\|\gamma V x_k\| + \bigl\|\mu F T_2^k T_1^k x_k\bigr\|\bigr) \\ &\le \|x_{k+1} - x_k\| + \|S_{k+1}x_k - S_k x_k\| \\ &\quad + \frac{\alpha_{k+1}}{t_{k+1}}\bigl(\|\gamma V x_{k+1}\| + \bigl\|\mu F T_2^{k+1} T_1^{k+1} x_{k+1}\bigr\|\bigr) \\ &\quad + \frac{\alpha_k}{t_k}\bigl(\|\gamma V x_k\| + \bigl\|\mu F T_2^k T_1^k x_k\bigr\|\bigr) \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \|S_{k+1}x_k - S_k x_k\| &= \biggl\|\frac{1}{t_{k+1}}T_2^{k+1}T_1^{k+1}x_k - \frac{1}{t_k}T_2^k T_1^k x_k - \frac{1-t_{k+1}}{t_{k+1}}x_k + \frac{1-t_k}{t_k}x_k\biggr\| \\ &\le \biggl|\frac{t_{k+1}-t_k}{t_{k+1}t_k}\biggr|\bigl(\bigl\|T_2^{k+1}T_1^{k+1}x_k\bigr\| + \|x_k\|\bigr) + \frac{1}{t_k}\bigl\|T_2^{k+1}T_1^{k+1}x_k - T_2^k T_1^k x_k\bigr\| \\ &\le \biggl|\frac{t_{k+1}-t_k}{t_{k+1}t_k}\biggr| M + \frac{1}{t_k}\bigl(\bigl\|T_2^{k+1}T_1^{k+1}x_k - T_2^{k+1}T_1^k x_k\bigr\| \\ &\quad + \bigl\|T_2^{k+1}T_1^k x_k - T_2^k T_1^k x_k\bigr\|\bigr) \\ &\le \biggl|\frac{t_{k+1}-t_k}{t_{k+1}t_k}\biggr| M + \frac{1}{\xi^*}\bigl(\bigl\|T_1^{k+1}x_k - T_1^k x_k\bigr\| \\ &\quad + \bigl\|T_2^{k+1}T_1^k x_k - T_2^k T_1^k x_k\bigr\|\bigr), \end{aligned} \tag{3.8}$$

where M is a fixed constant satisfying

$$M \ge \sup_{k\ge 0}\bigl\{\bigl\|T_2^{k+1}T_1^{k+1}x_k\bigr\| + \|x_k\|\bigr\}.$$

Note that

$$\begin{aligned}\left\|T_1^{k+1}x_k - T_1^k x_k\right\| &= \left\|\left(1-\beta_{k+1}^1\right)x_k + \beta_{k+1}^1 T_1 x_k - \left(1-\beta_k^1\right)x_k - \beta_k^1 T_1 x_k\right\| \\ &\leq \left|\beta_{k+1}^1 - \beta_k^1\right|\left(\|x_k\| + \|T_1 x_k\|\right).\end{aligned}$$

Since $\lim_{k\to\infty}|\beta_{k+1}^i - \beta_k^i| = 0$ for $i = 1,2$, and $\{x_k\}$ and $\{T_1 x_k\}$ are bounded, we easily obtain

$$\lim_{k\to\infty}\left\|T_1^{k+1}x_k - T_1^k x_k\right\| = 0. \tag{3.9}$$

Similarly,

$$\left\|T_2^{k+1}T_1^k x_k - T_2^k T_1^k x_k\right\| \leq \left|\beta_{k+1}^2 - \beta_k^2\right|\left(\left\|T_1^k x_k\right\| + \left\|T_2 T_1^k x_k\right\|\right),$$

from which it follows that

$$\lim_{k\to\infty}\left\|T_2^{k+1}T_1^k x_k - T_2^k T_1^k x_k\right\| = 0. \tag{3.10}$$

Using (3.4), (3.9), and (3.10), from (3.8) we have

$$\lim_{k\to\infty}\|S_{k+1}x_k - S_k x_k\| = 0. \tag{3.11}$$

Since $\lim_{k\to\infty}\alpha_k = 0$ and $0 < \xi^* < t_k < \zeta^* < 1$, combining (3.7) and (3.11) we get

$$\limsup_{k\to\infty}\left(\|z_{k+1} - z_k\| - \|x_{k+1} - x_k\|\right) \leq 0.$$

By Lemma 2.5, we conclude that $\lim_{k\to\infty}\|z_k - x_k\| = 0$, which implies that $\lim_{k\to\infty}\|x_{k+1} - x_k\| = 0$ by (3.6). Thus from (3.3), it is true that

$$\lim_{k\to\infty}\left\|x_k - T_2^k T_1^k x_k\right\| = 0. \tag{3.12}$$

From [8, Theorem 3.2], we know that the solution of the variational inequality (3.2) is unique. We use x^* to denote the unique solution of (3.2). Since $\{x_k\}_{k\geq 0}$ is bounded, there exists a subsequence $\{x_{k_j}\}_{j\geq 1}$ of $\{x_k\}_{k\geq 0}$ such that $x_{k_j} \rightharpoonup \hat{x}$ as $j\to\infty$ and

$$\limsup_{k\to\infty}\left\langle(\mu F - \gamma V)x^*, x^* - x_k\right\rangle = \lim_{j\to\infty}\left\langle(\mu F - \gamma V)x^*, x^* - x_{k_j}\right\rangle.$$

Since $\{\beta_k^i\}$ is bounded for $i = 1,2$, we can assume that $\beta_{k_j}^i \to \beta_\infty^i$ as $j\to\infty$, where $0 < \xi \leq \beta_\infty^i \leq \zeta < 1$ for $i = 1,2$. Define $T_i^\infty = (1-\beta_\infty^i)I + \beta_\infty^i T_i$ ($i = 1,2$). Then we have $\mathrm{Fix}(T_i^\infty) = \mathrm{Fix}(T_i)$ for $i = 1,2$. Note that

$$\left\|T_i^{k_j}x - T_i^\infty x\right\| \leq \left|\beta_{k_j}^i - \beta_\infty^i\right|\left(\|x\| + \|T_i x\|\right).$$

Hence, we deduce that

$$\lim_{j\to\infty}\sup_{x\in D}\left\|T_i^{k_j}x - T_i^\infty x\right\| = 0, \tag{3.13}$$

where D is an arbitrary bounded subset of H.

Since $\mathrm{Fix}(T_1^\infty)\cap\mathrm{Fix}(T_2^\infty)=\mathrm{Fix}(T_1)\cap\mathrm{Fix}(T_2)=C\neq\emptyset$ and T_i^∞ is β_∞^i-averaged for $i=1,2$, by Lemma 2.1, we know that $\mathrm{Fix}(T_2^\infty T_1^\infty)=\mathrm{Fix}(T_2^\infty)\cap\mathrm{Fix}(T_1^\infty)=C$. Combining (3.12) and (3.13), we obtain

$$\begin{aligned}\|x_{k_j}-T_2^\infty T_1^\infty x_{k_j}\| &\le \|x_{k_j}-T_2^{k_j}T_1^{k_j}x_{k_j}\|+\|T_2^{k_j}T_1^{k_j}x_{k_j}-T_2^\infty T_1^{k_j}x_{k_j}\|\\ &\quad+\|T_2^\infty T_1^{k_j}x_{k_j}-T_2^\infty T_1^\infty x_{k_j}\|\\ &\le \|x_{k_j}-T_2^{k_j}T_1^{k_j}x_{k_j}\|+\|T_2^{k_j}T_1^{k_j}x_{k_j}-T_2^\infty T_1^{k_j}x_{k_j}\|\\ &\quad+\|T_1^{k_j}x_{k_j}-T_1^\infty x_{k_j}\|\\ &\le \|x_{k_j}-T_2^{k_j}T_1^{k_j}x_{k_j}\|+\sup_{x\in D'}\|T_2^{k_j}x-T_2^\infty x\|\\ &\quad+\sup_{x\in D''}\|T_1^{k_j}x-T_1^\infty x\|,\end{aligned}$$

where D' is a bounded subset including $\{T_1^{k_j}x_{k_j}\}$ and D'' is a bounded subset including $\{x_{k_j}\}$. Hence $\lim_{j\to\infty}\|x_{k_j}-T_2^\infty T_1^\infty x_{k_j}\|=0$. From Lemma 2.4, we have $\hat{x}\in\mathrm{Fix}(T_2^\infty T_1^\infty)=C$. It follows that

$$\begin{aligned}\limsup_{k\to\infty}\langle(\mu F-\gamma V)x^*,x^*-T_2^kT_1^kx_k\rangle &= \limsup_{k\to\infty}\langle(\mu F-\gamma V)x^*,x^*-x_k\rangle\\ &= \lim_{j\to\infty}\langle(\mu F-\gamma V)x^*,x^*-x_{k_j}\rangle\\ &= \langle(\mu F-\gamma V)x^*,x^*-\hat{x}\rangle\le 0.\end{aligned}\tag{3.14}$$

Finally, we show that $x_k\to x^*$ as $k\to\infty$. From (3.1), we have

$$\begin{aligned}\|x_{k+1}-x^*\|^2 &= \|\alpha_k\gamma Vx_k+(I-\mu\alpha_kF)T_2^kT_1^kx_k-x^*\|^2\\ &= \|(I-\mu\alpha_kF)T_2^kT_1^kx_k-(I-\mu\alpha_kF)x^*+\alpha_k(\gamma Vx_k-\mu Fx^*)\|^2\\ &= \|(I-\mu\alpha_kF)T_2^kT_1^kx_k-(I-\mu\alpha_kF)x^*\|^2+\alpha_k^2\|\gamma Vx_k-\mu Fx^*\|^2\\ &\quad+2\alpha_k\langle(I-\mu\alpha_kF)T_2^kT_1^kx_k-(I-\mu\alpha_kF)x^*,\gamma Vx_k-\mu Fx^*\rangle\\ &\le (1-\alpha_k\tau)^2\|x_k-x^*\|^2+\alpha_k^2\|\gamma Vx_k-\mu Fx^*\|^2\\ &\quad+2\alpha_k\langle T_2^kT_1^kx_k-x^*,\gamma Vx_k-\mu Fx^*\rangle\\ &\quad-2\mu\alpha_k^2\langle FT_2^kT_1^kx_k-Fx^*,\gamma Vx_k-\mu Fx^*\rangle\\ &\le (1-\alpha_k\tau)^2\|x_k-x^*\|^2+\alpha_k^2\|\gamma Vx_k-\mu Fx^*\|^2\\ &\quad+2\alpha_k\gamma\langle T_2^kT_1^kx_k-x^*,Vx_k-Vx^*\rangle\\ &\quad+2\alpha_k\langle T_2^kT_1^kx_k-x^*,\gamma Vx^*-\mu Fx^*\rangle\\ &\quad-2\mu\alpha_k^2\langle FT_2^kT_1^kx_k-Fx^*,\gamma Vx_k-\mu Fx^*\rangle\\ &\le \big[(1-\alpha_k\tau)^2+2\alpha\alpha_k\gamma\big]\|x_k-x^*\|^2+\alpha_k\big[2\langle T_2^kT_1^kx_k-x^*,(\gamma V-\mu F)x^*\rangle\\ &\quad+\alpha_k\|\gamma Vx_k-\mu Fx^*\|^2+2\alpha_kL\|T_2^kT_1^kx_k-x^*\|\|\gamma Vx_k-\mu Fx^*\|\big]\\ &= \big[1-2\alpha_k(\tau-\alpha\gamma)\big]\|x_k-x^*\|^2+\alpha_k\big[2\langle T_2^kT_1^kx_k-x^*,(\gamma V-\mu F)x^*\rangle\\ &\quad+\alpha_k\big(\|\gamma Vx_k-\mu Fx^*\|^2+2L\|x_k-x^*\|\|\gamma Vx_k-\mu Fx^*\|+\tau^2\|x_k-x^*\|^2\big)\big]\end{aligned}$$

$$\le \bigl[1 - 2\alpha_k(\tau - \alpha\gamma)\bigr]\|x_k - x^*\|^2 + \alpha_k\bigl[2\bigl\langle T_2^k T_1^k x_k - x^*, (\gamma V - \mu F)x^*\bigr\rangle + \alpha_k M'\bigr],$$

where M' is a constant satisfying

$$M' \ge \sup_{k\ge 0}\bigl\{\|\gamma V x_k - \mu F x^*\|^2 + 2L\|T_2^k T_1^k x_k - x^*\|\|\gamma V x_k - \mu F x^*\| + \tau^2\|x_k - x^*\|^2\bigr\}.$$

Consequently, according to the conditions (i) and (ii), (3.14), and Lemma 2.6, we conclude that $x_k \to x^*$ as $k \to \infty$. This completes the proof. □

4 An extension of our result

In this section, we extend our result to the more broad family of λ-strictly pseudo-contractive mappings. Now let us recall that a mapping $S: H \to H$ is said to be λ-strictly pseudo-contractive if there exists a constant $\lambda \in [0,1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \lambda\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in H.$$

Let $\{S_i\}_{i=1}^N$ be a family of λ_i-strictly pseudo-contractive self-mappings of H with $0 \le \lambda_i < 1$. For $i = 1, 2, \ldots, N$, define

$$\hat{T}_i = \omega_i I + (1 - \omega_i)S_i, \tag{4.1}$$

where $0 \le \lambda_i \le \omega_i < 1$. By virtue of Lemma 2.7, we know that $\{\hat{T}_i\}_{i=1}^N$ is a family of nonexpansive mappings. Thus we extend Theorem 3.1 to the family of λ_i-strictly pseudo-contractions.

Theorem 4.1 *Let H be a real Hilbert space, $F: H \to H$ be an L-Lipschitizian continuous and η-strongly monotone operator on H with $L > 0$ and $\eta > 0$, V be an α-Lipschitzian continuous on H with $\alpha > 0$. Let $\{S_i\}_{i=1}^N$ be N λ_i-strictly pseudo-contractive mappings on H such that $C = \bigcap_{i=1}^N \mathrm{Fix}(S_i) \neq \emptyset$. Suppose $0 < \mu < \frac{\tau}{\alpha}$, $0 < \gamma < \frac{\tau}{\alpha}$ with $\tau = \mu(\eta - \frac{1}{2}\mu L^2)$, $\alpha_k \in (0,1)$, $\beta_k^i \in (\xi, \zeta)$ for some $\xi, \zeta \in (0,1)$ and $0 \le \lambda_i \le \omega_i < 1$ for $i = 1, 2, \ldots, N$. If the conditions* (i)-(iii) *of Theorem* 3.1 *are satisfied, the sequence $\{x_k\}_{k\ge 0}$ defined by* (3.1) *with T_i replaced by* (4.1), *converges strongly to the unique solution x^* of the following variational inequality*:

$$\bigl\langle(\mu F - \gamma V)x^*, x - x^*\bigr\rangle \ge 0, \quad \forall x \in \bigcap_{i=1}^N \mathrm{Fix}(S_i).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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References

1. Xu, HK: An iterative approach to quadratic optimization. J. Optim. Theory Appl. **116**, 659-678 (2003)
2. Marino, G, Xu, HK: A general iterative method for nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. **318**, 43-52 (2006)
3. Yamada, I: The hybrid steepest descent for the variational inequality problems over the intersection of fixed points sets of nonexpansive mapping. In: Butnariu, D, Censor, Y, Reich, S (eds.) Inherently Parallel Algorithms in Feasibility and Optimization and Their Application, pp. 473-504. Elsevier, New York (2001)
4. Tian, M: A general iterative algorithm for nonexpansive mappings in Hilbert spaces. Nonlinear Anal. **73**, 689-694 (2010)
5. Tian, M: A general iterative method based on the hybrid steepest descent scheme for nonexpansive mappings in Hilbert spaces. In: 2010 International Conference on Computational Intelligence and Software Engineering (CiSE) (2010)
6. Atsushiba, S, Takahashi, W: Strong convergence theorems for a finite family of nonexpansive mappings and applications. Indian J. Math. **116**(3), 659-678 (2003)
7. Yao, Y: A general iterative method for a finite family of nonexpansive mappings. Nonlinear Anal. **66**, 2676-2687 (2007)
8. Shang, M, Su, Y, Qin, X: Strong convergence theorems for a finite family of nonexpansive mappings. Fixed Point Theory Appl. **2007**, Article ID 76971 (2007). doi:10.1155/2007/76971
9. Buong, N, Duong, LT: An explicit iterative algorithm for a class of variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **151**, 513-524 (2011)
10. Zhou, Y, Wang, P: A simpler explicit iterative algorithm for a class of variational inequalities in Hilbert spaces. J. Optim. Theory Appl. (2013). doi:10.1007/s10957-013-0470-x (forthcoming)
11. López, G, Martin, V, Xu, HK: Iterative algorithm for the multiple-sets split feasibility problem. In: Biomedical Mathematics: Promising Directions in Imaging, Therapy Planning and Inverse Problems, pp. 243-279 (2009)
12. Marino, G, Xu, HK: Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. **329**, 336-346 (2007)
13. Goebel, K, Kirk, WA: Topics in Metric Fixed-Point Theory. Cambridge University Press, Cambridge (1990)
14. Suzuki, T: Strong convergence of Krasnoselskii and Mann's type sequences for one parameter nonexpansive semigroups without Bochner integrals. J. Math. Anal. Appl. **35**, 227-239 (2005)
15. Zhou, Y: Convergence theorems of fixed points for *k*-strict pseudo-contractions in Hilbert spaces. Nonlinear Anal. **69**, 456-462 (2008)

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