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Best proximity points of generalized almost ψ-Geraghty contractive non-self-mappings

Hassen Aydi[1], Erdal Karapınar[2,3], İnci M Erhan[2*] and Peyman Salimi[4]

*Correspondence: ierhan@atilim.edu.tr; incierhan@gmail.com
[2] Department of Mathematics, Atilim University, İncek, Ankara 06836, Turkey
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce the new notion of almost ψ-Geraghty contractive mappings and investigate the existence of a best proximity point for such mappings in complete metric spaces via the weak *P*-property. We provide an example to validate our best proximity point theorem. The obtained results extend, generalize, and complement some known fixed and best proximity point results from the literature.

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1 Introduction and preliminaries

Non-self-mappings are among the intriguing research directions in fixed point theory. This is evident from the increase of the number of publications related with such maps. A great deal of articles on the subject investigate the non-self-contraction mappings on metric spaces. Let (X,d) be a metric space and A and B be nonempty subsets of X. A mapping $T: A \to B$ is said to be a k-contraction if there exists $k \in [0,1)$ such that $d(Tx,Ty) \le kd(x,y)$ for any $x,y \in A$. It is clear that a k-contraction coincides with the celebrated Banach fixed point theorem (Banach contraction principle) [1] if one takes $A = B$ where the induced metric space $(A, d|_A)$ is complete.

In nonlinear analysis, the theory of fixed points is an essential instrument to solve the equation $Tx = x$ for a self-mapping T defined on a subset of an abstract space such as a metric space, a normed linear space or a topological vector space. Following the Banach contraction principle, most of the fixed point results have been proved for a self-mapping defined on an abstract space. It is quite natural to investigate the existence and uniqueness of a non-self-mapping $T: A \to B$ which does not possess a fixed point. If a non-self-mapping $T: A \to B$ has no fixed point, then the answer of the following question makes sense: Is there a point $x \in X$ such that the distance between x and Tx is closest in some sense? Roughly speaking, best proximity theory investigates the existence and uniqueness of such a closest point x. We refer the reader to [2–9] and [10–32] for further discussion of best proximity.

Definition 1.1 Let (X,d) be a metric space and $A,B \subset X$. We say that $x^* \in A$ is a best proximity point of the non-self-mapping $T : A \to B$ if the following equality holds:

$$d\bigl(x^*, Tx^*\bigr) = d(A,B), \tag{1}$$

where $d(A,B) = \inf\{d(x,y) : x \in A, y \in B\}$.

It is clear that the notion of a fixed point coincided with the notion of a best proximity point when the underlying mapping is a self-mapping.

Let (X,d) be a metric space. Suppose that A and B are nonempty subsets of a metric space (X,d). We define the following sets:

$$\begin{aligned} A_0 &= \bigl\{x \in A : d(x,y) = d(A,B) \text{ for some } y \in B\bigr\}, \\ B_0 &= \bigl\{y \in B : d(x,y) = d(A,B) \text{ for some } x \in A\bigr\}. \end{aligned} \tag{2}$$

In [17], the authors presented sufficient conditions for the sets A_0 and B_0 to be nonempty.

In 1973 Geraghty [33] introduced the class S of functions $\beta : [0,\infty) \to [0,1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \quad \text{implies} \quad t_n \to 0. \tag{3}$$

The author defined contraction mappings via functions from this class and proved the following result.

Theorem 1.1 (Geraghty [33]) *Let (X,d) be a complete metric space and $T : X \to X$ be an operator. If T satisfies the following inequality*:

$$d(Tx, Ty) \le \beta\bigl(d(x,y)\bigr)d(x,y) \quad \textit{for any } x,y \in X, \tag{4}$$

where $\beta \in S$, then T has a unique fixed point.

Recently, Caballero *et al.* [6] introduced the following contraction.

Definition 1.2 ([6]) Let A, B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a Geraghty-contraction if there exists $\beta \in S$ such that

$$d(Tx, Ty) \le \beta\bigl(d(x,y)\bigr)d(x,y) \quad \textit{for any } x,y \in A. \tag{5}$$

Based on Definition 1.2, the authors [6] obtained the following result.

Theorem 1.2 (See [6]) *Let (A,B) be a pair of nonempty closet subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T : A \to B$ be a continuous, Geraghty-contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair (A,B) has the P-property, then there exists a unique x^* in A such that $d(x^*, Tx^*) = d(A,B)$.*

The P-property mentioned in the theorem above has been introduced in [29].

Definition 1.3 Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1,y_1) = d(A,B) \quad \textit{and} \quad d(x_2,y_2) = d(A,B) \quad \Rightarrow \quad d(x_1,x_2) = d(y_1,y_2). \tag{6}$$

It is easily seen that for any nonempty subset A of (X,d), the pair (A,A) has the P-property. In [29], the author proved that any pair (A,B) of nonempty closed convex subsets of a real Hilbert space H satisfies the P-property.

Recently, Zhang *et al.* [34] defined the following notion, which is weaker than the P-property.

Definition 1.4 Let (A,B) be a pair of nonempty subsets of a metric space (X,d) with $A_0 \neq \emptyset$. Then the pair (A,B) is said to have the weak P-property if and only if for any $x_1, x_2 \in A_0$ and $y_1, y_2 \in B_0$,

$$d(x_1,y_1) = d(A,B) \quad \textit{and} \quad d(x_2,y_2) = d(A,B) \quad \Rightarrow \quad d(x_1,x_2) \leq d(y_1,y_2). \tag{7}$$

Let Ψ denote the class of functions $\psi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(a) ψ is nondecreasing;
(b) ψ is subadditive, that is, $\psi(s+t) \leq \psi(s) + \psi(t)$;
(c) ψ is continuous;
(d) $\psi(t) = 0 \Leftrightarrow t = 0$.

The notion of ψ-Geraghty contraction has been introduced very recently in [11], as an extension of Definition 1.2.

Definition 1.5 Let A, B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a ψ-Geraghty contraction if there exist $\beta \in S$ and $\psi \in \Psi$ such that

$$\psi\bigl(d(Tx,Ty)\bigr) \leq \beta\bigl(\psi\bigl(d(x,y)\bigr)\bigr)\psi\bigl(d(x,y)\bigr) \quad \textit{for any } x,y \in A. \tag{8}$$

Remark 1.1 Notice that since $\beta : [0,\infty) \to [0,1)$, we have

$$\psi\bigl(d(Tx,Ty)\bigr) \leq \beta\bigl(\psi\bigl(d(x,y)\bigr)\bigr)\psi\bigl(d(x,y)\bigr) < \psi\bigl(d(x,y)\bigr) \quad \text{for any } x,y \in A \text{ with } x \neq y. \tag{9}$$

In [11], the author also proved the following best proximity point theorem.

Theorem 1.3 (See [11]) *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T : A \to B$ be a ψ-Geraghty contraction satisfying $T(A_0) \subseteq B_0$. Suppose that the pair (A,B) has the P-property. Then there exists a unique x^* in A such that $d(x^*,Tx^*) = d(A,B)$.*

2 Main results

Our main results are based on the following definition which is a generalization of Definition 1.5.

Definition 2.1 Let A, B be two nonempty subsets of a metric space (X,d). A mapping $T : A \to B$ is said to be a generalized almost ψ-Geraghty contraction if there exist $\beta \in S$ and $\psi \in \Psi$ such that

$$\psi\bigl(d(Tx,Ty)\bigr) \leq \beta\bigl(\psi\bigl(M(x,y)\bigr)\bigr)\psi\bigl(M(x,y)-d(A,B)\bigr)+L\psi\bigl(N(x,y)-d(A,B)\bigr) \tag{10}$$

for all $x, y \in A$ where $L \geq 0$,

$$M(x,y) = \max\bigl\{d(x,y), d(x,Tx), d(y,Ty)\bigr\},$$
$$N(x,y) = \min\bigl\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\bigr\}.$$

Now, we state and prove our main theorem about existence and uniqueness of a best proximity point for a non-self-mapping satisfying a generalized almost ψ-Geraghty contraction.

Theorem 2.1 *Let (A,B) be a pair of nonempty closed subsets of a complete metric space (X,d) such that A_0 is nonempty. Let $T : A \to B$ be a generalized almost ψ-Geraghty contraction satisfying $T(A_0) \subseteq B_0$. Assume that the pair (A,B) has the weak P-property. Then T has a unique best proximity point in A.*

Proof Since the subset A_0 is not empty, we can take x_0 in A_0. Taking into account that $Tx_0 \in T(A_0) \subseteq B_0$, we can find $x_1 \in A_0$ such that $d(x_1, Tx_0) = d(A,B)$. Further, since $Tx_1 \in T(A_0) \subseteq B_0$, it follows that there is an element x_2 in A_0 such that $d(x_2, Tx_1) = d(A,B)$. Recursively, we obtain a sequence $\{x_n\}$ in A_0 satisfying

$$d(x_{n+1}, Tx_n) = d(A,B) \quad \text{for any } n \in \mathbb{N}. \tag{11}$$

Since the pair (A,B) has the weak P-property, we deduce

$$d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n) \quad \text{for any } n \in \mathbb{N}. \tag{12}$$

Due to the triangle inequality together with the equality (11) we have

$$d(x_{n-1}, Tx_{n-1}) \leq d(x_{n-1}, x_n) + d(x_n, Tx_{n-1}) = d(x_{n-1}, x_n) + d(A,B).$$

Analogously, combining the equalities (11) and (12) with the triangle inequality we obtain

$$d(x_n, Tx_n) \leq d(x_n, x_{n+1}) + d(x_{n+1}, Tx_n) = d(x_n, x_{n+1}) + d(A,B). \tag{13}$$

Consequently, we have

$$\begin{aligned} M(x_{n-1}, x_n) &= \max\bigl\{d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\bigr\} \\ &\leq \max\bigl\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\bigr\} + d(A,B). \end{aligned} \tag{14}$$

Also note that

$$\begin{aligned} &N(x_{n-1},x_n) - d(A,B) \\ &\quad = \min\bigl\{d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_n),d(x_n,Tx_{n-1})\bigr\} - d(A,B) \\ &\quad \le \min\bigl\{d(x_{n-1},Tx_{n-1}),d(x_n,Tx_n),d(x_{n-1},Tx_n),d(A,B)\bigr\} - d(A,B) \\ &\quad = d(A,B) - d(A,B) = 0. \end{aligned} \tag{15}$$

If there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0},x_{n_0+1}) = 0$, then the proof is completed. Indeed,

$$0 = d(x_{n_0},x_{n_0+1}) = d(Tx_{n_0-1},Tx_{n_0}), \tag{16}$$

and consequently, $Tx_{n_0-1} = Tx_{n_0}$. Therefore, we conclude that

$$d(A,B) = d(x_{n_0},Tx_{n_0-1}) = d(x_{n_0},Tx_{n_0}). \tag{17}$$

For the rest of the proof, we suppose that $d(x_n,x_{n+1}) > 0$ for all $n \in \mathbb{N}$. In view of the fact that T is a generalized almost ψ-Geraghty contraction, we have

$$\begin{aligned} \psi\bigl(d(x_n,x_{n+1})\bigr) &\le \psi\bigl(d(Tx_{n-1},Tx_n)\bigr) \\ &\le \beta\bigl(\psi\bigl(M(x_{n-1},x_n)\bigr)\bigr)\psi\bigl(M(x_{n-1},x_n) - d(A,B)\bigr) \\ &\qquad + L\psi\bigl(N(x_{n-1},x_n) - d(A,B)\bigr) \\ &= \beta\bigl(\psi\bigl(M(x_{n-1},x_n)\bigr)\bigr)\psi\bigl(M(x_{n-1},x_n) - d(A,B)\bigr) + L\psi(0) \\ &= \beta\bigl(\psi\bigl(M(x_{n-1},x_n)\bigr)\bigr)\psi\bigl(M(x_{n-1},x_n) - d(A,B)\bigr) \\ &< \psi\bigl(M(x_{n-1},x_n) - d(A,B)\bigr). \end{aligned} \tag{18}$$

Taking into account the inequalities (14) and (18), we deduce that

$$\psi\bigl(d(x_n,x_{n+1})\bigr) < \psi\bigl(M(x_{n-1},x_n) - d(A,B)\bigr) \le \psi\bigl(\max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\}\bigr).$$

If for some n, $\max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then we get

$$\psi\bigl(d(x_n,x_{n+1})\bigr) < \psi\bigl(d(x_n,x_{n+1})\bigr),$$

which is a contradiction. Therefore, we must have

$$M(x_{n-1},x_n) \le \max\bigl\{d(x_{n-1},x_n),d(x_n,x_{n+1})\bigr\} + d(A,B) = d(x_{n-1},x_n) + d(A,B) \tag{19}$$

for all $n \in \mathbb{N}$. Regarding the inequality (18), we see that

$$\begin{aligned} \psi\bigl(d(x_n,x_{n+1})\bigr) &= \psi\bigl(d(Tx_{n-1},Tx_n)\bigr) \\ &\le \beta\bigl(\psi\bigl(M(x_{n-1},x_n)\bigr)\bigr)\psi\bigl(d(x_{n-1},x_n)\bigr) \\ &< \psi\bigl(d(x_{n-1},x_n)\bigr) \end{aligned} \tag{20}$$

holds for all $n \in \mathbb{N}$. Since ψ is nondecreasing, then $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all n. Consequently, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and is bounded below and hence $\lim_{n\to\infty} d(x_n, x_{n+1}) = s \geq 0$ exists. Assume that $s > 0$. Rewrite (20) as

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \leq \beta\big(\psi\big(M(x_n, x_{n+1})\big)\big) \leq 1$$

for each $n \geq 1$. Taking the limit of both sides as $n \to \infty$, we find

$$\lim_{n\to\infty} \beta\big(\psi\big(M(x_n, x_{n+1})\big)\big) = 1.$$

On the other hand, since $\beta \in S$, we conclude $\lim_{n\to\infty} \psi(M(x_n, x_{n+1})) = 0$, that is,

$$s = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{21}$$

Since $d(x_n, Tx_{n-1}) = d(A, B)$ holds for all $n \in \mathbb{N}$ and (A, B) satisfies the weak P-property, then for all $m, n \in \mathbb{N}$, we can write

$$d(x_m, x_n) \leq d(Tx_{m-1}, Tx_{n-1}). \tag{22}$$

From (13), we deduce

$$\begin{aligned} M(x_m, x_n) &= \max\big\{d(x_m, x_n), d(x_m, Tx_m), d(x_n, Tx_n)\big\} \\ &\leq \max\big\{d(x_m, x_n), d(x_m, x_{m+1}), d(x_n, x_{n+1})\big\} + d(A, B). \end{aligned}$$

By using $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we get

$$\lim_{m,n\to\infty} \big(M(x_m, x_n) - d(A, B)\big) \leq \lim_{m,n\to\infty} d(x_m, x_n). \tag{23}$$

On the other hand,

$$\begin{aligned} 0 &\leq N(x_m, x_n) - d(A, B) \\ &= \min\big\{d(x_m, Tx_m), d(x_n, Tx_n), d(x_m, Tx_n), d(x_n, Tx_m)\big\} - d(A, B) \\ &\leq \min\big\{d(x_m, x_{m+1}) + d(A, B), d(x_n, Tx_n), d(x_m, Tx_n), d(x_n, Tx_m)\big\} - d(A, B). \end{aligned} \tag{24}$$

Due to the fact that $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$, we obtain

$$\lim_{m,n\to\infty} \big[N(x_m, x_n) - d(A, B)\big] = 0. \tag{25}$$

We shall show next that $\{x_n\}$ is a Cauchy sequence. Assume on the contrary that

$$\varepsilon = \limsup_{m,n\to\infty} d(x_n, x_m) > 0. \tag{26}$$

Employing the triangular inequality and (22), we get

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_{m+1}, x_m) \\ &\leq d(x_n, x_{n+1}) + d(Tx_n, Tx_m) + d(x_{m+1}, x_m). \end{aligned} \tag{27}$$

Combining (10) and (27), and regarding the properties of ψ, we obtain

$$\begin{aligned}\psi\bigl(d(x_n,x_m)\bigr) &\le \psi\bigl(d(x_n,x_{n+1})+d(Tx_n,Tx_m)+d(x_{m+1},x_m)\bigr)\\ &\le \psi\bigl(d(x_n,x_{n+1})\bigr)+\psi\bigl(d(Tx_n,Tx_m)\bigr)+\psi\bigl(d(x_{m+1},x_m)\bigr)\\ &\le \psi\bigl(d(x_n,x_{n+1})\bigr)+\beta\bigl(\psi\bigl(M(x_n,x_m)\bigr)\bigr)\psi\bigl(M(x_n,x_m)-d(A,B)\bigr)\\ &\quad + L\psi\bigl(N(x_n,x_m)-d(A,B)\bigr)+\psi\bigl(d(x_{m+1},x_m)\bigr). \end{aligned} \tag{28}$$

From (23), (25), (28), and by using $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, we have

$$\begin{aligned}\lim_{m,n\to\infty}\psi\bigl(d(x_n,x_m)\bigr) &\le \lim_{m,n\to\infty}\beta\bigl(\psi\bigl(M(x_n,x_m)\bigr)\bigr)\lim_{m,n\to\infty}\psi\bigl(M(x_m,x_n)-d(A,B)\bigr)\\ &\le \lim_{m,n\to\infty}\beta\bigl(\psi\bigl(M(x_n,x_m)\bigr)\bigr)\lim_{m,n\to\infty}\psi\bigl(d(x_m,x_n)\bigr).\end{aligned}$$

So by (26), we get

$$1 \le \lim_{m,n\to\infty}\beta\bigl(\psi\bigl(M(x_n,x_m)\bigr)\bigr),$$

that is, $\lim_{m,n\to\infty}\beta(\psi(M(x_n,x_m))) = 1$. Therefore, $\lim_{m,n\to\infty} M(x_n,x_m) = 0$. This implies that $\lim_{m,n\to\infty} d(x_n,x_m) = 0$, which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence.

Since $\{x_n\}\subset A$ and A is a closed subset of the complete metric space (X,d), we can find $x^*\in A$ such that $x_n\to x^*$ as $n\to\infty$. We shall show that $d(x^*,Tx^*) = d(A,B)$. If $x^* = Tx^*$, then $A\cap B\neq\emptyset$, and $d(x^*,Tx^*) = d(A,B) = 0$, *i.e.*, x^* is a best proximity point of T. Hence, we assume that $d(x^*,Tx^*) > 0$. Suppose on the contrary that x^* is not a best proximity point of T, that is, $d(x^*,Tx^*) > d(A,B)$. First note that

$$\begin{aligned}d\bigl(x^*,Tx^*\bigr) &\le d\bigl(x^*,Tx_n\bigr)+d\bigl(Tx_n,Tx^*\bigr)\\ &\le d\bigl(x^*,x_{n+1}\bigr)+d(x_{n+1},Tx_n)+d\bigl(Tx_n,Tx^*\bigr)\\ &\le d\bigl(x^*,x_{n+1}\bigr)+d(A,B)+d\bigl(Tx_n,Tx^*\bigr).\end{aligned}$$

Taking the limit as $n\to\infty$ in the above inequality, we obtain

$$d\bigl(x^*,Tx^*\bigr)-d(A,B)\le\lim_{n\to\infty} d\bigl(Tx_n,Tx^*\bigr).$$

Since ψ is nondecreasing and continuous, then

$$\psi\bigl(d\bigl(x^*,Tx^*\bigr)-d(A,B)\bigr)\le\psi\Bigl(\lim_{n\to\infty} d\bigl(Tx_n,Tx^*\bigr)\Bigr)=\lim_{n\to\infty}\psi\bigl(d\bigl(Tx_n,Tx^*\bigr)\bigr). \tag{29}$$

Also, letting $n\to\infty$ in (13) results in

$$\lim_{n\to\infty} d(x_n,Tx_n)\le d(A,B),$$

that is, $\lim_{n\to\infty} d(x_n,Tx_n) = d(A,B)$. Then we get

$$\lim_{n\to\infty} M\bigl(x_n,x^*\bigr)=\max\Bigl\{\lim_{n\to\infty} d\bigl(x^*,x_n\bigr),\lim_{n\to\infty} d(x_n,Tx_n),d\bigl(x^*,Tx^*\bigr)\Bigr\}=d\bigl(x^*,Tx^*\bigr),$$

and therefore

$$\lim_{n\to\infty} \psi\left(M\left(x_n, x^*\right) - d(A,B)\right) = \psi\left(d\left(x^*, Tx^*\right) - d(A,B)\right). \tag{30}$$

Further,

$$\begin{aligned}&\lim_{n\to\infty} N\left(x_n, x^*\right) - d(A,B)\\&\quad= \min\left\{\lim_{n\to\infty} d(x_n, Tx_n), d\left(x^*, Tx^*\right), \lim_{n\to\infty} d\left(x_n, Tx^*\right), \lim_{n\to\infty} d\left(x^*, Tx_n\right)\right\} - d(A,B),\end{aligned}$$

which implies

$$\lim_{n\to\infty} N\left(x_n, x^*\right) - d(A,B) = 0. \tag{31}$$

Therefore, combining (10), (29), (30), and (31) we deduce

$$\begin{aligned}\psi\left(d\left(x^*, Tx^*\right) - d(A,B)\right) &\le \lim_{n\to\infty} \psi\left(d\left(Tx_n, Tx^*\right)\right)\\&\le \lim_{n\to\infty} \beta\left(\psi\left(M\left(x_n, x^*\right)\right)\right) \lim_{n\to\infty} \psi\left(M\left(x_n, x^*\right) - d(A,B)\right)\\&\quad + L \lim_{n\to\infty} \psi\left(N\left(x_n, x^*\right) - d(A,B)\right)\\&= \lim_{n\to\infty} \beta\left(\psi\left(M\left(x_n, x^*\right)\right)\right) \psi\left(d\left(x^*, Tx^*\right) - d(A,B)\right).\end{aligned} \tag{32}$$

Now, since $\psi(d(x^*, Tx^*) - d(A,B)) > 0$, and making use of (32), we get

$$1 \le \lim_{n\to\infty} \beta\left(\psi\left(M\left(x_n, x^*\right)\right)\right),$$

that is,

$$\lim_{n\to\infty} \beta\left(\psi\left(M\left(x_n, x^*\right)\right)\right) = 1,$$

which implies

$$\lim_{n\to\infty} M\left(x_n, x^*\right) = d\left(x^*, Tx^*\right) = 0,$$

and so $d(x^*, Tx^*) = 0 > d(A,B)$, which is a contradiction. Therefore, $d(x^*, Tx^*) \le d(A,B)$, that is, $d(x^*, Tx^*) = d(A,B)$. In other words, x^* is a best proximity point of T. This completes the proof of the existence of a best proximity point.

We shall show next the uniqueness of the best proximity point of T. Suppose that x^* and y^* are two best proximity points of T, such that $x^* \neq y^*$. This implies that

$$d\left(x^*, Tx^*\right) = d(A,B) = d\left(y^*, Ty^*\right), \tag{33}$$

where $d(x^*, y^*) > 0$. Due to the weak P-property of the pair (A,B), we have

$$d\left(x^*, y^*\right) \le d\left(Tx^*, Ty^*\right). \tag{34}$$

Observe that in this case

$$\begin{aligned} M(x^*,y^*) &= \max\{d(x^*,y^*),d(x^*,Tx^*),d(y^*,Ty^*)\} \\ &= \max\{d(x^*,y^*),d(A,B),d(A,B)\}. \end{aligned}$$

Also, note that

$$\begin{aligned} N(x^*,y^*)-d(A,B) &= \min\{d(x^*,Tx^*),d(y^*,Ty^*),d(x^*,Ty^*),d(y^*,Tx^*)\}-d(A,B) \\ &= \min\{d(A,B),d(A,B),d(x^*,Ty^*),d(y^*,Tx^*)\}-d(A,B) \\ &= d(A,B)-d(A,B)=0. \end{aligned}$$

Using the fact that T is a generalized almost ψ-Geraghty contraction, we derive

$$\begin{aligned} \psi(d(x^*,y^*)) &\le \psi(d(Tx^*,Ty^*)) \\ &\le \beta(\psi(M(x^*,y^*)))\psi(M(x^*,y^*)-d(A,B))+L\psi(N(x^*,y^*)-d(A,B)) \\ &= \beta(\psi(M(x^*,y^*)))\psi(M(x^*,y^*)-d(A,B)) \\ &< \psi(M(x^*,y^*)-d(A,B)). \end{aligned}$$

If $M(x^*,y^*)=d(A,B)$, due to the fact that $d(x^*,y^*)>0$, the inequality above becomes

$$0<\psi(d(x^*,y^*))<\psi(0), \tag{35}$$

which implies $d(x^*,y^*)=0$ and contradicts the assumption $d(x^*,y^*)>0$. Else, if $M(x^*,y^*)=d(x^*,y^*)$, we deduce

$$0<\psi(d(x^*,y^*))<\psi(d(x^*,y^*)-d(A,B)), \tag{36}$$

which is not possible, since ψ is nondecreasing. Therefore, we must have $d(x^*,y^*)=0$. This completes the proof. □

To illustrate our result given in Theorem 2.1, we present the following example, which shows that Theorem 2.1 is a proper generalization of Theorem 1.2.

Example 2.1 Consider the space $X=\mathbb{R}$ with Euclidean metric. Take the sets

$$A=(-\infty,-1] \quad \text{and} \quad B=[1,+\infty).$$

Obviously, $d(A,B)=2$. Let $T:A\to B$ be defined by $Tx=-x$. Notice that $A_0=\{-1\}$, $B_0=\{1\}$ and $T(A_0)\subseteq B_0$. Also, it is clear that the pair (A,B) has the weak P-property.

Consider

$$\beta(t)=\begin{cases} \frac{1}{1+t}, & \text{if } 0\le t<1, \\ \frac{t}{1+t}, & \text{if } t\ge 1, \end{cases}$$

and $\psi(t) = \alpha t$ (with $\alpha \geq \frac{1}{2}$) for all $t \geq 0$. Note that $\beta \in S$ and $\psi \in \Psi$. For all $x, y \in A$, we have

$$d(Tx, Ty) = |x - y| \quad \text{and} \quad M(x, y) = \max\{|x - y|, -2x, -2y\}.$$

We shall show that T is a generalized almost ψ-Geraghty contraction. Without loss of generality, consider the case where $x \geq y$. Then we have $M(x, y) = -2y$ and $d(Tx, Ty) = x - y$.

In this case, we see that

$$\begin{aligned}
\psi\big(d(Tx, Ty)\big) &= \alpha(x - y) \leq \alpha(-x - y - 2) \\
&\leq 2\alpha^2(-x - y - 2) \leq [-2\alpha y]\big[\alpha(-x - y - 2)\big] \\
&= [-2\alpha y]\big[\alpha(-2y - 2) - \alpha(x - y)\big] \\
&= \psi\big(M(x, y)\big)\big[\psi\big(M(x, y) - d(A, B)\big) - \psi\big(d(Tx, Ty)\big)\big].
\end{aligned}$$

Therefore

$$\psi\big(d(Tx, Ty)\big) \leq \frac{\psi(M(x, y))}{1 + \psi(M(x, y))}\psi\big(M(x, y) - d(A, B)\big). \tag{37}$$

On the other hand, we know that $\psi(M(x, y)) = -2\alpha y \geq 1$ for all $x, y \in A$ with $x \geq y$. Hence,

$$\beta\big(\psi\big(M(x, y)\big)\big) = \frac{\psi(M(x, y))}{1 + \psi(M(x, y))},$$

and from (37) we deduce

$$\psi\big(d(Tx, Ty)\big) \leq \beta\big(\psi\big(M(x, y)\big)\big)\psi\big(M(x, y) - d(A, B)\big).$$

Thus, all hypotheses of Theorem 2.1 are satisfied, and $x^* = -1$ is the unique best proximity point of the map T.

On the other hand, T is not a Geraghty contraction. Indeed, taking $x = -1$ and $y = -2$, we get

$$d(Tx, Ty) = 1 > \frac{1}{2} = \beta\big(d(x, y)\big)d(x, y).$$

Then Theorem 1.2 (the main result of Caballero *et al.* [6]) is not applicable.

Similarly, we cannot apply Theorem 1.3 because T is not a ψ-Geraghty contraction. Let $x = -1$, $y = -2$ and $\psi(t) = \alpha t$ with $\alpha < 2$. Then T does not satisfy (8).

If in Theorem 2.1 we take $\psi(t) = t$ for all $t \geq 0$, then we deduce the following corollary.

Corollary 2.1 *Let (A, B) be a pair of nonempty closed subsets of a complete metric space (X, d) such that A_0 is nonempty. Let $T : A \to B$ be a non-self-mapping satisfying $T(A_0) \subseteq B_0$ and*

$$d(Tx, Ty) \leq \beta\big(M(x, y)\big)\big[M(x, y) - d(A, B)\big] + L\big[N(x, y) - d(A, B)\big]$$

for all $x, y \in A$ *where* $\beta \in S$, $L \geq 0$,

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$
$$N(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Assume that the pair (A,B) *has the weak P-property. Then* T *has a unique best proximity point in* A.

If further in the above corollary we take $\beta(t) = r$ where $0 \leq r < 1$, then we deduce another particular result.

Corollary 2.2 *Let* (A,B) *be a pair of nonempty closed subsets of a complete metric space* (X,d) *such that* A_0 *is nonempty. Let* $T : A \to B$ *be a non-self-mapping satisfying* $T(A_0) \subseteq B_0$ *and*

$$d(Tx,Ty) \leq r\big[M(x,y) - d(A,B)\big] + L\big[N(x,y) - d(A,B)\big]$$

for all $x, y \in A$ *where* $0 \leq r < 1$, $L \geq 0$,

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\},$$
$$N(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Assume that the pair (A,B) *has the weak P-property. Then* T *has a unique best proximity point in* A.

3 Application to fixed point theory

The case $A = B$ in Theorem 2.1 corresponds to a self-mapping and results in an existence and uniqueness theorem for a fixed point of the map T. We state this case in the next theorem.

Theorem 3.1 *Let* (X,d) *be a complete metric space. Suppose that* A *is a nonempty closed subset of* X. *Let* $T : A \to A$ *be a mapping such that*

$$\psi\big(d(Tx,Ty)\big) \leq \beta\big(\psi\big(M(x,y)\big)\big)\psi\big(M(x,y)\big) + L\psi\big(N(x,y)\big) \quad \text{for any } x, y \in A, \tag{38}$$

where $\psi \in \Psi$, $\beta \in S$, $L \geq 0$,

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\} \quad \text{and}$$
$$N(x,y) = \min\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}.$$

Then T *has a unique fixed point.*

Finally, taking $\psi(t) = t$ in Theorem 3.1, we get another fixed point result.

Corollary 3.1 *Let (X,d) be a complete metric space. Suppose that A is a nonempty closed subset of X. Let $T : A \to A$ be a mapping such that*

$$d(Tx, Ty) \leq \beta\big(M(x,y)\big)M(x,y) + LN(x,y) \quad \text{for any } x, y \in A, \tag{39}$$

where $\beta \in S$, $L \geq 0$,

$$M(x,y) = \max\big\{d(x,y), d(x,Tx), d(y,Ty)\big\} \quad \text{and}$$
$$N(x,y) = \min\big\{d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\big\}.$$

Then T has a unique fixed point.

Remark 3.1 The best proximity theorem given in this work, more precisely Theorem 2.1, is a quite general result. It is a generalization of Theorem 2.1 in [14], Theorem 8 in [5], and also Theorem 1.2 given in Section 1. In addition, Corollary 3.1 improves Theorem 1.1.

Remark 3.2 Very recently, Karapınar and Samet [15] proved that the function $d_\varphi = \varphi \circ d$ on the set X, where $\varphi \in \Psi$ is also a metric on X. Therefore, some of the fixed theorems regarding contraction mappings defined via auxiliary functions from the set Ψ can be in fact deduced from the existing ones in the literature. However, our main result given in Theorem 2.1 is not a consequence of any existing theorems due to the fact that the contraction condition contains the term $d(A,B)$.

On the other hand, the definition of $d_\varphi = \varphi \circ d$ can be used to show that Theorem 3.1 follows from Corollary 3.1. Nevertheless, Corollary 3.1 and hence Theorem 3.1 are still new results.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the manuscript.

Author details
[1] Department of Mathematics, Jubail College of Education, Dammam University, P.O. Box 12020, Industrial Jubail, 31961, Saudi Arabia. [2] Department of Mathematics, Atilim University, İncek, Ankara 06836, Turkey. [3] Nonlinear Analysis and Applied Mathematics Research Group (NAAM), King Abdulaziz University, Jeddah, Saudi Arabia. [4] Young Researchers and Elite Club, Rasht Branch, Islamic Azad University, Rasht, Iran.

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