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Boundary point algorithms for minimum norm fixed points of nonexpansive mappings

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Abstract

Let H be a real Hilbert space and C be a closed convex subset of H. Let $T: C \to C$ be a nonexpansive mapping with a nonempty set of fixed points $\mathrm{Fix}(T)$. If $0 \notin C$, then Halpern's iteration process $x_{n+1} = (1 - t_n)Tx_n$ cannot be used for finding a minimum norm fixed point of T since x_n may not belong to C. To overcome this weakness, Wang and Xu introduced the iteration process $x_{n+1} = P_C(1 - t_n)Tx_n$ for finding the minimum norm fixed point of T, where the sequence $\{t_n\} \subset (0, 1)$, $x_0 \in C$ arbitrarily and P_C is the metric projection from H onto C. However, it is difficult to implement this iteration process in actual computing programs because the specific expression of P_C cannot be obtained, in general. In this paper, three new algorithms (called boundary point algorithms due to using certain boundary points of C at each iterative step) for finding the minimum norm fixed point of T are proposed and strong convergence theorems are proved under some assumptions. Since the algorithms in this paper do not involve P_C, they are easy to implement in actual computing programs.

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Keywords: minimum norm fixed point; nonexpansive mapping; metric projection; boundary point algorithm; Hilbert space

1 Introduction and preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$, and let C be a nonempty closed convex subset of H. Recall that a mapping $T : C \to C$ is nonexpansive if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in C$. We use $\mathrm{Fix}(T)$ to denote a set of fixed points of T, *i.e.*, $\mathrm{Fix}(T) \triangleq \{x \in C \mid Tx = x\}$. Throughout this article, $\mathrm{Fix}(T)$ is always assumed to be nonempty.

For every nonempty closed convex subset K of H, the metric (or nearest point) projection indicated by P_K from H onto K can be defined, that is, for each $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| \mid z \in K\}$. It is well known (*e.g.*, see [1]) that P_K is nonexpansive and a characteristic inequality holds.

Lemma 1.1 *Let K be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if there holds the relation*

$$\langle x - z, y - z \rangle \le 0, \quad \forall y \in K.$$

Since $\mathrm{Fix}(T)$ is a closed convex subset of H, so the metric projection $P_{\mathrm{Fix}(T)}$ is valid and thus there exists a unique element, denoted by $x^\dagger$, in $\mathrm{Fix}(T)$ such that $\|x^\dagger\| = \inf_{x \in \mathrm{Fix}(T)} \|x\|$, that is, $x^\dagger = P_{\mathrm{Fix}(T)}0$. $x^\dagger$ is called a minimum norm fixed point of T. Because the minimum norm fixed point of a nonexpansive mapping is closely related to convex optimization problems, it is favored by people.

An extensive literature on iteration methods for fixed point problems of nonexpansive mappings has been published (for example, see [1–17]). Many iteration processes are often used to approximate a fixed point of a nonexpansive mapping in a Hilbert space or a Banach space. One of them is now known as Halpern's iteration process [2] and is defined as follows: take an initial guess $x_0 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = t_n u + (1 - t_n)Tx_n, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $\{t_n\}$ is a sequence in the interval $[0, 1]$ and u is some given element in C. For Halpern's iteration process, a classical result is as follows.

Theorem 1.2 ([13, 14]) *If $\{t_n\}$ satisfies the conditions*:
(i) $t_n \to 0$ $(n \to \infty)$;
(ii) $\sum_{n=1}^{\infty} t_n = \infty$;
(iii) $\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1$ *or* $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$;
then the sequence $\{x_n\}$ generated by (1.1) *converges strongly to a fixed point x^* of T such that $x^* = P_{\mathrm{Fix}(T)}u$, that is,*

$$\left\| u - x^* \right\| = \inf_{x\in\mathrm{Fix}(T)} \|u - x\|.$$

Now we consider how to get the minimum norm fixed point of T. In the case where $0 \in C$, taking $u = 0$ in (1.1), we assert by using Theorem 1.2 that $\{x_n\}$ generated by (1.1) converges strongly to $x^\dagger$ under conditions (i)-(iii) above. But, in the case where $0 \notin C$, the iteration process $x_{n+1} = (1 - t_n)Tx_n$ becomes invalid because x_n may not belong to C. In order to overcome this weakness, Wang and Xu [15] introduced the iteration process

$$x_{n+1} = P_C(1 - t_n)Tx_n, \quad n = 1, 2, \ldots. \tag{1.2}$$

They proved that if $\{t_n\}$ satisfies the same conditions in Theorem 1.2, then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $x^\dagger$.

However, it is difficult to implement the iteration process (1.2) in actual computing programs because the specific expression of P_C cannot be obtained, in general.

The purpose of this paper is to propose three new algorithms for finding the minimum norm fixed point of T. The strong convergence theorems are proved under some assumptions. The main advantage of the algorithms in this paper is that they have nothing to do with the metric projection P_C and thus they are easy to implement in actual computing programs. Because the key of our algorithms is replacing a fixed element u in (1.1) by a certain sequence $\{u_n\}$ in the boundary of C, they are called boundary point algorithms.

We will use the following notations:

1. $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.
2. $\omega_w(x_n) = \{x \mid \exists\{x_{n_k}\} \subset \{x_n\} \text{such that} x_{n_k} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.
3. $A \triangleq B$ means that B is the definition of A.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 1.3 ([18]) *Let C be a closed convex subset of a real Hilbert space H, and let $T: C \to C$ be a nonexpansive mapping such that* $\mathrm{Fix}(T) \neq \emptyset$. *If a sequence $\{x_n\}$ in C is such that $x_n \rightharpoonup z$ and $\|x_n - Tx_n\| \to 0$, then $z = Tz$.*

Lemma 1.4 *There holds the identity in a real Hilbert space H*:

$$\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle v, u - v\rangle, \quad u, v \in H.$$

Lemma 1.5 ([12, 19]) *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers satisfying the property*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\delta_n + \sigma_n, \quad n = 0, 1, 2, \ldots.$$

If $\{\gamma_n\}_{n=1}^{\infty} \subset (0,1)$, $\{\delta_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty}$ satisfy the conditions:

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,
(ii) $\limsup_{n\to\infty} \delta_n \leq 0$,
(iii) $\sum_{n=1}^{\infty} |\sigma_n| < \infty$,

then $\lim_{n\to\infty} a_n = 0$.

2 Main results

In this section, C is always assumed to be a nonempty closed convex subset of H such that $0 \notin C$. We use ∂C to denote the boundary of C. In order to give our main results, we first introduce a function $h: C \to (0,1]$ by the definition

$$h(x) = \inf\bigl\{\lambda \in (0,1] \mid \lambda x \in C\bigr\}, \quad \forall x \in C.$$

It is easy to see that $h(x)x \in \partial C$ and $h(x) > 0$ hold for each $x \in C$ due to the assumption $0 \notin C$.

Since our iteration processes will involve the function $h(x)$, it is necessary to explain how to calculate $h(x)$ for any given $x \in C$ in actual computing programs. In order to get the value $h(x)$ for a given $x \in C$, we often need to deal with an algebraic equation. But dealing with an algebraic equation is easier than calculating the metric projection P_C, in general. To illustrate this viewpoint, let us consider the following simple example.

Example 1 Let H be a real Hilbert space. Define a convex function $\varphi: H \to R^1$ by

$$\varphi(x) = \|x - x_0\|^2 + \langle x, u\rangle, \quad \forall x \in H,$$

where x_0 and u are two given points in H such that $\langle x_0, u\rangle < 0$. Setting $C = \{x \in H \mid \varphi(x) \leq 0\}$, then it is easy to show that C is a nonempty convex closed subset of H such that $0 \notin C$ (note that $\varphi(x_0) = \langle x_0, u\rangle < 0$ and $\varphi(0) = \|x_0\|^2 > 0$). For a given $x \in C$, we have $\varphi(x) \leq 0$. In order to get $h(x)$, let $\varphi(\lambda x) = 0$, where $\lambda \in (0,1]$ is an unknown number. Thus we obtain an algebraic equation

$$\|x\|^2\lambda^2 + \bigl(\langle x, u\rangle - 2\langle x, x_0\rangle\bigr)\lambda + \|x_0\|^2 = 0.$$

Consequently, we get

$$\lambda = \frac{2\langle x, x_0\rangle - \langle x, u\rangle \pm \sqrt{(\langle x, u\rangle - 2\langle x, x_0\rangle)^2 - 4\|x\|^2\|x_0\|^2}}{2\|x\|^2}.$$

By the definition of h, we have

$$h(x) = \frac{2\langle x, x_0\rangle - \langle x, u\rangle - \sqrt{(\langle x, u\rangle - 2\langle x, x_0\rangle)^2 - 4\|x\|^2\|x_0\|^2}}{2\|x\|^2}.$$

Next we give our first iteration process for finding the minimum norm fixed point of T: take $u_0 \in \partial C$ arbitrarily and define $\{x_n\}$ recursively by

$$\begin{cases} x_n = P_{\mathrm{Fix}(T)} u_n, \\ u_n = \lambda_n x_{n-1}, \end{cases} \tag{2.1}$$

where $\lambda_n = h(x_{n-1})$ ($n \geq 1$).

Remark 1 How to implement the iteration process (2.1)? In actual computing programs, we can use the standard Halpern's iteration process to get x_n from u_n for each $n \geq 0$. Indeed, taking $x_n^{(0)} = u_n$ and $\{x_n^{(m)}\}$ is generated inductively by

$$x_n^{(m+1)} = t_m u_n + (1 - t_m) T x_n^{(m)}, \quad m \geq 0,$$

then, using Theorem 1.2, $x_n^{(m)} \to x_n \triangleq P_{\mathrm{Fix}(T)} u_n$ as $m \to \infty$. Thus we can take $x_n = x_n^{(M_n)}$ approximately for a sufficiently large integer M_n in actual computing programs.

Geometric intuition seems to encourage us to guess $x_n \to P_{\mathrm{Fix}(T)} 0$ as $n \to \infty$ under some certain assumptions. As a matter of fact, it is true.

Theorem 2.1 *If* $\{\lambda_n\}$ *satisfies* $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$, *then* $\{x_n\}$ *generated by* (2.1) *converges strongly to* $x^\dagger = P_{\mathrm{Fix}(T)} 0$.

Proof Noticing the fact that $x^\dagger = P_{\mathrm{Fix}(T)} 0 = P_{\mathrm{Fix}(T)} \lambda x^\dagger$ holds for all $\lambda \in [0,1]$, we have from (2.1) that

$$\|x_n - x^\dagger\| = \|P_{\mathrm{Fix}(T)} u_n - x^\dagger\| = \|P_{\mathrm{Fix}(T)} \lambda_n x_{n-1} - P_{\mathrm{Fix}(T)} \lambda_n x^\dagger\| \leq \lambda_n \|x_{n-1} - x^\dagger\|,$$

consequently,

$$\|x_n - x^\dagger\| \leq \lambda_n \lambda_{n-1} \cdots \lambda_2 \lambda_1 \|x_0 - x^\dagger\|. \tag{2.2}$$

Thus this together with the condition $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$ leads to the conclusion. □

Remark 2 Is the condition $\sum_{n=1}^{\infty}(1 - \lambda_n) = \infty$ reasonable? In other words, can we find an example which satisfies this condition? The answer is yes. The following result implies that this condition is not harsh.

Corollary 1 *If* $d(\mathrm{Fix}(T), \partial C) \triangleq \inf\{\|x-y\| \mid x \in \mathrm{Fix}(T), y \in \partial C\} > 0$, *then* $\{x_n\}$ *generated by* (2.1) *converges strongly to* $x^\dagger = P_{\mathrm{Fix}(T)}0$.

Proof Obviously, it suffices to verify that if $d(\mathrm{Fix}(T), \partial C) > 0$, then $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$. In fact, setting $d \triangleq d(\mathrm{Fix}(T), \partial C) > 0$, we have from (2.1) and (2.2) that

$$\lambda_n = \frac{\|u_n\|}{\|x_{n-1}\|} = \frac{\|x_{n-1}\| - \|x_{n-1} - u_n\|}{\|x_{n-1}\|} \leq 1 - \frac{d}{\|x_0\| + \|x_0 - x^\dagger\|},$$

hence

$$1 - \lambda_n \geq \frac{d}{\|x_0\| + \|x_0 - x^\dagger\|}.$$

This implies that $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$ holds. □

Our second iteration process for finding the minimum norm fixed point of T is defined by

$$x_n = t_n\lambda_n x_{n-1} + (1-t_n)Tx_n, \quad n \geq 1, \tag{2.3}$$

where $\{t_n\} \subset (0,1)$, $\lambda_n = h(x_{n-1})$ $(n \geq 1)$ and x_0 is taken in C arbitrarily.

Remark 3 Equation (2.3) is an implicit iteration process. A natural question is how to get x_n from x_{n-1}. Indeed, suppose that we have got x_{n-1}, define the mapping $T_n : C \to C$ by $T_n : x \mapsto t_n\lambda_n x_{n-1} + (1-t_n)Tx$ $(\forall x \in C)$, then T_n is $(1-t_n)$-contractive and x_n is just its unique fixed point. So we can use Picard's iteration process

$$x_n^{(m+1)} = t_n\lambda_n x_{n-1} + (1-t_n)Tx_n^{(m)}, \quad m \geq 0,$$

to calculate x_n approximately since $x_n^{(m)} \to x_n$ as $m \to \infty$, where $x_n^{(0)}$ can be taken in C arbitrarily, for example, $x_n^{(0)} = x_{n-1}$.

Theorem 2.2 *Assume that* $\sum_{n=1}^{\infty}(1-\lambda_n) = \infty$ *and* $\sum_{n=1}^{\infty} t_n < \infty$, *then* $\{x_n\}$ *generated by* (2.3) *converges strongly to* $x^\dagger = P_{\mathrm{Fix}(T)}0$.

Proof We first show that $\{x_n\}$ is bounded. Indeed, take a $p \in \mathrm{Fix}(T)$ to derive that

$$\begin{aligned}\|x_n - p\| &= \bigl\|t_n\lambda_n(x_{n-1}-p) + (1-t_n)(Tx_n - p) - t_n(1-\lambda_n)p\bigr\| \\ &\leq t_n\lambda_n\|x_{n-1}-p\| + (1-t_n)\|x_n - p\| + t_n(1-\lambda_n)\|p\|.\end{aligned}$$

It follows that

$$\|x_n - p\| \leq \lambda_n\|x_{n-1}-p\| + (1-\lambda_n)\|p\|.$$

By induction,

$$\|x_n - p\| \leq \max\bigl\{\|x_1 - p\|, \|p\|\bigr\} \tag{2.4}$$

and $\{x_n\}$ is bounded, so are $\{Tx_n\}$. This together with (2.3) implies that $\|x_n - Tx_n\| \to 0$ ($n \to \infty$). Thus it follows from Lemma 1.3 that $\omega_w(x_n) \subset \operatorname{Fix}(T)$.

Next we show that

$$\lim_{n\to\infty} \sup\langle -x^\dagger, x_n - x^\dagger\rangle \le 0. \tag{2.5}$$

Indeed, take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} \sup\langle -x^\dagger, x_n - x^\dagger\rangle = \lim_{k\to\infty}\langle -x^\dagger, x_{n_k} - x^\dagger\rangle,$$

without loss of generality, we may assume that $x_{n_k} \rightharpoonup \bar{x}$. Noticing $x^\dagger = P_{\operatorname{Fix}(T)}0$, we obtain from $\bar{x} \in \operatorname{Fix}(T)$ and Lemma 1.1 that

$$\lim_{n\to\infty} \sup\langle -x^\dagger, x_n - x^\dagger\rangle = \langle -x^\dagger, \bar{x} - x^\dagger\rangle \le 0.$$

Finally, we show that $\|x_n - x^\dagger\| \to 0$ ($n \to \infty$). As a matter of fact, we have by using Lemma 1.4 that

$$\begin{aligned}
\|x_n - x^\dagger\|^2 &= \|t_n\lambda_n(x_{n-1} - x^\dagger) + (1-t_n)(Tx_n - x^\dagger) - t_n(1-\lambda_n)x^\dagger\|^2 \\
&\le \|t_n\lambda_n(x_{n-1} - x^\dagger) + (1-t_n)(Tx_n - x^\dagger)\|^2 \\
&\quad + 2t_n(1-\lambda_n)\langle -x^\dagger, x_n - x^\dagger\rangle \\
&\le t_n^2\lambda_n^2\|x_{n-1} - x^\dagger\|^2 + (1-t_n)^2\|x_n - x^\dagger\|^2 \\
&\quad + 2t_n\lambda_n(1-t_n)\|x_{n-1} - x^\dagger\| \cdot \|x_n - x^\dagger\| \\
&\quad + 2t_n(1-\lambda_n)\langle -x^\dagger, x_n - x^\dagger\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
(2-t_n)\|x_n - x^\dagger\|^2 &\le t_n\lambda_n^2\|x_{n-1} - x^\dagger\|^2 + 2\lambda_n(1-t_n)\|x_{n-1} - x^\dagger\| \cdot \|x_n - x^\dagger\| \\
&\quad + 2(1-\lambda_n)\langle -x^\dagger, x_n - x^\dagger\rangle \\
&\le t_n\lambda_n^2\|x_{n-1} - x^\dagger\|^2 + \lambda_n^2\|x_{n-1} - x^\dagger\|^2 + (1-t_n)^2\|x_n - x^\dagger\|^2 \\
&\quad + 2(1-\lambda_n)\langle -x^\dagger, x_n - x^\dagger\rangle.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\|x_n - x^\dagger\|^2 &\le [1-(1-\lambda_n)]\|x_{n-1} - x^\dagger\|^2 + 2(1-\lambda_n)\langle -x^\dagger, x_n - x^\dagger\rangle \\
&\quad + t_n\|x_{n-1} - x^\dagger\|^2.
\end{aligned}$$

Using Lemma 1.5, we conclude from (2.5) and conditions $\sum_{n=1}^\infty (1-\lambda_n) = \infty$ and $\sum_{n=1}^\infty t_n < \infty$ that $x_n \to x^\dagger$. □

By a similar argument as above, we easily get the following result.

Corollary 2 *If* $d(R(T), \partial C) \triangleq \inf\{\|x-y\| \mid x \in \mathrm{Fix}(T), y \in \partial C\} > 0$ *and* $\sum_1^\infty t_n < \infty$, *then* $\{x_n\}$ *generated by* (2.3) *converges strongly to* $x^\dagger = P_{\mathrm{Fix}(T)}0$, *where* $R(T)$ *is the range of* T.

Proof It suffices to verify that $d(R(T), \partial C) > 0$ implies $\sum_{n=1}^\infty (1-\lambda_n) = \infty$. Indeed,

$$\lambda_n = \frac{\|u_n\|}{\|x_{n-1}\|} = \frac{\|x_{n-1}\| - \|x_{n-1} - u_n\|}{\|x_{n-1}\|} = 1 - \frac{\|x_{n-1} - Tx_{n-1} + Tx_{n-1} - u_n\|}{\|x_{n-1}\|}.$$

Setting $d \triangleq d(R(T), \partial C) > 0$, we have from (2.4) that

$$1 - \lambda_n \geq \frac{\|Tx_{n-1} - u_n\|}{\|x_{n-1}\|} - \frac{\|x_{n-1} - Tx_{n-1}\|}{\|x_{n-1}\|} \geq \frac{d}{\|x_1 - x^\dagger\| + 2\|x^\dagger\|} - \frac{\|x_{n-1} - Tx_{n-1}\|}{d(0, C)}.$$

Note that $\|x_{n-1} - Tx_{n-1}\| \to 0$, it follows that $\sum_{n=1}^\infty (1-\lambda_n) = \infty$. □

Finally, we propose an explicit iteration process for finding the minimum norm fixed point of T which is defined by

$$x_{n+1} = t_n \lambda_n x_n + (1-t_n) T x_n, \quad n \geq 0, \tag{2.6}$$

where $\{t_n\} \subset (0,1)$, $\lambda_n = h(x_n)$ $(n \geq 0)$ and x_0 is taken in C arbitrarily.

Theorem 2.3 *Assume that* $\{t_n\}$ *and* $\{\lambda_n\}$ *satisfy the following conditions*:

(i) $t_n \to 0$ *and* $\sum_{n=0}^\infty t_n = \infty$;
(ii) $\limsup_{n\to\infty} \lambda_n \leq \bar{\lambda} < 1$;
(iii) $\sum_{n=1}^\infty |t_n - t_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{t_n}{t_{n-1}} = 1$;
(iv) $\sum_{n=1}^\infty t_n |\lambda_n - \lambda_{n-1}| < \infty$ *or* $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n-1}} = 1$.

Then $\{x_n\}$ *generated by* (2.6) *converges strongly to* $x^\dagger = P_{\mathrm{Fix}(T)}0$.

Proof We first show that $\{x_n\}$ is bounded. Indeed, we have by taking $p \in \mathrm{Fix}(T)$ arbitrarily that

$$\begin{aligned}
\|x_{n+1} - p\| &\leq t_n \|\lambda_n x_n - p\| + (1-t_n)\|Tx_n - p\| \\
&\leq t_n \big[\lambda_n \|x_n - p\| + (1-\lambda_n)\|p\|\big] + (1-t_n)\|x_n - p\| \\
&\leq t_n \max\big\{\|x_n - p\|, \|p\|\big\} + (1-t_n)\|x_n - p\| \\
&\leq \max\big\{\|x_n - p\|, \|p\|\big\}.
\end{aligned}$$

Inductively,

$$\|x_n - p\| \leq \max\big\{\|x_0 - p\|, \|p\|\big\}, \quad n \geq 0.$$

This means that $\{x_n\}$ is bounded, so are $\{Tx_n\}$.

We next show that $\|x_{n+1} - x_n\| \to 0$. Using (2.6), it follows from a direct calculation that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|\big[t_n\lambda_n x_n + (1-t_n)Tx_n\big] - \big[t_{n-1}\lambda_{n-1}x_{n-1} + (1-t_{n-1})Tx_{n-1}\big]\big\| \\
&= \big\|(1-t_n)(Tx_n - Tx_{n-1}) - (t_n - t_{n-1})Tx_{n-1} + t_n\lambda_n(x_n - x_{n-1})
\end{aligned}$$

$$
\begin{aligned}
&+ (t_n\lambda_n - t_{n-1}\lambda_{n-1})x_{n-1}\big\| \\
&\le \big[1 - t_n(1-\lambda_n)\big]\|x_n - x_{n-1}\| + |t_n - t_{n-1}|\big(\|Tx_{n-1}\| + \lambda_{n-1}\|x_{n-1}\|\big) \\
&\quad + t_n|\lambda_n - \lambda_{n-1}| \cdot \|x_{n-1}\|.
\end{aligned}
$$

Using Lemma 1.5, we conclude from conditions (i)-(iv) that $\|x_{n+1} - x_n\| \to 0$. Noticing the boundedness of $\{x_n\}$ and $\{Tx_n\}$ and condition (i), we have from (2.6) that $\|x_{n+1} - Tx_n\| \to 0$. Consequently, $\|x_n - Tx_n\| \to 0$. Using Lemma 1.3, we derive that $\omega_w(x_n) \subset \mathrm{Fix}(T)$.

Then we show that

$$\lim_{n\to\infty}\sup\langle -x^\dagger, x_{n-1} - x^\dagger\rangle \le 0. \tag{2.7}$$

As a matter of fact, this is derived by the same argument as in the proof of Theorem 2.3.

Finally, we show that $\|x_n - x^\dagger\| \to 0$. Using Lemma 1.4 and (2.6), it is easy to verify that

$$
\begin{aligned}
\|x_{n+1} - x^\dagger\|^2 &= \big\|t_n(\lambda_n x_n - x^\dagger) + (1-t_n)(Tx_n - x^\dagger)\big\|^2 \\
&\le (1-t_n)^2\|Tx_n - x^\dagger\|^2 + 2t_n\langle\lambda_n x_n - x^\dagger, x_{n+1} - x^\dagger\rangle \\
&\le (1-t_n)^2\|x_n - x^\dagger\|^2 + 2t_n\lambda_n\langle x_n - x^\dagger, x_{n+1} - x^\dagger\rangle \\
&\quad + 2t_n(1-\lambda_n)\langle -x^\dagger, x_{n+1} - x^\dagger\rangle \\
&\le (1-t_n)^2\|x_n - x^\dagger\|^2 + 2t_n\lambda_n\|x_n - x^\dagger\| \cdot \|x_{n+1} - x^\dagger\| \\
&\quad + 2t_n(1-\lambda_n)\langle -x^\dagger, x_{n+1} - x^\dagger\rangle.
\end{aligned}
$$

Hence,

$$\|x_{n+1} - x^\dagger\|^2 \le (1-\gamma_n)\|x_n - x^\dagger\|^2 + \gamma_n\sigma_n,$$

where

$$
\begin{aligned}
\gamma_n &= t_n\frac{2(1-\lambda_n) - t_n}{1 - t_n\lambda_n}, \\
\sigma_n &= \frac{2(1-\lambda_n)}{2(1-\lambda_n) - t_n}\langle -x^\dagger, x_{n+1} - x^\dagger\rangle.
\end{aligned}
$$

It is easily seen that $\gamma_n \to 0$, $\sum_{n=0}^\infty \gamma_n = \infty$ by conditions (i) and (ii), and $\lim_{n\to\infty}\sup\sigma_n \le 0$ by (2.7). By Lemma 1.5, we conclude that $x_n \to x^\dagger$. □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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