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A strong convergence theorem for equilibrium problems and split feasibility problems in Hilbert spaces

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Abstract
The main purpose of this paper is to introduce an iterative algorithm for equilibrium problems and split feasibility problems in Hilbert spaces. Under suitable conditions we prove that the sequence converges strongly to a common element of the set of solutions of equilibrium problems and the set of solutions of split feasibility problems. Our result extends and improves the corresponding results of some others.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$, let $F : H \times H \to \mathbb{R}$ be a bifunction. Then we consider the following equilibrium problem (EP): find $z \in H$ such that

$$F(z, y) \geq 0, \quad \forall y \in H. \tag{1.1}$$

The set of the EP is denoted by Ω, *i.e.*,

$$\Omega = \{z \in H : F(z, y) \geq 0, \forall y \in H\}.$$

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequality problems, the Nash equilibrium problems and others, see, for instance, [1–3]. Some methods have been proposed to solve the EP, see, *e.g.*, [4–6] and [7, 8].

The split feasibility problem (SFP) was proposed by Censer and Elfving in [9]. It can be formulated as the problem of finding a point x satisfying the property:

$$x \in C, \quad Ax \in Q, \tag{1.2}$$

where A is a given $M \times N$ real matrix, and C and Q are nonempty, closed and convex subsets in $\mathbb{R}^N$ and $\mathbb{R}^M$, respectively.

Due to its extraordinary utility and broad applicability in many areas of applied mathematics (most notably, fully discretized models of problems in image reconstruction from projections, in image processing, and in intensity-modulated radiation therapy), algorithms for solving convex feasibility problems have been received great attention (see, for instance [10–13] and also [14–18]).

We assume the SFP (1.2) is consistent, and let Γ be the solution set, *i.e.*,

$$\Gamma = \{x \in C : Ax \in Q\}.$$

It is not hard to see that Γ is closed convex and $x \in \Gamma$ if and only if it solves the fixed-point equation

$$x = P_C\bigl(I - \gamma A^*(I - P_Q)A\bigr)x, \tag{1.3}$$

where P_C and P_Q are the orthogonal projection onto C and Q, respectively, $\gamma > 0$ is any positive constant and A^* denotes the adjoint of A.

Recently, for the purpose of generality, the SFP (1.2) has been studied in a more general setting. For instance, see [16, 19]. However, the algorithms in these references have only weak convergence in the setting of infinite-dimensional Hilbert spaces. Very recently, He and Zhao [20] introduce a new relaxed CQ algorithm (1.4) such that the strong convergence is guaranteed in infinite-dimensional Hilbert spaces:

$$x_{n+1} = P_{C_n}\bigl(\alpha_n u + (1-\alpha_n)\bigl(x_n - \tau_n \nabla f_n(x_n)\bigr)\bigr). \tag{1.4}$$

Motivated and inspired by the research going on in the sections of equilibrium problems and split feasibility problems, the purpose of this article is to introduce an iterative algorithm for equilibrium problems and split feasibility problems in Hilbert spaces. Under suitable conditions we prove the sequence converges strongly to a common element of the set of solutions of equilibrium problems and the set of solutions of split feasibility problems. Our result extends and improves the corresponding results of He *et al.* [20] and some others.

2 Preliminaries and lemmas

Throughout this paper, we assume that H, H_1 or H_2 is a real Hilbert space, A is a bounded linear operator from H_1 to H_2, and I is the identity operator on H, H_1 or H_2. If $f : H \to \mathbb{R}$ is a differentiable function, then we denote by ∇f the gradient of the function f. We will also use the notations: $\to$ to denote strong convergence, $\rightharpoonup$ to denote weak convergence and to denote by

$$w_\omega(x_n) = \bigl\{x \mid \exists \{x_{n_k}\} \subset \{x_n\} \text{ such that } x_{n_k} \rightharpoonup x\bigr\}$$

the weak ω-limit set of $\{x_n\}$.

Recall that a mapping $T : H \to H$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad x, y \in H.$$

$T: H \to H$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \big\|(I - T)x - (I - T)y\big\|^2, \quad x, y \in H.$$

A mapping $T: H \to H$ is said to be demi-closed at origin, if for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$ and $\lim_{n\to\infty} \|(I - T)x_n\| = 0$, then $x^* = Tx^*$.

It is easy to prove that if $T: H \to H$ is a firmly nonexpansive mapping, then T is demi-closed at the origin.

A function $f: H \to \mathbb{R}$ is called convex if

$$f\big(\lambda x + (1 - \lambda)y\big) \le \lambda f(x) + (1 - \lambda)f(y), \quad \forall \lambda \in (0, 1), \forall x, y \in H.$$

Lemma 2.1 *Let $T: H_2 \to H_2$ be a firmly nonexpansive mapping such that $\|(I - T)x\|$ is a convex function from H_2 to $\bar{\mathbb{R}} = [-\infty, +\infty]$. Let $A: H_1 \to H_2$ be a bounded linear operator and*

$$f(x) := \frac{1}{2}\big\|(I - T)Ax\big\|^2, \quad \forall x \in H_1.$$

Then

(i) $\nabla f(x) = A^*(I - T)Ax$, $x \in H_1$.

(ii) ∇f *is* $\|A\|^2$*-Lipschitz, i.e.*, $\|\nabla f(x) - \nabla f(y)\| \le \|A\|^2 \|x - y\|$, $x, y \in H_1$.

Proof (i) From the definition of f, we know that f is convex. First we prove that the limit

$$\big\langle \nabla f(x), v\big\rangle = \lim_{h\to 0+} \frac{f(x + hv) - f(x)}{h}$$

exists in $\bar{\mathcal{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$ and satisfies

$$\big\langle \nabla f(x), v\big\rangle \le f(x + v) - f(x), \quad \forall v \in H_1.$$

If fact, if $0 < h_1 \le h_2$, then

$$f(x + h_1 v) - f(x) = f\left(\frac{h_1}{h_2}(x + h_2 v) + \left(1 - \frac{h_1}{h_2}\right)x\right) - f(x).$$

Since f is convex and $\frac{h_1}{h_2} \le 1$, it follows that

$$f(x + h_1 v) - f(x) \le \frac{h_1}{h_2} f(x + h_2 v) + \left(1 - \frac{h_1}{h_2}\right) f(x) - f(x),$$

and hence that

$$\frac{f(x + h_1 v) - f(x)}{h_1} \le \frac{f(x + h_2 v) - f(x)}{h_2}.$$

This shows that this difference quotient is increasing, therefore it has a limit in $\bar{\mathcal{R}}$ as $h \to 0+$:

$$\big\langle \nabla f(x), v\big\rangle = \inf_{h>0} \frac{f(x + hv) - f(x)}{h} = \lim_{h\to 0+} \frac{f(x + hv) - f(x)}{h}. \tag{2.1}$$

This implies that f is differential. Taking $h = 1$, (2.1) implies that

$$\langle \nabla f(x), v \rangle \leq f(x+v) - f(x).$$

Next we prove that

$$\nabla f(x) = A^*(I-T)Ax, \quad x \in H_1.$$

In fact, since

$$\lim_{h\to 0+} \frac{f(x+hv)-f(x)}{h} = \lim_{h\to 0+} \frac{\|Ax + hAv - TA(x+hv)\|^2 - \|(I-T)Ax\|^2}{2h} \tag{2.2}$$

and

$$\begin{aligned} &\|Ax + hAv - TA(x+hv)\|^2 - \|(I-T)Ax\|^2 \\ &\quad = \|Ax\|^2 + h^2\|Av\|^2 + 2h\langle A^*Ax, v\rangle + \|TA(x+hv)\|^2 - \|Ax\|^2 - \|TAx\|^2 \\ &\qquad - 2\langle Ax, TA(x+hv) - TAx\rangle - 2h\langle A^*TA(x+hv), v\rangle. \end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2), simplifying and then letting $h \to 0+$ and taking the limit we have

$$\begin{aligned} \lim_{h\to 0+} \frac{f(x+hv)-f(x)}{h} &= \lim_{h\to 0+} \frac{2h\{\langle A^*Ax, v\rangle - \langle A^*TA(x+hv), v\rangle\}}{2h} \\ &= \langle A^*(I-T)Ax, v\rangle, \quad \forall v \in H_1. \end{aligned}$$

It follows from (2.1) that

$$\nabla f(x) = A^*(I-T)Ax, \quad x \in H_1.$$

(ii) From (i) we have

$$\begin{aligned} \|\nabla f(x) - \nabla f(y)\| &= \|A^*(I-T)Ax - A^*(I-T)Ay\| \\ &= \|A^*[(I-T)Ax - (I-T)Ay]\| \\ &\leq \|A\|\|Ax - Ay\| \leq \|A\|^2\|x-y\|, \quad x, y \in H_1. \end{aligned}$$

□

Lemma 2.2 (See, for example, [21]) *Let $T: H \to H$ be an operator. The following statements are equivalent.*

(i) *T is firmly nonexpansive.*
(ii) $\|Tx - Ty\|^2 \leq \langle x-y, Tx-Ty\rangle$, $\forall x, y \in H$.
(iii) *$I - T$ is firmly nonexpansive.*

Proof (i) ⇒ (ii): Since T is firmly nonexpansive, for all $x, y \in H$ we have

$$\begin{aligned} \|Tx - Ty\|^2 &\leq \|x-y\|^2 - \|(I-T)x - (I-T)y\|^2 \\ &= \|x-y\|^2 - \|x-y\|^2 - \|Tx-Ty\|^2 + 2\langle x-y, Tx-Ty\rangle \\ &= 2\langle x-y, Tx-Ty\rangle - \|Tx-Ty\|^2, \end{aligned}$$

hence

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in H.$$

(ii) ⇒ (iii): From (ii), we know that for all $x, y \in H$

$$\begin{aligned}\|(I - T)x - (I - T)y\|^2 &= \|(x - y) - (Tx - Ty)\|^2 \\ &= \|x - y\|^2 - 2\langle x - y, Tx - Ty\rangle + \|Tx - Ty\|^2 \\ &\leq \|x - y\|^2 - \langle x - y, Tx - Ty\rangle \\ &= \langle x - y, (I - T)x - (I - T)y\rangle.\end{aligned}$$

This implies that $I - T$ is firmly nonexpansive.

(iii) ⇒ (i): From (iii) we immediately know that T is firmly nonexpansive.

Let C be a nonempty closed convex subset of H. Recall that for every point $x \in H$, there exists a unique nearest point of C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection from H onto C. We know that P_C is a firmly nonexpansive mapping from H onto C, *i.e.*,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

Further, for any $x \in H$ and $z \in C$, $z = P_C x$ if and only if

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C. \tag{2.4}$$

Throughout this paper, let us assume that a bifunction $F : H \times H \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$, $\forall x \in H$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in H$;

(A3) $\lim_{t\downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$, $\forall x, y, z \in H$;

(A4) for each $x \in H$, $y \mapsto F(x, y)$ is convex and lower semicontinuous. □

Lemma 2.3 ([1, 4]) *Let H be a Hilbert space and let $F : H \times H \to \mathbb{R}$ satisfy* (A1), (A2), (A3), *and* (A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in H$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in H.$$

Furthermore, if

$$T_r x = \left\{z \in H : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in H\right\},$$

then the following hold:

(1) *T_r is single-valued*;

(2) *T_r is firmly nonexpansive*;

(3) $F(T_r) = \Omega$;

(4) *Ω is closed and convex.*

The following results play an important role in this paper.

Lemma 2.4 ([22]) *Let X be a real Hilbert space, then we have*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall x, y \in X.$$

Lemma 2.5 ([23]) *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X. Let $\{\beta_n\}$ be a sequence in $[0,1]$ satisfying $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that*

$$x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$$

for all integer $n \geq 0$ and

$$\limsup_{n\to\infty}\bigl(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\bigr) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 2.6 ([24]) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\sigma_n, \quad n = 0, 1, 2, \ldots,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$, and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=0}^{\infty} \gamma_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$, *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

3 Main results

We are now in a position to prove the following theorem.

Theorem 3.1 *Let H_1, H_2 be two real Hilbert spaces, $F : H_1 \times H_1 \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3), and (A4). Let $A : H_1 \to H_2$ be a bounded linear operator, $S : H_1 \to H_1$ be a firmly nonexpansive mapping, and let $T : H_2 \to H_2$ be a firmly nonexpansive mapping such that $\|(I-T)x\|$ is a convex function from H_2 to $\mathbb{R}$. Assume that $C := F(S) \cap \Omega \neq \emptyset$ and $Q := F(T) \neq \emptyset$. Let $u \in H_1$ and $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 \in H_1 \text{ chosen arbitrarily}, \\ x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \\ F(y_n, x) + \frac{1}{\lambda_n}\langle x - y_n, y_n - z_n\rangle \geq 0, \quad \forall x \in H_1, \\ z_n = S(\alpha_n u + (1-\alpha_n)(x_n - \xi_n \nabla f(x_n))), \end{cases} \tag{3.1}$$

where

$$f(x_n) = \frac{1}{2}\bigl\|(I-T)Ax_n\bigr\|^2, \qquad \nabla f(x_n) = A^*(I-T)Ax_n \neq 0 \quad \forall n \geq 1,$$
$$\xi_n = \frac{\rho_n f(x_n)}{\|\nabla f(x_n)\|^2}.$$

If the solution set Γ *of SPF* (1.2) *is not empty, and the sequences* $\{\rho_n\} \subset (0,4)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ *satisfy the following conditions*:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $\lambda_n \in (a,b) \subset (0,+\infty)$ *and* $\lim_{n\to\infty}(\lambda_{n+1} - \lambda_n) = 0$,

then the sequence $\{x_n\}$ *converges strongly to* $P_\Gamma u$.

Proof Since the solution set Ω of EP and the solution set of SPF (1.2) are both closed and convex, Γ $(\neq \emptyset)$ is closed and convex. Thus, the metric projection P_Γ is well defined.

Letting $p = P_\Gamma u$, it follows from Lemma 2.3 that $y_n = T_{\lambda_n} z_n$ and

$$\|y_n - p\| = \|T_{\lambda_n} z_n - T_{\lambda_n} p\| \leq \|z_n - p\|. \tag{3.2}$$

Observing that S is firmly nonexpansive, we have

$$\begin{aligned} \|z_n - p\| &= \big\|S\big(\alpha_n u + (1-\alpha_n)\big(x_n - \xi_n \nabla f(x_n)\big)\big) - p\big\| \\ &\leq \big\|\alpha_n(u-p) + (1-\alpha_n)\big(x_n - \xi_n \nabla f(x_n) - p\big)\big\| \\ &\leq \alpha_n\|u-p\| + (1-\alpha_n)\big\|x_n - \xi_n \nabla f(x_n) - p\big\|. \end{aligned} \tag{3.3}$$

Since $p \in \Gamma \subset C$, $\nabla f(p) = 0$. Observe that $I - T$ is firmly nonexpansive, from Lemma 2.2(ii) we have

$$\begin{aligned} \big\langle \nabla f(x_n), x_n - p\big\rangle &= \big\langle (I-T)Ax_n, Ax_n - Ap\big\rangle \\ &\geq \big\|(I-T)Ax_n\big\|^2 = 2f(x_n). \end{aligned} \tag{3.4}$$

This implies that

$$\begin{aligned} \big\|x_n - \xi_n \nabla f(x_n) - p\big\|^2 &= \|x_n - p\|^2 + \big\|\xi_n \nabla f(x_n)\big\|^2 - 2\xi_n\big\langle \nabla f(x_n), x_n - p\big\rangle \\ &\leq \|x_n - p\|^2 + \xi_n^2\big\|\nabla f(x_n)\big\|^2 - 4\xi_n f(x_n) \\ &= \|x_n - p\|^2 - \rho_n(4-\rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2} \\ &\leq \|x_n - p\|^2. \end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.3), we get

$$\|z_n - p\| \leq \alpha_n\|u-p\| + (1-\alpha_n)\|x_n - p\|. \tag{3.6}$$

Thus, from (3.2) and (3.6) we have

$$\begin{aligned} \|x_{n+1} - p\| &= \big\|\beta_n x_n + (1-\beta_n)y_n - p\big\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\ &\leq \beta_n\|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &\leq \big(1 - \alpha_n(1-\beta_n)\big)\|x_n - p\| + \alpha_n(1-\beta_n)\|u-p\|. \end{aligned}$$

It turns out that

$$\|x_{n+1} - p\| \le \max\{\|x_n - p\|, \|u - p\|\}.$$

By induction, we have

$$\|x_n - p\| \le \max\{\|x_0 - p\|, \|u - p\|\}.$$

This implies that the sequence $\{x_n\}$ is bounded. From (3.2) and (3.6) we know that $\{y_n\}$ and $\{z_n\}$ both are bounded.

From Lemma 2.4 and (3.5), we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|S(\alpha_n u + (1-\alpha_n)(x_n - \xi_n \nabla f(x_n))) - p\|^2 \\
&\le \|\alpha_n(u-p) + (1-\alpha_n)(x_n - \xi_n \nabla f(x_n) - p)\|^2 \\
&\le (1-\alpha_n)\|x_n - \xi_n \nabla f(x_n) - p\|^2 + 2\alpha_n\langle u-p, z_n - p\rangle \\
&\le (1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n\langle u-p, z_n-p\rangle \\
&\quad - (1-\alpha_n)\rho_n(4-\rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2}.
\end{aligned} \tag{3.7}$$

Therefore, from Lemma 2.6 and (3.2), (3.7) we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\beta_n x_n + (1-\beta_n)y_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\
&\le \beta_n\|x_n - p\|^2 + (1-\beta_n)(1-\alpha_n)\|x_n - p\|^2 + 2\alpha_n(1-\beta_n)\langle u-p, z_n-p\rangle \\
&\quad - (1-\alpha_n)(1-\beta_n)\rho_n(4-\rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2} \\
&= \|x_n - p\|^2 - \alpha_n(1-\beta_n)\|x_n - p\|^2 + 2\alpha_n(1-\beta_n)\langle u-p, z_n-p\rangle \\
&\quad - (1-\alpha_n)(1-\beta_n)\rho_n(4-\rho_n)\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2}.
\end{aligned} \tag{3.8}$$

On the other hand, without loss of generality, we may assume that there is a constant $\sigma > 0$ such that

$$(1-\alpha_n)(1-\beta_n)\rho_n(4-\rho_n) > \sigma, \quad \forall n \ge 1.$$

Setting $s_n = \|x_n - p\|^2$, we get the following inequality:

$$s_{n+1} - s_n + \alpha_n(1-\beta_n)s_n + \frac{\sigma f^2(x_n)}{\|\nabla f(x_n)\|^2} \le 2\alpha_n(1-\beta_n)\langle u-p, z_n-p\rangle. \tag{3.9}$$

Now, we prove $s_n \to 0$ by employing the technique studied by Maingé [25]. For the purpose we consider two cases.

Case 1: $\{s_n\}$ is eventually decreasing, *i.e.*, there exists a sufficient large positive integer $k \geq 1$ such that $s_n > s_{n+1}$ holds for all $n \geq k$. In this case, $\{s_n\}$ must be convergent, and from (3.9) it follows that

$$\frac{\sigma f^2(x_n)}{\|\nabla f(x_n)\|^2} \leq (s_n - s_{n+1}) + \alpha_n(1-\beta_n)M, \tag{3.10}$$

where M is a constant such that $M \geq 2\|z_n - p\|\|u - p\|$ for all $n \in \mathbb{N}$. Using the condition (i) and (3.10), we have

$$\frac{f^2(x_n)}{\|\nabla f(x_n)\|^2} \to 0 \quad (n \to \infty). \tag{3.11}$$

Moreover, it follows from Lemma 2.1(ii) that for all $n \in \mathbb{N}$

$$\|\nabla f(x_n)\| = \|\nabla f(x_n) - \nabla f(p)\| \leq \|A\|^2\|x_n - p\|.$$

This implies that $\{\|\nabla f(x_n)\|\}$ is bounded. From (3.11) it yields $f(x_n) \to 0$, namely

$$\|(I-T)Ax_n\| \to 0. \tag{3.12}$$

Furthermore, we have

$$\lim_{n\to\infty} \xi_n = 0. \tag{3.13}$$

For any $x^* \in w_\omega(x_n)$, and if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in H_1$, then

$$Ax_{n_k} \rightharpoonup Ax^*. \tag{3.14}$$

On the other hand, from (3.12), we have

$$\|(I-T)Ax_{n_k}\| \to 0. \tag{3.15}$$

Since T is demi-closed at origin, from (3.14) and (3.15) we have $Ax^* \in F(T)$, *i.e.*, $Ax^* \in Q$.

In order to prove $x^* \in C = F(S) \cap \Omega$, we need to prove $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty}\|x_n - z_n\| = 0$. In fact, from (3.1) we have

$$F(y_n, x) + \frac{1}{\lambda_n}\langle x - y_n, y_n - z_n\rangle \geq 0, \quad \forall x \in H_1.$$

Taking $x = y_{n+1}$, we get

$$F(y_n, y_{n+1}) + \frac{1}{\lambda_n}\langle y_{n+1} - y_n, y_n - z_n\rangle \geq 0.$$

Similarly, we also have

$$F(y_{n+1}, y_n) + \frac{1}{\lambda_{n+1}}\langle y_n - y_{n+1}, y_{n+1} - z_{n+1}\rangle \geq 0.$$

Adding up the above two inequalities, we get

$$F(y_n, y_{n+1}) + F(y_{n+1}, y_n) + \left\langle y_{n+1} - y_n, \frac{y_n - z_n}{\lambda_n} - \frac{y_{n+1} - z_{n+1}}{\lambda_{n+1}} \right\rangle \geq 0.$$

From (A2), we have

$$\left\langle y_{n+1} - y_n, \frac{y_n - z_n}{\lambda_n} - \frac{y_{n+1} - z_{n+1}}{\lambda_{n+1}} \right\rangle \geq 0.$$

Multiplying the above inequality by λ_n and simplifying, we have

$$\left\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - z_n - \frac{\lambda_n}{\lambda_{n+1}}(y_{n+1} - z_{n+1}) \right\rangle \geq 0.$$

Hence we have

$$\begin{aligned}
\|y_{n+1} - y_n\|^2 &\leq \left\langle y_{n+1} - y_n, y_{n+1} - z_n - \frac{\lambda_n}{\lambda_{n+1}}(y_{n+1} - z_{n+1}) \right\rangle \\
&= \left\langle y_{n+1} - y_n, z_{n+1} - z_n + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)(y_{n+1} - z_{n+1}) \right\rangle \\
&\leq \|y_{n+1} - y_n\| \left(\|z_{n+1} - z_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \cdot \|y_{n+1} - z_{n+1}\| \right)
\end{aligned}$$

and hence

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \|z_{n+1} - z_n\| + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right| \|y_{n+1} - z_{n+1}\| \\
&\leq \|z_{n+1} - z_n\| + \frac{1}{a}|\lambda_{n+1} - \lambda_n| \cdot \|y_{n+1} - z_{n+1}\|.
\end{aligned}$$

By (3.1) we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \big\| S\big(\alpha_{n+1}u + (1 - \alpha_{n+1})\big(x_{n+1} - \xi_{n+1}\nabla f(x_{n+1})\big)\big) \\
&\quad - S\big(\alpha_n u + (1 - \alpha_n)\big(x_n - \xi_n \nabla f(x_n)\big)\big) \big\| \\
&\leq \big\| (\alpha_{n+1} - \alpha_n)u \\
&\quad + (1 - \alpha_{n+1})\big\{ \big(x_{n+1} - \xi_{n+1}\nabla f(x_{n+1})\big) - \big(x_n - \xi_n \nabla f(x_n)\big) \big\} \\
&\quad - (\alpha_{n+1} - \alpha_n)\big(x_n - \xi_n \nabla f(x_n)\big) \big\| \\
&\leq (1 - \alpha_{n+1})\|x_{n+1} - x_n\| + N_n \leq \|x_{n+1} - x_n\| + N_n,
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
N_n &= |\alpha_{n+1} - \alpha_n| \cdot \|u\| + (1 - \alpha_{n+1})\big(\xi_{n+1}\big\|\nabla f(x_{n+1})\big\| + \xi_n \big\|\nabla f(x_n)\big\|\big) \\
&\quad + |\alpha_{n+1} - \alpha_n| \cdot \big\|x_n - \xi_n \nabla f(x_n)\big\| \to 0 \quad (n \to \infty).
\end{aligned} \tag{3.17}$$

This implies that

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + \frac{1}{a}|\lambda_{n+1} - \lambda_n| \cdot \|y_{n+1} - z_{n+1}\| + N_n.$$

It follows that

$$\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\| \le \frac{1}{a}|\lambda_{n+1}-\lambda_n|\cdot\|y_{n+1}-z_{n+1}\|+N_n.$$

In view of condition (iii) and (3.17) we get

$$\limsup_{n\to\infty}\bigl(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|\bigr)\le 0.$$

By Lemma 2.5, we obtain

$$\lim_{n\to\infty}\|y_n-x_n\|=0. \tag{3.18}$$

Consequently

$$\begin{aligned}\|x_{n+1}-x_n\| &= \bigl\|\beta_n x_n+(1-\beta_n)y_n-x_n\bigr\| \\ &= (1-\beta_n)\|y_n-x_n\|\to 0 \quad (n\to\infty).\end{aligned} \tag{3.19}$$

Since S is firmly nonexpansive, it follows from (3.1) that

$$\begin{aligned}2\|z_n-p\|^2 &= 2\bigl\|S\bigl(\alpha_n u+(1-\alpha_n)\bigl(x_n-\xi_n\nabla f(x_n)\bigr)\bigr)-Sp\bigr\|^2 \\ &\le 2\bigl\langle\alpha_n u+(1-\alpha_n)\bigl(x_n-\xi_n\nabla f(x_n)\bigr)-p, z_n-p\bigr\rangle \\ &= \bigl\|\alpha_n u+(1-\alpha_n)\bigl(x_n-\xi_n\nabla f(x_n)\bigr)-p\bigr\|^2+\|z_n-p\|^2 \\ &\quad -\bigl\|\alpha_n u+(1-\alpha_n)\bigl(x_n-\xi_n\nabla f(x_n)\bigr)-p-z_n+p\bigr\|^2 \\ &= \bigl\|\alpha_n(u-p)+(1-\alpha_n)\bigl(x_n-\xi_n\nabla f(x_n)-p\bigr)\bigr\|^2+\|z_n-p\|^2 \\ &\quad -\bigl\|\alpha_n(u-z_n)+(1-\alpha_n)\bigl(x_n-\xi_n\nabla f(x_n)-z_n\bigr)\bigr\|^2 \\ &\le (1-\alpha_n)\|x_n-p\|^2+\|z_n-p\|^2-\|x_n-z_n\|^2+M_n,\end{aligned}$$

where

$$\begin{aligned}M_n :={}& \alpha_n\|u-p\|^2+(1-\alpha_n)\bigl\|\xi_n\nabla f(x_n)\bigr\|^2-2(1-\alpha_n)\xi_n\bigl\langle x_n-p,\nabla f(x_n)\bigr\rangle \\ &-\alpha_n\|u-z_n\|^2-(1-\alpha_n)\bigl\{\bigl\|\xi_n\nabla f(x_n)\bigr\|^2-2\bigl\langle x_n-z_n,\xi_n\nabla f(x_n)\bigr\rangle\bigr\} \\ &+\alpha_n\|x_n-z_n\|^2+\alpha_n(1-\alpha_n)\bigl\|x_n-u-\xi_n\nabla f(x_n)\bigr\|^2 \\ \to{}& 0 \quad (\text{as } n\to\infty).\end{aligned}$$

Therefore we have

$$\|z_n-p\|^2\le\|x_n-p\|^2-\|x_n-z_n\|^2+M_n.$$

This together with (3.8) shows that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|z_n-p\|^2 \\ &\le \|x_n-p\|^2-(1-\beta_n)\|x_n-z_n\|^2+(1-\beta_n)M_n.\end{aligned}$$

Then we obtain

$$\begin{aligned}(1-\beta_n)\|x_n-z_n\|^2 &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+(1-\beta_n)M_n \\ &= s_n-s_{n+1}+(1-\beta_n)M_n.\end{aligned}$$

Therefore, we get

$$\lim_{n\to\infty}\|x_n-z_n\|=0. \tag{3.20}$$

By virtue of (3.18), we have

$$\lim_{n\to\infty}\|y_n-z_n\|=0. \tag{3.21}$$

Now, we turn to a proof that $x^* \in C = F(S)\cap\Omega$. For this purpose, we denote

$$v_n := \alpha_n u+(1-\alpha_n)\bigl(x_n-\xi_n\nabla f(x_n)\bigr).$$

In view of condition (i) and (3.13) we have

$$\begin{aligned}\|v_n-x_n\| &= \bigl\|\alpha_n u+(1-\alpha_n)\bigl(x_n-\xi_n\nabla f(x_n)\bigr)-x_n\bigr\| \\ &= \bigl\|\alpha_n(u-x_n)-(1-\alpha_n)\xi_n\nabla f(x_n)\bigr\| \\ &\le \alpha_n\|u-x_n\|+(1-\alpha_n)\xi_n\bigl\|\nabla f(x_n)\bigr\| \to 0.\end{aligned} \tag{3.22}$$

Since S is firmly nonexpansive (and so it is also nonexpansive), it follows from Lemma 2.4 that

$$\begin{aligned}\|z_{n+1}-p\|^2 &= \|Sv_{n+1}-Sx_n+Sx_n-Sp\|^2 \\ &\le \|Sx_n-Sp\|^2+2\langle Sv_{n+1}-Sx_{n+1}+Sx_{n+1}-Sx_n, z_{n+1}-p\rangle \\ &\le \|x_n-p\|^2-\bigl\|(I-S)x_n\bigr\|^2+2\bigl(\|v_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|\bigr)\|z_{n+1}-p\|.\end{aligned}$$

Thus, we have

$$\begin{aligned}&\bigl\|(I-S)x_n\bigr\|^2 \\ &\quad\le \|x_n-p\|^2-\|z_{n+1}-p\|^2+2\bigl(\|v_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|\bigr)\|z_{n+1}-p\| \\ &\quad\le \|x_n-p\|^2-\bigl(\|z_{n+1}-x_{n+1}\|-\|x_{n+1}-p\|\bigr)^2 \\ &\qquad+2\bigl(\|v_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|\bigr)\|z_{n+1}-p\| \\ &\quad\le \|x_n-p\|^2-\|x_{n+1}-p\|^2-\|z_{n+1}-x_{n+1}\|^2+2\|x_{n+1}-p\|\cdot\|z_{n+1}-x_{n+1}\| \\ &\qquad+2\bigl(\|v_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|\bigr)\|z_{n+1}-p\| \\ &\quad= s_n-s_{n+1}-\|z_{n+1}-x_{n+1}\|^2+2\|x_{n+1}-p\|\cdot\|z_{n+1}-x_{n+1}\| \\ &\qquad+2\bigl(\|v_{n+1}-x_{n+1}\|+\|x_{n+1}-x_n\|\bigr)\|z_{n+1}-p\|.\end{aligned} \tag{3.23}$$

It follows from (3.19), (3.20), and (3.22) that $\|(I-S)x_n\|\to 0$. In view of $x_{n_k}\rightharpoonup x^*$ and that S is demi-closed at origin, we get $x^*\in F(S)$.

On the other hand, from $x_{n_k} \rightharpoonup x^*$ and (3.18), we obtain $y_{n_k} \rightharpoonup x^*$. From (3.1), for any $x \in H_1$, we have

$$F(y_n, x) + \frac{1}{\lambda_n}\langle x - y_n, y_n - z_n\rangle \geq 0.$$

From (A2), we have

$$\frac{1}{\lambda_n}\langle x - y_n, y_n - z_n\rangle \geq F(x, y_n), \quad \forall x \in H_1.$$

Replacing n by n_k, we have

$$\left\langle x - y_{n_k}, \frac{y_{n_k} - z_{n_k}}{\lambda_{n_k}}\right\rangle \geq F(x, y_{n_k}), \quad \forall x \in H_1.$$

Since $\|\frac{y_{n_k} - z_{n_k}}{\lambda_{n_k}}\| \to 0$ and $y_{n_k} \rightharpoonup x^*$, from (A4) we have

$$F\bigl(x, x^*\bigr) \leq 0, \quad \forall x \in H_1. \tag{3.24}$$

Put $w_t = tx + (1-t)x^*$ for all $t \in (0,1]$ and $x \in H_1$. Then we get $w_t \in H_1$. So, from (3.24) we have

$$F\bigl(w_t, x^*\bigr) \leq 0, \quad \forall x \in H_1.$$

From (A4), we have

$$\begin{aligned} 0 = F(w_t, w_t) &\leq tF(w_t, x) + (1-t)F\bigl(w_t, x^*\bigr) \\ &\leq tF(w_t, x), \end{aligned}$$

and hence $F(w_t, x) \geq 0$. Letting $t \to 0$, we have

$$F\bigl(x^*, x\bigr) \geq 0, \quad \forall x \in H_1.$$

This implies $x^* \in \Omega$. Consequently, $x^* \in C$, and hence $w_w(x_n) \subset \Gamma$. Furthermore, in view of (3.20) we have

$$\begin{aligned} \limsup_{n\to\infty}\langle u - p, z_n - p\rangle &= \limsup_{n\to\infty}\langle u - p, x_n - p\rangle \\ &= \max_{w \in w_w(x_n)}\langle u - P_\Gamma u, w - P_\Gamma u\rangle \leq 0. \end{aligned}$$

On the other hand, from (3.9), we have

$$s_{n+1} \leq \bigl(1 - \alpha_n(1-\beta_n)\bigr)s_n + 2\alpha_n(1-\beta_n)\langle u - p, z_n - p\rangle. \tag{3.25}$$

Applying Lemma 2.6 to (3.25), from the condition (i) we obtain $s_n \to 0$, that is, $x_n \to p$.

Case 2: $\{s_n\}$ is not eventually decreasing, that is, we can find a positive integer n_0 such that $s_{n_0} \leq s_{n_0+1}$. Now we define

$$U_n := \{n_0 \leq k \leq n : s_k \leq s_{k+1}\}, \quad n > n_0.$$

It easy to see that U_n is nonempty and satisfies $U_n \subseteq U_{n+1}$. Let

$$\psi(n) := \max U_n, \quad n > n_0.$$

It is clear that $\psi(n) \to \infty$ as $n \to \infty$ (otherwise, $\{s_n\}$ is eventually decreasing). It is also clear that $s_{\psi(n)} \leq s_{\psi(n)+1}$ for all $n > n_0$. Moreover, we prove that

$$s_n \leq s_{\psi(n)+1}, \quad \forall n > n_0. \tag{3.26}$$

In fact, if $\psi(n) = n$, then the inequality (3.26) is trivial; if $\psi(n) < n$, from the definition of $\psi(n)$, there exists some $i \in \mathbb{N}$ such that $\psi(n) + i = n$, we deduce that

$$s_{\psi(n)+1} > s_{\psi(n)+2} > \cdots > s_{\psi(n)+i} = s_n,$$

and the inequality (3.26) holds again. Since $s_{\psi(n)} \leq s_{\psi(n)+1}$ for all $n > n_0$, it follows from (3.10) that

$$\frac{\sigma f^2(x_{\psi(n)})}{\|\nabla f(x_{\psi(n)})\|^2} \leq \alpha_{\psi(n)}(1 - \beta_{\psi(n)})M \to 0.$$

Noting that $\{\|\nabla f(x_{\psi(n)})\|\}$ is bounded, we get $f(x_{\psi(n)}) \to 0$. By the same argument to the proof in case 1, we have $w_w(x_{\psi(n)}) \subset \Gamma$. From (3.19) we have

$$\lim_{n\to\infty} \|x_{\psi(n)} - x_{\psi(n)+1}\| = 0. \tag{3.27}$$

Furthermore, in view of (3.20), we can deduce that

$$\begin{aligned}
&\limsup_{n\to\infty} \langle u - p, z_{\psi(n)} - p\rangle \\
&\quad = \limsup_{n\to\infty} \langle u - p, x_{\psi(n)} - p\rangle \\
&\quad = \max_{w \in w_w(x_{\psi(n)})} \langle u - P_\Gamma u, w - P_\Gamma u\rangle \leq 0.
\end{aligned} \tag{3.28}$$

Since $s_{\psi(n)} \leq s_{\psi(n)+1}$, it follows from (3.9) that

$$s_{\psi(n)} \leq 2\langle u - p, z_{\psi(n)} - p\rangle, \quad n > n_0. \tag{3.29}$$

Combining (3.28) and (3.29) we have

$$\limsup_{n\to\infty} s_{\psi(n)} \leq 0, \tag{3.30}$$

and hence $s_{\psi(n)} \to 0$, which together with (3.27) implies that

$$\begin{aligned}\sqrt{s_{\psi(n)+1}} &\leq \left\|(x_{\psi(n)} - p) + (x_{\psi(n)+1} - x_{\psi(n)})\right\| \\ &\leq \sqrt{s_{\psi(n)}} + \|x_{\psi(n)+1} - x_{\psi(n)}\| \to 0.\end{aligned}$$

Noting the inequality (3.26), this shows that $s_n \to 0$, that is, $x_n \to p$. This completes the proof of Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly to this research work. All authors read and approved the final manuscript.

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