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Some generalizations of Mizoguchi-Takahashi's fixed point theorem with new local constraints

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Abstract
In this paper, motivated by Kikkawa-Suzuki's fixed point theorem, we establish some new generalizations of Mizoguchi-Takahashi's fixed point theorem with new local constraints on discussion maps.
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1 Introduction and preliminaries

Let (X,d) be a metric space. Denote by $\mathcal{N}(X)$ the family of all nonempty subsets of X, $\mathcal{C}(X)$ the class of all nonempty closed subsets of X and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. For each $x \in X$ and $A \subseteq X$, let $d(x,A) = \inf_{y\in A} d(x,y)$. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0,\infty)$ defined by

$$\mathcal{H}(A,B) = \max\left\{\sup_{x\in B} d(x,A), \sup_{x\in A} d(x,B)\right\}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X. Let $T : X \to \mathcal{N}(X)$ be a multivalued map. A point v in X is said to be a *fixed point* of T if $v \in Tv$. The set of fixed points of T is denoted by $\mathcal{F}(T)$. The map T is said to have the *approximate fixed point property* [1–3] on X provided $\inf_{x\in X} d(x,Tx) = 0$. It is obvious that $\mathcal{F}(T) \neq \emptyset$ implies that T has the approximate fixed point property. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the sets of positive integers and real numbers, respectively.

A function $\varphi : [0,\infty) \to [0,1)$ is said to be an $\mathcal{MT}$*-function* (or $\mathcal{R}$*-function*) [2–5] if $\limsup_{s\to t^+} \varphi(s) < 1$ for all $t \in [0,\infty)$. It is evident that if $\varphi : [0,\infty) \to [0,1)$ is a nondecreasing function or a nonincreasing function, then φ is a $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class.

Recently, Du [5] first proved the following characterizations of $\mathcal{MT}$-functions.

Theorem 1.1 ([5]) *Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*

(b) *For each* $t \in [0,\infty)$, *there exist* $r_t^{(1)} \in [0,1)$ *and* $\varepsilon_t^{(1)} > 0$ *such that* $\varphi(s) \le r_t^{(1)}$ *for all* $s \in (t, t+\varepsilon_t^{(1)})$.

(c) *For each* $t \in [0,\infty)$, *there exist* $r_t^{(2)} \in [0,1)$ *and* $\varepsilon_t^{(2)} > 0$ *such that* $\varphi(s) \le r_t^{(2)}$ *for all* $s \in [t, t+\varepsilon_t^{(2)}]$.

(d) *For each* $t \in [0,\infty)$, *there exist* $r_t^{(3)} \in [0,1)$ *and* $\varepsilon_t^{(3)} > 0$ *such that* $\varphi(s) \le r_t^{(3)}$ *for all* $s \in (t, t+\varepsilon_t^{(3)}]$.

(e) *For each* $t \in [0,\infty)$, *there exist* $r_t^{(4)} \in [0,1)$ *and* $\varepsilon_t^{(4)} > 0$ *such that* $\varphi(s) \le r_t^{(4)}$ *for all* $s \in [t, t+\varepsilon_t^{(4)})$.

(f) *For any nonincreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $[0,\infty)$, *we have* $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.

(g) φ *is a function of contractive factor*; *that is*, *for any strictly decreasing sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* $[0,\infty)$, *we have* $0 \le \sup_{n\in\mathbb{N}} \varphi(x_n) < 1$.

In 1989, Mizoguchi and Takahashi [6] proved a famous generalization of Nadler's fixed point theorem, which gives a partial answer of Problem 9 in Reich [7].

Theorem 1.2 (Mizoguchi and Takahashi [6]) *Let* (X,d) *be a complete metric space*, $\varphi : [0,\infty) \to [0,1)$ *be a* $\mathcal{MT}$*-function and* $T : X \to \mathcal{CB}(X)$ *be a multivalued map. Assume that*

$$\mathcal{H}(Tx, Ty) \le \varphi\big(d(x,y)\big)d(x,y),$$

for all $x, y \in X$. *Then* $\mathcal{F}(T) \ne \emptyset$.

A number of generalizations in various different directions of research of Mizoguchi-Takahashi's fixed point theorem were investigated by several authors; see, *e.g.*, [2–5, 8–12] and references therein.

In 2008, Suzuki [13] presented a new type of generalization of the celebrated Banach contraction principle [14] which characterized the metric completeness.

Theorem 1.3 (Suzuki [13]) *Define a nonincreasing function* θ *from* $[0,1)$ *onto* $(\frac{1}{2},1]$ *by*

$$\theta(r) = \begin{cases} 1, & \text{if } 0 \le r \le \frac{1}{2}(\sqrt{5}-1), \\ \frac{1-r}{r^2}, & \text{if } \frac{1}{2}(\sqrt{5}-1) \le r \le \frac{1}{\sqrt{2}}, \\ \frac{1}{1+r}, & \text{if } \frac{1}{\sqrt{2}} \le r < 1. \end{cases}$$

Then for a metric space (X,d), *the following are equivalent*:

(1) X *is complete*.

(2) *Every mapping* T *on* X *satisfying the following has a fixed point*:
   - *There exists* $r \in [0,1)$ *such that* $\theta(r)d(x,Tx) \le d(x,y)$ *implies* $d(Tx,Ty) \le rd(x,y)$ *for all* $x, y \in X$.

(3) *There exists* $r \in [0,1)$ *such that every mapping* T *on* X *satisfying the following has a fixed point*:
   - $\frac{1}{10{,}000}d(x,Tx) \le d(x,y)$ *implies* $d(Tx,Ty) \le rd(x,y)$ *for all* $x, y \in X$.

Remark 1.1 ([13]) For every $r \in [0,1)$, $\theta(r)$ is the best constant.

Later, Kikkawa and Suzuki [15] proved an interesting generalization of both Theorem 1.1 and Nadler's fixed point theorem. In fact, Kikkawa-Suzuki's fixed point theorem can be regarded as a generalization of Nadler fixed point theorem with a local constraint on the discussion map.

Theorem 1.4 (Kikkawa and Suzuki [15]) *Define a strictly decreasing function η from $[0,1)$ onto $(\frac{1}{2},1]$ by*

$$\eta(r)=\frac{1}{1+r}.$$

Let (X,d) be a complete metric space and let T be a map from X into $\mathcal{CB}(X)$. Assume that there exists $r\in[0,1)$ such that

$$\eta(r)d(x,Tx)\leq d(x,y) \quad \textit{implies} \quad \mathcal{H}(Tx,Ty)\leq rd(x,y)$$

for all $x,y\in X$. Then $\mathcal{F}(T)\neq\emptyset$.

In this paper, motivated by Kikkawa-Suzuki's fixed point theorem, we establish some new generalizations of Mizoguchi-Takahashi's fixed point theorem with new local constraints on discussion maps. Our new results generalize and improve Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem and Banach contraction principle.

2 Main results

Very recently, Du and Khojasteh [12] first introduced the concept of manageable functions.

Definition 2.1 ([12]) A function $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is called *manageable* if the following conditions hold:

(η1) $\eta(t,s)<s-t$ for all $s,t>0$.
(η2) For any bounded sequence $\{t_n\}\subset(0,+\infty)$ and any nonincreasing sequence $\{s_n\}\subset(0,+\infty)$, we have

$$\limsup_{n\to\infty}\frac{t_n+\eta(t_n,s_n)}{s_n}<1.$$

We denote the sets of all manageable functions by $\widehat{\mathrm{Man}(\mathbb{R})}$.

Remark 2.1 If $\eta\in\widehat{\mathrm{Man}(\mathbb{R})}$, then $\eta(t,t)<0$ for all $t>0$.

Example 2.1 Let $\gamma\in[0,1)$ and $a\geq 0$. Then the function $\eta_\gamma:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by $\eta_\gamma(t,s)=\gamma s-t-a$ is manageable.

Example 2.2 ([12]) Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be any function and $\varphi:[0,\infty)\to[0,1)$ be an $\mathcal{MT}$-function. Define $\eta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$\eta(t,s)=\begin{cases} s\varphi(s)-t, & \text{if } (t,s)\in[0,+\infty)\times[0,+\infty),\\ f(t,s), & \text{otherwise.}\end{cases}$$

Then η is a manageable function. Indeed, one can verify easily that (η1) holds. Next, we verify that η satisfies (η2). Let $\{t_n\} \subset (0,+\infty)$ be a bounded sequence and $\{s_n\} \subset (0,+\infty)$ be a nonincreasing sequence. Then $\lim_{n\to\infty} s_n = \inf_{n\in\mathbb{N}} s_n = a$ for some $a \in [0,+\infty)$. Since φ is an $\mathcal{MT}$-function, by Theorem 1.1, there exist $r_a \in [0,1)$ and $\varepsilon_a > 0$ such that $\varphi(s) \leq r_a$ for all $s \in [a, a+\varepsilon_a)$. Since $\lim_{n\to\infty} s_n = \inf_{n\in\mathbb{N}} s_n = a$, there exists $n_a \in \mathbb{N}$, such that

$$a \leq s_n < a + \varepsilon_a \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq n_a.$$

Hence we have

$$\limsup_{n\to\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} = \limsup_{n\to\infty} \varphi(s_n) \leq r_a < 1,$$

which means that (η2) holds. Thus we prove $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$.

In this paper, we first introduce the concepts of weakly transmitted functions and (λ)-strongly transmitted functions.

Definition 2.2 A function $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called
(i) *weakly transmitted* if $\xi(t,s) > s - t$ for all $s, t > 0$;
(ii) (λ)-*strongly transmitted* if there exists $\lambda > 2$, such that $\xi(t,s) \geq s - \frac{1}{\lambda}t$ for all $s, t \geq 0$.

We denote by $\widetilde{\mathrm{TRA}}_{(w)}$ and $\widetilde{\mathrm{TRA}}(\lambda)$, the sets of all weakly transmitted functions and (λ)-strongly transmitted functions, respectively. It is quite obvious that $\widetilde{\mathrm{TRA}}(\lambda) \subseteq \widetilde{\mathrm{TRA}}_{(w)}$ for all $\lambda > 2$.

Example 2.3 Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $g : [0,+\infty) \to [1,+\infty)$ be functions and $\lambda > 2$. Define $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\xi(t,s) = \begin{cases} sg(s) - \frac{1}{\lambda}t, & \text{if } (t,s) \in [0,+\infty) \times [0,+\infty), \\ f(t,s), & \text{otherwise.} \end{cases}$$

Then $\xi \in \widetilde{\mathrm{TRA}}(\lambda)$.

The following simple example shows that there exists a weakly transmitted function which is not (λ)-strongly transmitted for all $\lambda > 2$. In other words, $\widetilde{\mathrm{TRA}}(\lambda) \subsetneq \widetilde{\mathrm{TRA}}_{(w)}$ for all $\lambda > 2$.

Example 2.4 Let $\xi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\xi(t,s) = \begin{cases} 2s - t, & \text{if } (t,s) \in [0,+\infty) \times [0,+\infty), \\ 0, & \text{otherwise.} \end{cases}$$

Then ξ is a weakly transmitted function which is not (λ)-strongly transmitted for all $\lambda > 2$.

The following result is simple, but it is very crucial in our proofs.

Lemma 2.1 *Let (X,d) be a metric space, $\mathcal{W}$ be a nonempty subset of $X \times X$ and $T : X \to \mathcal{CB}(X)$ be a multivalued map. Suppose that there exists $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$ such that*

$$\eta\big(\mathcal{H}(Tx,Ty),d(x,y)\big) \geq 0 \quad \text{for all } (x,y) \in \mathcal{W}.$$

If $(p,q) \in \mathcal{W}$ with $p \neq q$, then $H(Tp,Tq) < d(p,q)$.

Proof Since $p \neq q$, $d(p,q) > 0$. If $\mathcal{H}(Tp,Tq) = d(p,q) > 0$, then, by Remark 2.2, we have $0 \leq \eta(\mathcal{H}(Tp,Tq),d(p,q)) < 0$, a contradiction. If $\mathcal{H}(Tp,Tq) > d(p,q) > 0$, then, by $(\eta 1)$, we have

$$0 \leq \eta\big(\mathcal{H}(Tp,Tq),d(p,q)\big) < d(p,q) - \mathcal{H}(Tp,Tq) < 0,$$

which also leads a contradiction. Therefore $\mathcal{H}(Tp,Tq) < d(p,q)$. □

Now, we establish an existence theorem for approximate fixed point property and fixed points by using manageable functions and transmitted functions which is one of the main results of this paper.

Theorem 2.1 *Let (X,d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multivalued map. Assume that there exist $\xi \in \widetilde{\mathrm{TRA}}_{(w)}$ and $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$ such that*

$$\eta\big(\mathcal{H}(Tx,Ty),d(x,y)\big) \geq 0 \quad \text{for all } (x,y) \in \mathcal{W}, \tag{2.1}$$

where

$$\mathcal{W} = \big\{(x,y) \in X \times X : \xi\big(d(x,Tx),d(x,y)\big) \geq 0\big\}.$$

Then T has the approximate fixed property on X.

Moreover, if (X,d) is complete and $\xi \in \widetilde{\mathrm{TRA}}(\lambda)$, then $\mathcal{F}(T) \neq \emptyset$.

Proof Let $x_0 \in X$. If $x_0 \in Tx_0$, then x_0 is a fixed point of T and we are done. Suppose that $x_0 \notin Tx_0$. Then $d(x_0,Tx_0) > 0$. Since $Tx_0 \neq \emptyset$, we can find $x_1 \in Tx_0$ with $x_1 \neq x_0$. Thus

$$d(x_0,x_1) > 0.$$

Since $\xi \in \widetilde{\mathrm{TRA}}_{(w)}$, we get

$$\xi\big(d(x_0,Tx_0),d(x_0,x_1)\big) > d(x_0,x_1) - d(x_0,Tx_0) \geq 0,$$

which means that $(x_0,x_1) \in \mathcal{W}$. Therefore, by (2.1), we obtain

$$\eta\big(\mathcal{H}(Tx_0,Tx_1),d(x_0,x_1)\big) \geq 0.$$

By Lemma 2.1, we have

$$\mathcal{H}(Tx_0,Tx_1) < d(x_0,x_1).$$

If $x_1 \in Tx_1$, then we have nothing to prove. So we assume that $x_1 \notin Tx_1$. Hence we have

$$0 < d(x_1, Tx_1) \leq \mathcal{H}(Tx_0, Tx_1). \tag{2.2}$$

Define $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$h(t,s) = \begin{cases} \frac{t+\eta(t,s)}{s}, & \text{if } t,s > 0, \\ 0, & \text{otherwise.} \end{cases}$$

By $(\eta 1)$, we know that

$$0 < h(t,s) < 1 \quad \text{for all } t,s > 0. \tag{2.3}$$

Since $\mathcal{H}(Tx_0, Tx_1) > 0$ and $d(x_0, x_1) > 0$, by the definition of h and (2.2), we have

$$d(x_1, Tx_1) \leq \mathcal{H}(Tx_0, Tx_1) \leq d(x_0, x_1) h\bigl(\mathcal{H}(Tx_0, Tx_1), d(x_0, x_1)\bigr). \tag{2.4}$$

Take

$$\epsilon_1 = \left(\frac{1}{\sqrt{h(\mathcal{H}(Tx_0, Tx_1), d(x_0, x_1))}} - 1 \right) d(x_1, Tx_1).$$

Then $\epsilon_1 > 0$. Since

$$\begin{aligned} d(x_1, Tx_1) &< d(x_1, Tx_1) + \epsilon_1 \\ &= \frac{1}{\sqrt{h(\mathcal{H}(Tx_0, Tx_1), d(x_0, x_1))}} d(x_1, Tx_1), \end{aligned}$$

there exists $x_2 \in Tx_1$ such that $x_2 \neq x_1$ and

$$\begin{aligned} d(x_1, x_2) &< \frac{1}{\sqrt{h(\mathcal{H}(Tx_0, Tx_1), d(x_0, x_1))}} d(x_1, Tx_1) \\ &\leq d(x_0, x_1) \sqrt{h\bigl(\mathcal{H}(Tx_0, Tx_1), d(x_0, x_1)\bigr)}. \end{aligned}$$

If $x_2 \in Tx_2$, then the proof is finished. Otherwise, we have

$$0 < d(x_2, Tx_2) \leq \mathcal{H}(Tx_1, Tx_2). \tag{2.5}$$

By (2.2) and $x_2 \neq x_1$, we get

$$\xi\bigl(d(x_1, Tx_1), d(x_1, x_2)\bigr) \geq d(x_1, x_2) - d(x_1, Tx_2) \geq 0,$$

which implies $(x_1, x_2) \in \mathcal{W}$. By (2.1), we obtain

$$\eta\bigl(\mathcal{H}(Tx_1, Tx_2), d(x_1, x_2)\bigr) \geq 0.$$

By Lemma 2.1 and (2.5), we have

$$0 < \mathcal{H}(Tx_1, Tx_2) < d(x_1, x_2). \tag{2.6}$$

Taking into account (2.5), (2.6) and the definition of h conclude that

$$d(x_2, Tx_2) \leq d(x_1, x_2) h\big(\mathcal{H}(Tx_1, Tx_2), d(x_1, x_2)\big).$$

By taking

$$\epsilon_2 = \left(\frac{1}{\sqrt{h(\mathcal{H}(Tx_1, Tx_2), d(x_1, x_2))}} - 1\right) d(x_2, Tx_2),$$

there exists $x_3 \in Tx_2$ with $x_3 \neq x_2$ such that

$$d(x_2, x_3) < d(x_1, x_2)\sqrt{h\big(\mathcal{H}(Tx_1, Tx_2), d(x_1, x_2)\big)}.$$

Hence, by induction, we can establish a sequences $\{x_n\}$ in X satisfying for each $n \in \mathbb{N}$,

$$\begin{aligned} & x_n \in Tx_{n-1}, \\ & d(x_{n-1}, x_n) > 0, \\ & 0 < d(x_n, Tx_n) \leq \mathcal{H}(Tx_{n-1}, Tx_n) < d(x_{n-1}, x_n), \\ & \eta\big(\mathcal{H}(Tx_{n-1}, Tx_n), d(x_{n-1}, x_n)\big) \geq 0, \end{aligned} \tag{2.7}$$

and

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)\sqrt{h\big(\mathcal{H}(Tx_{n-1}, Tx_n), d(x_{n-1}, x_n)\big)}. \tag{2.8}$$

We claim that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. For each $n \in \mathbb{N}$, let

$$\rho_n := \sqrt{h\big(\mathcal{H}(Tx_{n-1}, Tx_n), d(x_{n-1}, x_n)\big)}.$$

By (2.3), we know that

$$0 < h\big(\mathcal{H}(Tx_{n-1}, Tx_n), d(x_{n-1}, x_n)\big) < 1 \quad \text{for all } n \in \mathbb{N}, \tag{2.9}$$

so, from (2.8) and (2.9), we obtain $\rho_n \in (0, 1)$ and

$$d(x_n, x_{n+1}) < \rho_n d(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N}. \tag{2.10}$$

Hence the sequence $\{d(x_{n-1}, x_n)\}_{n\in\mathbb{N}}$ is strictly decreasing in $(0, +\infty)$. Thus

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) \geq 0 \quad \text{exists.} \tag{2.11}$$

By (2.7), we get

$$\mathcal{H}(Tx_{n-1}, Tx_n) \leq d(x_{n-1}, x_n) \quad \text{for all } n \in \mathbb{N},$$

which means that $\{\mathcal{H}(Tx_{n-1}, Tx_n)\}_{n\in\mathbb{N}}$ is a bounded sequence. By (η2) and the definition of h, we have

$$\limsup_{n\to\infty} h\big(\mathcal{H}(Tx_{n-1}, Tx_n), d(x_{n-1}, x_n)\big) < 1,$$

which implies $\limsup_{n\to\infty} \rho_n < 1$. So, there exists $c \in [0,1)$ and $n_0 \in \mathbb{N}$, such that

$$\rho_n \le c \quad \text{for all } n \in \mathbb{N} \text{ with } n \ge n_0. \tag{2.12}$$

For any $n \ge n_0$, since $\rho_n \in (0,1)$ for all $n \in \mathbb{N}$ and $c \in [0,1)$, taking into account (2.10) and (2.12), we conclude

$$\begin{aligned} d(x_n, x_{n+1}) &< \rho_n d(x_{n-1}, x_n) \\ &< \cdots \\ &< \rho_n \rho_{n-1} \rho_{n-2} \cdots \rho_{n_0} d(x_0, x_1) \\ &\le c^{n-n_0+1} d(x_0, x_1). \end{aligned}$$

Put $\alpha_n = \frac{c^{n-n_0+1}}{1-c} d(x_0, x_1)$, $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n \ge n_0$, from the last inequality, we have

$$d(x_n, x_m) \le \sum_{j=n}^{m-1} d(x_j, x_{j+1}) < \alpha_n.$$

Since $c \in [0,1)$, $\lim_{n\to\infty} \alpha_n = 0$ and hence

$$\lim_{n\to\infty} \sup\big\{d(x_n, x_m) : m > n\big\} = 0. \tag{2.13}$$

So $\{x_n\}$ is a Cauchy sequence in X. Combining (2.11) and (2.13), we get

$$\inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.14}$$

Since $x_n \in Tx_{n-1}$ for each $n \in \mathbb{N}$, we have

$$\inf_{x\in X} d(x, Tx) \le d(x_n, Tx_n) \le d(x_n, x_{n+1}) \quad \text{for all } n \in \mathbb{N}. \tag{2.15}$$

Combining (2.14) and (2.15) yields

$$\inf_{x\in X} d(x, Tx) = 0,$$

which means that T has the approximate fixed property on X.

Now, we assume that (X, d) is complete and $\xi \in \widetilde{\mathrm{TRA}}(\lambda)$. Since $\{x_n\}$ is a Cauchy sequence in X, by the completeness of X, there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$. We will proceed with the following claims to prove $v \in \mathcal{F}(T)$.

Claim 1. $d(v, Tx) \le d(v, x)$ for all $x \in X \setminus \{v\}$.

Given $x \in X$ with $x \neq v$. Let

$$S = \{n \in \mathbb{N} : x_n = x\}.$$

Suppose that $\sharp(S) = \infty$, where $\sharp(S)$ is the cardinal number of S. Then there exists $\{x_{n_j}\} \subset \{x_n\}$ such that $x_{n_j} = x$ for all $j \in \mathbb{N}$. So $x_{n_j} \to x$ as $j \to \infty$. By the uniqueness of the limit, we get $x = v$, a contradiction. Hence $\sharp(S) < \infty$ which deduces that there exists $\ell \in \mathbb{N}$ such that $x_n \neq x$ for all $n \in \mathbb{N}$ with $n \geq \ell$. For any $n \in \mathbb{N}$, put

$$w_n = x_{n+\ell-1}.$$

Thus we have

- $w_n \neq x$ for all $n \in \mathbb{N}$;
- $w_{n+1} \in Tw_n$ for all $n \in \mathbb{N}$;
- $w_n \to v$ as $n \to \infty$.

Since $d(x,v) > 0$, there exists $n_0 > 0$ such that

$$d(v, w_n) \leq \frac{1}{3} d(x,v) \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq n_0. \tag{2.16}$$

For $n \in \mathbb{N}$ with $n \geq n_0$, from (2.16), we have

$$\begin{aligned}
\xi\big(d(w_n, Tw_n), d(w_n, x)\big) &> d(w_n, x) - d(w_n, Tx_n) \\
&\geq d(x,v) - d(w_n, v) - d(w_n, Tw_n) \\
&\geq d(x,v) - d(w_n, v) - d(w_n, w_{n+1}) \\
&\geq d(x,v) - d(w_n, v) - d(w_n, v) - d(w_{n+1}, v) \\
&= d(x,v) - 2d(w_n, v) - d(w_{n+1}, v) \\
&> d(x,v) - \frac{2}{3} d(x,v) - \frac{1}{3} d(x,v) \\
&= 0,
\end{aligned}$$

which implies that $(w_n, x) \in \mathcal{W}$. Applying Lemma 2.1,

$$\mathcal{H}(Tw_n, Tx) < d(w_n, x) \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq n_0.$$

Since $w_n \to v$ as $n \to \infty$ and

$$d(w_{n+1}, Tx) \leq \mathcal{H}(Tw_n, Tx) < d(w_n, x) \quad \text{for all } n \in \mathbb{N} \text{ with } n \geq n_0, \tag{2.17}$$

by taking the limit from both sides of (2.17), we get

$$d(v, Tx) \leq d(v, x).$$

Claim 2. $v \in \mathcal{F}(T)$.

We first prove $(x,v) \in \mathcal{W}$ for all $x \in X \setminus \{v\}$. Suppose that there exists $u \in X$ with $u \neq v$ such that $(u,v) \notin \mathcal{W}$. So

$$\xi\big(d(u,Tu), d(u,v)\big) < 0.$$

Note that for any $n \in \mathbb{N}$, there exists $z_n \in Tu$ such that

$$d(v,z_n) < d(v,Tu) + \frac{1}{n} d(v,u).$$

Thus, for any $n \in \mathbb{N}$, by Claim 1, we have

$$\begin{aligned} d(u,Tu) &\leq d(u,z_n) \\ &\leq d(u,v) + d(v,z_n) \\ &\leq d(u,v) + d(v,Tu) + \frac{1}{n} d(v,u) \\ &\leq \left(2 + \frac{1}{n}\right) d(v,u). \end{aligned}$$

Hence we get $d(u,Tu) \leq 2d(v,u)$. Since $\xi \in \widetilde{\mathrm{TRA}(\lambda)}$, there exists $\lambda > 2$, such that $\xi(t,s) \geq s - \frac{1}{\lambda} t$ for all $s,t \geq 0$. So, we obtain

$$\begin{aligned} 0 &> \xi\big(d(u,Tu), d(u,v)\big) \\ &\geq d(u,v) - \frac{1}{\lambda} d(u,Tu) \\ &> d(u,v) - \frac{1}{2} d(u,Tu) \\ &\geq 0, \end{aligned}$$

a contradiction. Therefore $(x,v) \in \mathcal{W}$ for all $x \in X \setminus \{v\}$. By Lemma 2.1, we have

$$\mathcal{H}(Tx,Tv) < d(x,v) \quad \text{for all } x \in X \setminus \{v\}.$$

Therefore,

$$\mathcal{H}(Tx,Tv) \leq d(x,v) \quad \text{for all } x \in X. \tag{2.18}$$

From (2.18), we obtain

$$d(x_{n+1},Tv) \leq \mathcal{H}(Tx_n,Tv) \leq d(x_n,v) \quad \text{for all } n \in \mathbb{N}. \tag{2.19}$$

By taking limit from both side of (2.19), we get $d(v,Tv) = 0$. By the closedness of Tv, we have $v \in \mathcal{F}(T)$. The proof is completed. □

Theorem 2.2 *Let (X,d) be a complete metric space, $T : X \to \mathcal{CB}(X)$ be a multivalued map and $\lambda > 2$. Assume that there exist an $\mathcal{MT}$-function $\alpha : [0,\infty) \to [0,1)$ and a function*

$\beta : [0,+\infty) \to [1,+\infty)$ *such that, for* $x, y \in X$,

$$\frac{1}{\lambda} d(x,Tx) \le \beta\big(d(x,y)\big)d(x,y) \quad \textit{implies} \quad \mathcal{H}(Tx,Ty) \le \alpha\big(d(x,y)\big)d(x,y). \tag{2.20}$$

Then $\mathcal{F}(T) \neq \emptyset$.

Proof Define $\xi : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $\eta : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, respectively, by

$$\xi(t,s) = \begin{cases} s\beta(s) - \frac{1}{\lambda}t, & \text{if } (t,s) \in [0,+\infty)\times[0,+\infty), \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\eta(t,s) = \begin{cases} s\alpha(s) - t, & \text{if } (t,s) \in [0,+\infty)\times[0,+\infty), \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\xi \in \widetilde{\mathrm{TRA}(\lambda)}$. By Example 2.2, we know $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$. By (2.20), we obtain

$$\eta\big(\mathcal{H}(Tx,Ty), d(x,y)\big) \ge 0 \quad \text{for all } (x,y) \in \mathcal{W},$$

where

$$\mathcal{W} = \big\{(x,y) \in X \times X : \xi\big(d(x,Tx), d(x,y)\big) \ge 0\big\}.$$

Therefore the desired conclusion follows from Theorem 2.1 immediately. □

In Theorem 2.2, if we take $\beta(t) = 1$ for all $t \ge 0$, then we obtain the following new generalization of Mizoguchi-Takahashi's fixed point theorem.

Theorem 2.3 *Let* (X,d) *be a complete metric space,* $T : X \to \mathcal{CB}(X)$ *be a multivalued map and* $\lambda > 2$. *Assume that there exists an* $\mathcal{MT}$*-function* $\alpha : [0,\infty) \to [0,1)$ *such that for* $x, y \in X$,

$$d(x,Tx) \le \lambda d(x,y) \quad \textit{implies} \quad \mathcal{H}(Tx,Ty) \le \alpha\big(d(x,y)\big)d(x,y).$$

Then $\mathcal{F}(T) \neq \emptyset$.

Remark 2.2 Theorems 2.1, 2.2 and 2.3 generalize and improve Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, and Banach contraction principle.

Finally, a question arises naturally.

Question Can we give new generalizations of Mizoguchi-Takahashi's fixed point theorem with other new local constraints which also extend Kikkawa-Suzuki's fixed point theorem?

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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