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Relaxed and composite viscosity methods for variational inequalities, fixed points of nonexpansive mappings and zeros of accretive operators

Lu-Chuan Ceng[1], Abdullah Al-Otaibi[2], Qamrul Hasan Ansari[3,4] and Abdul Latif[2*]

*Correspondence: alatif@kau.edu.sa
[2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we present relaxed and composite viscosity methods for computing a common solution of a general systems of variational inequalities, common fixed points of infinitely many nonexpansive mappings and zeros of accretive operators in real smooth and uniformly convex Banach spaces. The relaxed and composite viscosity methods are based on Korpelevich's extragradient method, the viscosity approximation method and the Mann iteration method. Under suitable assumptions, we derive some strong convergence theorems for relaxed and composite viscosity algorithms not only in the setting of a uniformly convex and 2-uniformly smooth Banach space but also in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. The results presented in this paper improve, extend, supplement, and develop the corresponding results given in the literature.

1 Introduction

The theory of variational inequalities is well established and a tool to solve many problems arising from science, engineering, social sciences, *etc.*, see, for example, [1–4] and the references therein. One of the interesting directions, from the research view point, in the theory of variational inequalities is to develop some new iterative methods for computing the approximate solutions of different kinds of variational inequalities. In 1976, Korpelevich [5] proposed an iterative algorithm for solving variational inequalities (VI) in the finite dimensional space setting, It is now known as the extragradient method. Korpelevich's extragradient method has received great attention by many authors, who improved it in various ways and in different directions, see, for example [6–16] and the references therein. In the recent past, several iterative methods for solving VI were proposed and analyzed in [17–24] in the setting of Banach spaces. In the last three decades, the system of variational inequalities is used as a tool to study the Nash equilibrium problem for a finite or infinite number of players, see, for example, [2, 3, 25, 26] and the references therein. Cai and Bu [20] considered a system of two variational inequalities (SVI) in the setting of real smooth Banach spaces. They proposed and analyzed an iterative method for computing the approximate solutions of system of variational inequalities. Such a solution is also a common fixed point of a family of nonexpansive mappings.

One of the most interesting problems in nonlinear analysis is to find a zero of an accretive operator. In 2007, Aoyama *et al.* [27] suggested a Halpern type iterative method for finding a common fixed point of a countable family of nonexpansive mappings and a zero of an accretive operator. They studied the strong convergence of the sequence generated by the proposed method in the setting of a uniformly convex Banach space having a uniformly Gâreaux differentiable norm. Ceng *et al.* [28] introduced and analyzed the composite iterative scheme to compute a zero of m-accretive operator A defined on a uniformly smooth Banach space or a reflexive Banach space having a weakly sequentially continuous duality mapping. It is shown that the iterative process in each case converges strongly to a zero of A. Subsequently, Jung [29] studied a viscosity approximation method, which generalizes the composite method in [28], to investigate the zero of an accretive operator.

During the last decade, several iterative methods have been proposed and analyzed to find a common solution of two different fixed point problems, a fixed point problem and a variational inequality problem, a fixed point problem for a family of nonexpansive mappings and a variational inequality problem or a fixed point problem and a system of variational inequalities, *etc.* See, for example, [8, 16, 20, 30, 31] and the references therein.

In the present paper, we mainly propose two different methods, namely, relaxed viscosity method and composite viscosity method, to find a common fixed point of an infinite family of nonexpansive mappings, a system of variational inequalities and zero of an accretive operator in the setting of a uniformly convex and 2-uniformly smooth Banach spaces. These methods are based on Korpelevich's extragradient method, viscosity approximation method and Mann iteration method. Under suitable assumptions, we derive some strong convergence theorems for relaxed and composite viscosity algorithms not only in the setting of a uniformly convex and 2-uniformly smooth Banach space but also in the setting of uniformly convex Banach spaces having a uniformly Gâteaux differentiable norm. The results presented in this paper improve, extend, supplement, and develop the corresponding results in [10, 20, 24, 29, 30].

2 Preliminaries

Throughout the paper, unless otherwise specified, we adopt the following assumptions and notations.

Let X be a real Banach space whose dual space is denoted by X^*. Let C be a nonempty closed convex subset of X. We denote by Ξ_C the set of all contractive mappings from C into itself.

The *normalized duality mapping* $J : X \to 2^{X^*}$ is defined by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}, \quad \forall x \in X,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is an immediate consequence of the Hahn-Banach Theorem that $J(x)$ is nonempty for each $x \in X$.

Let $U = \{x \in X : \|x\| = 1\}$ denote the unite sphere in X. A Banach space X is said to be *uniformly convex* if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in U$,

$$\|x - y\| \geq \epsilon \quad \Rightarrow \quad \frac{\|x + y\|}{2} \leq 1 - \delta.$$

It is well known that a uniformly convex Banach space is reflexive and strictly convex. A Banach space X is said to be *smooth* if the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t},$$

exists for all $x,y\in U$; in this case, X is also said to have a *Gâteaux differentiable norm*. X is said to have a *uniformly Gâteaux differentiable norm* if for each $y\in U$, the limit is attained uniformly for all $x\in U$. Moreover, it is said to be *uniformly smooth* if this limit is attained uniformly for all $x,y\in U$. The norm of X is said to be *Fréchet differentiable* if, for each $x\in U$, this limit is attained uniformly for all $y\in U$. A function $\rho:[0,\infty)\to[0,\infty)$ defined by

$$\rho(\tau)=\sup\left\{\frac{1}{2}\bigl(\|x+y\|+\|x-y\|\bigr)-1:x,y\in X,\|x\|=1,\|y\|=\tau\right\}$$

is called the *modulus of smoothness* of X. It is well known that X is uniformly smooth if and only if $\lim_{\tau\to 0}\rho(\tau)/\tau=0$. Let q be a fixed real number with $1<q\leq 2$. Then a Banach space X is said to be q-uniformly smooth if there exists a constant $c>0$ such that $\rho(\tau)\leq c\tau^q$ for all $\tau>0$. As pointed out in [32], no Banach space is q-uniformly smooth for $q>2$. In addition, it is also known that J is single-valued if and only if X is smooth, whereas if X is uniformly smooth, then the mapping J is norm-to-norm uniformly continuous on bounded subsets of X. If X has a uniformly Gâteaux differentiable norm then the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X. For further details of the geometry of Banach spaces, we refer to [33–35].

Now, we present some lemmas which will be used in the sequel.

Lemma 2.1 [36] *Let X be a 2-uniformly smooth Banach space. Then*

$$\|x+y\|^2\leq\|x\|^2+2\bigl\langle y,J(x)\bigr\rangle+2\|\kappa y\|^2,\quad\forall x,y\in X,$$

where κ is the 2-uniformly smooth constant of X.

The following lemma is an immediate consequence of the subdifferential inequality of the function $\frac{1}{2}\|\cdot\|^2$.

Lemma 2.2 [37] *Let X be a real Banach space X. Then, for all $x,y\in X$,*
(a) $\|x+y\|^2\leq\|x\|^2+2\langle y,j(x+y)\rangle$, $\forall j(x+y)\in J(x+y)$;
(b) $\|x+y\|^2\geq\|x\|^2+2\langle y,j(x)\rangle$, $\forall j(x)\in J(x)$.

Lemma 2.3 [36] *Given a number $r>0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $g:[0,\infty)\to[0,\infty)$, $g(0)=0$, such that*

$$\bigl\|\lambda x+(1-\lambda)y\bigr\|^2\leq\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)g\bigl(\|x-y\|\bigr)$$

for all $\lambda\in[0,1]$ and $x,y\in X$ such that $\|x\|\leq r$ and $\|y\|\leq r$.

Lemma 2.4 [38] *Let X be a uniformly convex Banach space and $B_r = \{x \in X : \|x\| \leq r\}$, $r > 0$. Then there exists a continuous, strictly increasing, and convex function $g : [0,\infty] \to [0,\infty]$, $g(0) = 0$ such that*

$$\|\alpha x + \beta y + \gamma z\|^2 \leq \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta g\bigl(\|x-y\|\bigr)$$

for all $x,y,z \in B_r$ and all $\alpha,\beta,\gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Proposition 2.1 [22] *Let X be a real smooth and uniform convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0,2r] \to \mathbf{R}$, $g(0) = 0$ such that*

$$g\bigl(\|x-y\|\bigr) \leq \|x\|^2 - 2\bigl\langle x, J(y)\bigr\rangle + \|y\|^2, \quad \forall x,y \in B_r,$$

where $B_r = \{x \in X : \|x\| \leq r\}$.

Lemma 2.5 [39] *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive mappings from C into itself such that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Let $\{\lambda_n\}$ be a sequence of positive numbers with $\sum_{n=0}^{\infty} \lambda_n = 1$. Then a mapping $S : C \to C$ defined by $Sx = \sum_{n=0}^{\infty} \lambda_n T_n x$, for all $x \in C$, is well defined and nonexpansive, and $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$.*

Lemma 2.6 [40] *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and $\{\beta_n\}$ be a sequence of nonnegative numbers in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then $\lim_{n\to\infty} \|x_n - z_n\| = 0$.*

Lemma 2.7 [41] *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \geq 0,$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ satisfy the conditions:

(i) $\{\alpha_n\} \subset [0,1]$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \beta_n \leq 0$;
(iii) $\gamma_n \geq 0$, $\forall n \geq 0$, *and* $\sum_{n=0}^{\infty} \gamma_n < \infty$.

Then $\limsup_{n\to\infty} s_n = 0$.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for every $x,y \in C$. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. A mapping $A : C \to X$ is said to be

(a) *accretive* if for each $x,y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\bigl\langle Ax - Ay, j(x-y)\bigr\rangle \geq 0;$$

(b) *α-strongly accretive* if for each $x,y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\bigl\langle Ax - Ay, j(x-y)\bigr\rangle \geq \alpha\|x-y\|^2, \quad \text{for some } \alpha \in (0,1);$$

(c) β*-inverse strongly accretive* if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \geq \beta \|Ax - Ay\|^2, \quad \text{for some } \beta > 0;$$

(d) λ*-strictly pseudocontractive* [18, 42] if for each $x, y \in C$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Ax - Ay, j(x-y)\rangle \leq \|x-y\|^2 - \lambda \big\|x - y - (Ax - Ay)\big\|^2, \quad \text{for some } \lambda \in (0,1).$$

It is worth to emphasize that the definition of the inverse strongly accretive mapping is based on that of the inverse strongly monotone mapping [43].

Lemma 2.8 [20, Lemma 2.8] *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X and for each $i = 1, 2$, $B_i : C \to X$ be an α_i-inverse strongly accretive mapping. Then, for each $i = 1, 2$,*

$$\big\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\big\|^2 \leq \|x-y\|^2 + 2\mu_i\big(\mu_i \kappa^2 - \alpha_i\big)\|B_i x - B_i y\|^2, \quad \forall x, y \in C,$$

where $\mu_i > 0$. In particular, if $0 < \mu_i \leq \frac{\alpha_i}{\kappa^2}$, then $I - \mu_i B_i$ is nonexpansive for each $i = 1, 2$.

Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. For all $t \in (0,1)$ and $f \in \Xi_C$, let $x_t \in C$ be a unique fixed point of the contraction $x \mapsto tf(x) + (1-t)Tx$ on C, that is,

$$x_t = tf(x_t) + (1-t)Tx_t.$$

Lemma 2.9 [44, 45] *Let X be an uniformly smooth Banach space, or a reflexive and strictly convex Banach space with a uniformly Gâteaux differentiable norm. Let C be a nonempty closed convex subset of X, $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$, and $f \in \Xi_C$. Then the net $\{x_t\}$ defined by $x_t = tf(x_t) + (1-t)Tx_t$ converges strongly to a point in $\mathrm{Fix}(T)$. If we define a mapping $Q : \Xi_C \to \mathrm{Fix}(T)$ by $Q(f) := s - \lim_{t \to 0} x_t$, $\forall f \in \Xi_C$, then $Q(f)$ solves the VIP*

$$\big\langle (I - f)Q(f), J\big(Q(f) - p\big)\big\rangle \leq 0, \quad \forall f \in \Xi_C, p \in \mathrm{Fix}(T).$$

Recall that a (possibly set-valued mapping) operator $A \subset X \times X$ with domain $D(A)$ and range $R(A)$ in X is *accretive* if, for each $x_i \in D(A)$ and $y_i \in Ax_i$ $(i = 1, 2)$, there exists a $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$. An accretive operator A is said to satisfy the *range condition* if $\overline{D(A)} \subset R(I + rA)$ for all $r > 0$. An accretive operator A is *m-accretive* if $R(I + rA) = X$ for each $r > 0$. If A is an accretive operator which satisfies the range condition, then we define a mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the *resolvent* of A. It is well known that J_r is nonexpansive and $\mathrm{Fix}(J_r) = A^{-1}0$ for all $r > 0$. Therefore,

$$\mathrm{Fix}(J_r) = A^{-1}0 = \big\{z \in D(A) : 0 \in Az\big\}.$$

If $A^{-1}0 \neq \emptyset$, then the inclusion $0 \in Az$ is solvable.

Proposition 2.2 (Resolvent Identity [46]) *For* $\lambda > 0$, $\mu > 0$ *and* $x \in X$,

$$J_\lambda x = J_\mu\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda x\right).$$

Let D be a subset of C. A mapping $\Pi : C \to D$ is said to be *sunny* if

$$\Pi\bigl[\Pi(x) + t\bigl(x - \Pi(x)\bigr)\bigr] = \Pi(x),$$

whenever $\Pi(x) + t(x - \Pi(x)) \in C$ for all $x \in C$ and $t \geq 0$. A mapping $\Pi : C \to C$ is called a *retraction* if $\Pi^2 = \Pi$. If a mapping $\Pi : C \to C$ is a retraction, then $\Pi(z) = z$ for every $z \in R(\Pi)$ where $R(\Pi)$ is the range of Π. A subset D of C is called a *sunny nonexpansive retract* of C if there exists a sunny nonexpansive retraction from C onto D.

Lemma 2.10 [23] *Let C be a nonempty closed convex subset of a real smooth Banach space X, D be a nonempty subset of C and Π be a retraction of C onto D. Then the following statements are equivalent*:

(a) *Π is sunny and nonexpansive*;
(b) $\|\Pi(x) - \Pi(y)\|^2 \leq \langle x - y, J(\Pi(x) - \Pi(y))\rangle$, $\forall x, y \in C$;
(c) $\langle x - \Pi(x), J(y - \Pi(x))\rangle \leq 0$, $\forall x \in C$, $y \in D$.

It is well known that if $X = H$ a Hilbert space, then a sunny nonexpansive retraction Π_C is coincident with the metric projection from X onto C, that is, $\Pi_C = P_C$. If C is a nonempty closed convex subset of a strictly convex and uniformly smooth Banach space X and if $T : C \to C$ is a nonexpansive mapping with the fixed point set $\mathrm{Fix}(T) \neq \emptyset$, then the set $\mathrm{Fix}(T)$ is a sunny nonexpansive retract of C.

Lemma 2.11 [20, Lemma 2.9] *Let C be a nonempty closed convex subset of a real 2-uniformly smooth Banach space X and Π_C be a sunny nonexpansive retraction from X onto C. For each $i = 1, 2$, let $B_i : C \to X$ be an α_i-inverse strongly accretive mapping and $G : C \to C$ be defined by*

$$Gx = \Pi_C\bigl[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)\bigr], \quad \forall x \in C.$$

If $0 < \mu_i \leq \frac{\alpha_i}{\kappa^2}$ for each $i = 1, 2$, then $G : C \to C$ is nonexpansive.

Let $f \in \Xi_C$ with a contractive coefficient $\rho \in (0, 1)$, $\{T_n\}_{n=0}^\infty$ be a sequence of nonexpansive self-mappings on C and $\{\lambda_n\}_{n=0}^\infty$ be a sequence of nonnegative numbers in $[0, 1]$. For any $n \geq 0$, a self-mapping W_n on C defined by

$$\begin{cases} U_{n,n+1} = I, \\ U_{n,n} = \lambda_n T_n U_{n,n+1} + (1 - \lambda_n)I, \\ U_{n,n-1} = \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1})I, \\ \dots \\ U_{n,k} = \lambda_k T_k U_{n,k+1} + (1 - \lambda_k)I, \\ U_{n,k-1} = \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1})I, \\ \dots \\ U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1)I, \\ W_n = U_{n,0} = \lambda_0 T_0 U_{n,1} + (1 - \lambda_0)I \end{cases} \tag{2.1}$$

is called *W-mapping* [47] generated by $T_n, T_{n-1}, \dots, T_0$ and $\lambda_n, \lambda_{n-1}, \dots, \lambda_0$.

Lemma 2.12 [37, Lemma 3.2] *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$ and $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 0$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.*

B using Lemma 2.12, we define a W-mapping $W : C \to C$ generated by the sequences $\{T_n\}_{n=0}^{\infty}$ and $\{\lambda_n\}_{n=0}^{\infty}$ by

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,0}x, \quad \text{for every } x \in C.$$

Throughout this paper, we assume that $\{\lambda_n\}_{n=0}^{\infty}$ is a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$.

Lemma 2.13 [37, Lemma 3.3] *Let C be a nonempty closed convex subset of a strictly convex Banach space X. Let $\{T_n\}_{n=0}^{\infty}$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n) \neq \emptyset$ and let $\{\lambda_n\}_{n=0}^{\infty}$ be a sequence of positive numbers in $(0, b]$ for some $b \in (0, 1)$. Then $\mathrm{Fix}(W) = \bigcap_{n=0}^{\infty} \mathrm{Fix}(T_n)$.*

Let μ be a continuous linear functional on l^{∞} and $s = (a_0, a_1, \dots) \in l^{\infty}$. We write $\mu_n(a_n)$ instead of $\mu(s)$. μ is called a *Banach limit* if μ satisfies $\|\mu\| = \mu_n(1) = 1$ and $\mu_n(a_{n+1}) = \mu_n(a_n)$ for all $(a_0, a_1, \dots) \in l^{\infty}$. If μ is a Banach limit, then the following implications hold:

(a) for all $n \geq 0$, $a_n \leq c_n$ implies $\mu_n(a_n) \leq \mu_n(c_n)$;
(b) $\mu_n(a_{n+r}) = \mu_n(a_n)$ for any fixed positive integer r;
(c) $\liminf_{n\to\infty} a_n \leq \mu_n(a_n) \leq \limsup_{n\to\infty} a_n$ for all $(a_0, a_1, \dots) \in l^{\infty}$.

Lemma 2.14 [48] *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_n\} \in l^{\infty}$ satisfy the condition $\mu_n(a_n) \leq a$ for all Banach limits μ. If $\limsup_{n\to\infty}(a_{n+r} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

In particular, if $r = 1$ in Lemma 2.14, then we obtain the following corollary.

Corollary 2.1 [49] *Let $a \in \mathbf{R}$ be a real number and a sequence $\{a_n\} \in l^\infty$ satisfy the condition $\mu_n(a_n) \leq a$ for all Banach limits μ. If $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$, then $\limsup_{n\to\infty} a_n \leq a$.*

3 Formulations

Let C be a nonempty closed convex subset of a smooth Banach space X, $B_1, B_2 : C \to X$ be nonlinear mappings and μ_1 and μ_2 be two positive constants. The problem of system of variational inequalities (SVI) in the setting of a real smooth Banach space X is to find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, J(x - x^*)\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, J(x - y^*)\rangle \geq 0, & \forall x \in C. \end{cases} \tag{3.1}$$

The set of solutions of SVI (3.1) is denoted by $\mathrm{SVI}(C, B_1, B_2)$. Very recently, Cai and Bu [20] constructed an iterative algorithm for solving SVI (3.1) and a common fixed point problem of an infinite family of nonexpansive mappings in a uniformly convex and 2-uniformly smooth Banach space. They studied the strong convergence of the proposed algorithm.

In particular, if $X = H$, a real Hilbert space, then SVI (3.1) reduces to the following problem of SVI of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C. \end{cases} \tag{3.2}$$

Further, if $B_1 = B_2 = A$, where $A : C \to X$ is an operator, and $x^* = y^*$, then the SVI (3.2) reduces to the classical variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{3.3}$$

The solution set of the VIP (3.3) is denoted by $\mathrm{VI}(C, A)$. For details and applications of theory of variational inequalities, we refer to [1–4] and the references therein.

Recently, Ceng *et al.* [10] transformed problem (3.2) into a fixed point problem in the following way.

Lemma 3.1 [10] *For given $\bar{x}, \bar{y} \in C$, $(\bar{x}, \bar{y})$ is a solution of problem* (3.2) *if and only if $\bar{x}$ is a fixed point of the mapping $G : C \to C$ defined by*

$$G(x) = P_C\big[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)\big], \quad \forall x \in C, \tag{3.4}$$

where $\bar{y} = P_C(\bar{x} - \mu_2 B_2 \bar{x})$ and P_C is the projection of H onto C.

In particular, if for each $i = 1, 2$, $B_i : C \to H$ is a β_i-inverse strongly monotone mapping, then G is a nonexpansive mapping provided $\mu_i \in (0, 2\beta_i)$ for each $i = 1, 2$.

In particular, whenever X is a real smooth Banach space, $B_1 \equiv B_2 \equiv A$ and $x^* = y^*$, then SVI (3.1) reduces to the variational inequality problem (VIP) of finding $x^* \in C$ such that

$$\langle Ax^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C, \tag{3.5}$$

which was considered by Aoyama *et al.* [17]. Note that VIP (3.5) is connected with the fixed point problem for nonlinear mapping [44], the problem of finding a zero point of a nonlinear operator [50] and so on. It is clear that VIP (3.5) extends VIP (3.3) from Hilbert spaces to Banach spaces. For further study on VIP in the setting of Banach spaces, we refer to [17, 21] and the references therein.

Define a mapping $G : C \to C$ by

$$G(x) := \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x, \quad \forall x \in C. \tag{3.6}$$

The fixed point set of G is denoted by Ω.

Lemma 3.2 *Let C be a nonempty closed convex subset of a smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and $B_1, B_2 : C \to X$ be nonlinear mappings. Then $(x^*, y^*) \in C \times C$ is a solution of SVI* (3.1) *if and only if $x^* = \Pi_C(y^* - \mu_1 B_1 y^*)$, where $y^* = \Pi_C(x^* - \mu_2 B_2 x^*)$.*

Proof We rewrite SVI (3.1) as

$$\begin{cases} \langle x^* - (y^* - \mu_1 B_1 y^*), J(x - x^*)\rangle \ge 0, & \forall x \in C, \\ \langle y^* - (x^* - \mu_2 B_2 x^*), J(x - y^*)\rangle \ge 0, & \forall x \in C, \end{cases}$$

which is obviously equivalent to

$$\begin{cases} x^* = \Pi_C(y^* - \mu_1 B_1 y^*), \\ y^* = \Pi_C(x^* - \mu_2 B_2 x^*), \end{cases}$$

because of Lemma 2.10. This completes the proof. □

In terms of Lemma 3.2, we observe that

$$x^* = \Pi_C\bigl[\Pi_C\bigl(x^* - \mu_2 B_2 x^*\bigr) - \mu_1 B_1 \Pi_C\bigl(x^* - \mu_2 B_2 x^*\bigr)\bigr],$$

which implies that x^* is a fixed point of the mapping G.

Motivated and inspired by the research going on in this area, we introduce some relaxed and composite viscosity methods for finding a zero of an accretive operator $A \subset X \times X$ such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$, solving SVI (3.1) and the common fixed point problem of an infinite family $\{T_n\}$ of nonexpansive self-mappings on C. Our methods are based on Korpelevich's extragradient method, the viscosity approximation method, and Mann's iteration method. Under suitable assumptions, we derive some strong convergence theorems for relaxed and composite viscosity algorithms not only in the setting of uniformly convex and 2-uniformly smooth Banach space but also in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. The results presented in this paper improve, extend, supplement, and develop the corresponding results given in [10, 20, 24, 29, 48].

4 Relaxed viscosity algorithms and convergence criteria

In this section, we introduce relaxed viscosity algorithms in the setting of real smooth uniformly convex Banach spaces and study the strong convergence of the sequences generated by the proposed algorithms.

Throughout this paper, we denote by Ω the fixed point set of the mapping $G = \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)$.

Assumption 4.1 Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$ be the sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\{\gamma_n\}, \{\delta_n\} \subset [c,d]$ for some $c, d \in (0,1)$;
(iii) $\lim_{n\to\infty}(|\sigma_n - \sigma_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) = 0$;
(iv) $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ and $r_n \geq \varepsilon > 0$ for all $n \geq 0$;
(v) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$.

Theorem 4.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and $A \subset X \times X$ be an accretive operator such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. For each $i = 1, 2$, let $B_i : C \to X$ be α_i-inverse strongly accretive mapping and $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{T_i\}_{i=0}^{\infty}$ be an infinite family of nonexpansive mappings from C into itself such that $F := \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \Omega \cap A^{-1}0 \neq \emptyset$ with $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$. Assume that Assumption* 4.1 *holds. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = \sigma_n x_n + (1 - \sigma_n) J_{r_n} G x_n, \\ x_{n+1} = \alpha_n f(y_n) + \beta_n x_n + \gamma_n W_n y_n + \delta_n J_{r_n} G y_n, \quad \forall n \geq 0, \end{cases} \tag{4.1}$$

where W_n is the W-mapping generated by (2.1). *Then*

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;
(b) *the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $q \in F$ which is a unique solution of the following variational inequality problem* (*VIP*):

$$\langle (I - f)q, J(q - p) \rangle \leq 0, \quad \forall p \in F,$$

provided $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$.

Proof We first claim that the sequence $\{x_n\}$ is bounded. Indeed, take a fixed $p \in F$ arbitrarily. Then we get $p = Gp$, $p = W_n p$, and $p = J_{r_n} p$ for all $n \geq 0$. By Lemma 2.11, G is nonexpansive. Then, from (4.1), we have

$$\begin{aligned} \|y_n - p\| &\leq \sigma_n \|x_n - p\| + (1 - \sigma_n)\|J_{r_n} G x_n - p\| \\ &\leq \sigma_n \|x_n - p\| + (1 - \sigma_n)\|G x_n - p\| \\ &\leq \sigma_n \|x_n - p\| + (1 - \sigma_n)\|x_n - p\| \\ &= \|x_n - p\| \end{aligned} \tag{4.2}$$

and

$$\begin{aligned}
\|x_{n+1}-p\| &\le \alpha_n\|f(y_n)-p\| + \beta_n\|x_n-p\| + \gamma_n\|W_ny_n-p\| + \delta_n\|J_{r_n}Gy_n-p\| \\
&\le \alpha_n\bigl(\|f(y_n)-f(p)\| + \|f(p)-p\|\bigr) + \beta_n\|x_n-p\| + \gamma_n\|y_n-p\| + \delta_n\|Gy_n-p\| \\
&\le \alpha_n\bigl(\rho\|y_n-p\| + \|f(p)-p\|\bigr) + \beta_n\|x_n-p\| + \gamma_n\|y_n-p\| + \delta_n\|y_n-p\| \\
&\le \alpha_n\bigl(\rho\|x_n-p\| + \|f(p)-p\|\bigr) + \beta_n\|x_n-p\| + \gamma_n\|x_n-p\| + \delta_n\|x_n-p\| \\
&= \bigl(1-\alpha_n(1-\rho)\bigr)\|x_n-p\| + \alpha_n(1-\rho)\frac{\|f(p)-p\|}{1-\rho} \\
&\le \max\left\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-\rho}\right\}.
\end{aligned}$$

By induction, we obtain

$$\|x_n-p\| \le \max\left\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-\rho}\right\}, \quad \forall n \ge 0. \tag{4.3}$$

Hence, $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{Gx_n\}$, $\{Gy_n\}$, and $\{f(y_n)\}$.

Next we show that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{4.4}$$

We note that x_{n+1} can be rewritten as follows:

$$x_{n+1} = \beta_nx_n + (1-\beta_n)z_n,$$

where $z_n = \frac{\alpha_nf(y_n)+\gamma_nW_ny_n+\delta_nJ_{r_n}Gy_n}{1-\beta_n}$. Observe that

$$\begin{aligned}
&\|z_n-z_{n-1}\| \\
&\quad= \left\|\frac{\alpha_nf(y_n)+\gamma_nW_ny_n+\delta_nJ_{r_n}Gy_n}{1-\beta_n} - \frac{\alpha_{n-1}f(y_{n-1})+\gamma_{n-1}W_{n-1}y_{n-1}+\delta_{n-1}J_{r_{n-1}}Gy_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad= \left\|\frac{x_{n+1}-\beta_nx_n}{1-\beta_n} - \frac{x_n-\beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad= \left\|\frac{x_{n+1}-\beta_nx_n}{1-\beta_n} - \frac{x_n-\beta_{n-1}x_{n-1}}{1-\beta_n} + \frac{x_n-\beta_{n-1}x_{n-1}}{1-\beta_n} - \frac{x_n-\beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad\le \left\|\frac{x_{n+1}-\beta_nx_n}{1-\beta_n} - \frac{x_n-\beta_{n-1}x_{n-1}}{1-\beta_n}\right\| + \left\|\frac{x_n-\beta_{n-1}x_{n-1}}{1-\beta_n} - \frac{x_n-\beta_{n-1}x_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad= \frac{1}{1-\beta_n}\bigl\|x_{n+1}-\beta_nx_n-(x_n-\beta_{n-1}x_{n-1})\bigr\| + \left|\frac{1}{1-\beta_n} - \frac{1}{1-\beta_{n-1}}\right|\|x_n-\beta_{n-1}x_{n-1}\| \\
&\quad= \frac{1}{1-\beta_n}\bigl\|x_{n+1}-\beta_nx_n-(x_n-\beta_{n-1}x_{n-1})\bigr\| + \frac{|\beta_n-\beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|x_n-\beta_{n-1}x_{n-1}\| \\
&\quad= \frac{1}{1-\beta_n} \\
&\qquad\times\bigl\|\alpha_nf(y_n)+\gamma_nW_ny_n+\delta_nJ_{r_n}Gy_n-\alpha_{n-1}f(y_{n-1})-\gamma_{n-1}W_{n-1}y_{n-1}-\delta_{n-1}J_{r_{n-1}}Gy_{n-1}\bigr\| \\
&\qquad+ \frac{|\beta_n-\beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|x_n-\beta_{n-1}x_{n-1}\|
\end{aligned}$$

$$\begin{aligned}
&\le \frac{1}{1-\beta_n}\big[\alpha_n \big\|f(y_n)-f(y_{n-1})\big\| + \gamma_n\|W_n y_n - W_{n-1}y_{n-1}\| + \delta_n\|J_{r_n}Gy_n - J_{r_{n-1}}Gy_{n-1}\| \\
&\quad + |\alpha_n-\alpha_{n-1}|\big\|f(y_{n-1})\big\| + |\gamma_n-\gamma_{n-1}|\|W_{n-1}y_{n-1}\| + |\delta_n-\delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\|\big] \\
&\quad + \frac{|\beta_n-\beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|x_n-\beta_{n-1}x_{n-1}\|. \qquad (4.5)
\end{aligned}$$

On the other hand, if $r_{n-1} \le r_n$, using the resolvent identity in Proposition 2.2,

$$J_{r_n}x_n = J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_n + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}x_n\right),$$

we get

$$\begin{aligned}
\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| &= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}Gx_n + \left(1-\frac{r_{n-1}}{r_n}\right)J_{r_n}Gx_n\right) - J_{r_{n-1}}Gx_{n-1}\right\| \\
&\le \frac{r_{n-1}}{r_n}\|Gx_n - Gx_{n-1}\| + \left(1-\frac{r_{n-1}}{r_n}\right)\|J_{r_n}Gx_n - Gx_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + \frac{r_n - r_{n-1}}{r_n}\|J_{r_n}Gx_n - Gx_{n-1}\| \\
&\le \|x_n - x_{n-1}\| + \frac{1}{\varepsilon}|r_n - r_{n-1}|\|J_{r_n}Gx_n - Gx_{n-1}\|.
\end{aligned}$$

If $r_n \le r_{n-1}$, then it is easy to see that

$$\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \le \|x_{n-1} - x_n\| + \frac{1}{\varepsilon}|r_{n-1} - r_n|\|J_{r_{n-1}}Gx_{n-1} - Gx_n\|.$$

By combining the above cases, we obtain

$$\begin{aligned}
&\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \\
&\quad \le \|x_{n-1} - x_n\| + \frac{|r_{n-1}-r_n|}{\varepsilon}\sup_{n\ge 1}\big\{\|J_{r_n}Gx_n - Gx_{n-1}\| + \|J_{r_{n-1}}Gx_{n-1} - Gx_n\|\big\}, \quad \forall n \ge 1.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\|J_{r_n}Gy_n - J_{r_{n-1}}Gy_{n-1}\| \\
&\quad \le \|y_{n-1} - y_n\| + \frac{|r_{n-1}-r_n|}{\varepsilon}\sup_{n\ge 1}\big\{\|J_{r_n}Gy_n - Gy_{n-1}\| + \|J_{r_{n-1}}Gy_{n-1} - Gy_n\|\big\}, \quad \forall n \ge 1.
\end{aligned}$$

Therefore, we obtain

$$\begin{cases} \|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \le \|x_{n-1} - x_n\| + |r_{n-1} - r_n|M_0, \\ \|J_{r_n}Gy_n - J_{r_{n-1}}Gy_{n-1}\| \le \|y_{n-1} - y_n\| + |r_{n-1} - r_n|M_0, \quad \forall n \ge 1, \end{cases} \qquad (4.6)$$

where

$$\sup_{n\ge 1}\left\{\frac{1}{\varepsilon}\big(\|J_{r_n}Gx_n - Gx_{n-1}\| + \|J_{r_{n-1}}Gx_{n-1} - Gx_n\|\big)\right\} \le M_0,$$

and

$$\sup_{n\ge 1}\left\{\frac{1}{\varepsilon}\big(\|J_{r_n}Gy_n - Gy_{n-1}\| + \|J_{r_{n-1}}Gy_{n-1} - Gy_n\|\big)\right\} \le M_0,$$

for some $M_0 > 0$. Since T_i and $U_{n,i}$ are nonexpansive, from (2.1), we deduce that for each $n \geq 1$

$$\begin{aligned}
\|W_n y_{n-1} - W_{n-1} y_{n-1}\| &= \|\lambda_0 T_0 U_{n,1} y_{n-1} - \lambda_0 T_0 U_{n-1,1} y_{n-1}\| \\
&\leq \lambda_0 \|U_{n,1} y_{n-1} - U_{n-1,1} y_{n-1}\| \\
&= \lambda_0 \|\lambda_1 T_1 U_{n,2} y_{n-1} - \lambda_1 T_1 U_{n-1,2} y_{n-1}\| \\
&\leq \lambda_0 \lambda_1 \|U_{n,2} y_{n-1} - U_{n-1,2} y_{n-1}\| \\
&\cdots \\
&\leq \left(\prod_{i=0}^{n-1} \lambda_i\right) \|U_{n,n} y_{n-1} - U_{n-1,n} y_{n-1}\| \\
&\leq M \prod_{i=0}^{n-1} \lambda_i, \quad \text{for some constant } M > 0.
\end{aligned} \tag{4.7}$$

By simple computations, we obtain

$$\begin{aligned}
y_n - y_{n-1} = {} & \sigma_n (x_n - x_{n-1}) + (\sigma_n - \sigma_{n-1})(x_{n-1} - J_{r_{n-1}} G x_{n-1}) \\
& + (1 - \sigma_n)(J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}).
\end{aligned}$$

It follows from (4.6) that

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq \sigma_n \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|x_{n-1} - J_{r_{n-1}} G x_{n-1}\| \\
&\quad + (1 - \sigma_n) \|J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}\| \\
&\leq \sigma_n \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|x_{n-1} - J_{r_{n-1}} G x_{n-1}\| \\
&\quad + (1 - \sigma_n) \big[\|x_{n-1} - x_n\| + |r_{n-1} - r_n| M_0\big] \\
&\leq \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}| \|x_{n-1} - J_{r_{n-1}} G x_{n-1}\| + |r_{n-1} - r_n| M_0.
\end{aligned} \tag{4.8}$$

Taking into account that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, without loss of generality, we may assume that $\{\beta_n\} \subset [\hat{c}, \hat{d}]$. Utilizing (4.5)-(4.8), we have

$$\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\quad \leq \frac{1}{1 - \beta_n} \big[\alpha_n \big\|f(y_n) - f(y_{n-1})\big\| + \gamma_n \|W_n y_n - W_{n-1} y_{n-1}\| + \delta_n \|J_{r_n} G y_n - J_{r_{n-1}} G y_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| \big\|f(y_{n-1})\big\| + |\gamma_n - \gamma_{n-1}| \|W_{n-1} y_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} G y_{n-1}\|\big] \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1 - \beta_{n-1})(1 - \beta_n)} \|x_n - \beta_{n-1} x_{n-1}\| \\
&\quad \leq \frac{1}{1 - \beta_n} \big[\alpha_n \big\|f(y_n) - f(y_{n-1})\big\| + \gamma_n \|W_n y_n - W_n y_{n-1}\| + \delta_n \|J_{r_n} G y_n - J_{r_{n-1}} G y_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| \big\|f(y_{n-1})\big\| + |\gamma_n - \gamma_{n-1}| \|W_{n-1} y_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} G y_{n-1}\| \\
&\qquad + \gamma_n \|W_n y_{n-1} - W_{n-1} y_{n-1}\|\big] + \frac{|\beta_n - \beta_{n-1}|}{(1 - \beta_{n-1})(1 - \beta_n)} \|x_n - \beta_{n-1} x_{n-1}\|
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{1-\beta_n}\Bigg[\alpha_n\rho\|y_n - y_{n-1}\| + \gamma_n\|y_n - y_{n-1}\| + \delta_n\big(\|y_{n-1} - y_n\| + |r_{n-1} - r_n|M_0\big) \\
&\quad + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| + |\gamma_n - \gamma_{n-1}|\|W_{n-1}y_{n-1}\| + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\| \\
&\quad + \gamma_n M\prod_{i=0}^{n-1}\lambda_i\Bigg] + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|x_n - \beta_{n-1}x_{n-1}\| \\
&= \frac{1}{1-\beta_n}\Bigg[\big(1 - \beta_n - \alpha_n(1-\rho)\big)\|y_n - y_{n-1}\| \\
&\quad + \frac{1}{1-\beta_n}\big[\delta_n|r_{n-1} - r_n|M_0 + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\|\big] \\
&\quad + |\gamma_n - \gamma_{n-1}|\|W_{n-1}y_{n-1}\| + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\| + \gamma_n M\prod_{i=0}^{n-1}\lambda_i\Bigg] \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\big\|\alpha_{n-1}f(y_{n-1}) + \gamma_{n-1}W_{n-1}y_{n-1} + J_{r_{n-1}}Gy_{n-1}\big\| \\
&= \left(1 - \frac{\alpha_n(1-\rho)}{1-\beta_n}\right)\|y_n - y_{n-1}\| + \frac{1}{1-\beta_n}\Bigg[\delta_n|r_{n-1} - r_n|M_0 + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|W_{n-1}y_{n-1}\| + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\| + \gamma_n M\prod_{i=0}^{n-1}\lambda_i\Bigg] \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\big\|\alpha_{n-1}f(y_{n-1}) + \gamma_{n-1}W_{n-1}y_{n-1} + J_{r_{n-1}}Gy_{n-1}\big\| \\
&\le \|y_n - y_{n-1}\| + \frac{1}{1-\beta_n}\Bigg[\delta_n|r_{n-1} - r_n|M_0 + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|W_{n-1}y_{n-1}\| + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\| + \gamma_n M\prod_{i=0}^{n-1}\lambda_i\Bigg] \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\big\|\alpha_{n-1}f(y_{n-1}) + \gamma_{n-1}W_{n-1}y_{n-1} + J_{r_{n-1}}Gy_{n-1}\big\| \\
&\le \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|x_{n-1} - J_{r_{n-1}}Gx_{n-1}\| + |r_{n-1} - r_n|M_0 \\
&\quad + \frac{1}{1-\beta_n}\Bigg[\delta_n|r_{n-1} - r_n|M_0 + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| \\
&\quad + |\gamma_n - \gamma_{n-1}|\|W_{n-1}y_{n-1}\| + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\| + \gamma_n M\prod_{i=0}^{n-1}\lambda_i\Bigg] \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\big\|\alpha_{n-1}f(y_{n-1}) + \gamma_{n-1}W_{n-1}y_{n-1} + \delta_{n-1}J_{r_{n-1}}Gy_{n-1}\big\| \\
&\le \|x_n - x_{n-1}\| + \Bigg[|\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \\
&\quad + |\delta_n - \delta_{n-1}| + \prod_{i=0}^{n-1}\lambda_i\Bigg]M_1,
\end{aligned}
\tag{4.9}
$$

where $\sup_{n\geq 0}\{\frac{1}{(1-\hat{d})^2}(\|f(y_n)\| + \|W_n y_n\| + \|J_{r_n} G y_n\| + \|x_n - J_{r_n} G x_n\| + M + 2M_0)\} \leq M_1$ for some $M_1 > 0$. Thus, it follows from (4.9) and conditions (i), (iii), (iv) that

$$\lim_{n\to\infty}\bigl(\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|\bigr) \leq 0.$$

Since $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, by Lemma 2.6, we get

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Consequently,

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|z_n - x_n\| = 0.$$

Now we show that $\|x_n - Gx_n\| \to 0$ as $n \to \infty$. Indeed, by Lemma 2.3 and (4.1), we get

$$\begin{aligned}
\|y_n - p\|^2 &= \bigl\|\sigma_n(x_n - p) + (1 - \sigma_n)(J_{r_n} G x_n - p)\bigr\|^2 \\
&\leq \sigma_n\|x_n - p\|^2 + (1 - \sigma_n)\|J_{r_n} G x_n - p\|^2 - \sigma_n(1 - \sigma_n)g\bigl(\|x_n - J_{r_n} G x_n\|\bigr) \\
&\leq \sigma_n\|x_n - p\|^2 + (1 - \sigma_n)\|x_n - p\|^2 - \sigma_n(1 - \sigma_n)g\bigl(\|x_n - J_{r_n} G x_n\|\bigr) \\
&= \|x_n - p\|^2 - \sigma_n(1 - \sigma_n)g\bigl(\|x_n - J_{r_n} G x_n\|\bigr).
\end{aligned} \tag{4.10}$$

By Lemma 2.2(a), (4.1), and (4.10), we obtain

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \bigl\|\alpha_n\bigl(f(y_n) - f(p)\bigr) + \beta_n(x_n - p) + \gamma_n(W_n y_n - p) + \delta_n(J_{r_n} G y_n - p) + \alpha_n\bigl(f(p) - p\bigr)\bigr\|^2 \\
&\quad \leq \bigl\|\alpha_n\bigl(f(y_n) - f(p)\bigr) + \beta_n(x_n - p) + \gamma_n(W_n y_n - p) + \delta_n(J_{r_n} G y_n - p)\bigr\|^2 \\
&\qquad + 2\alpha_n\bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \\
&\quad \leq \alpha_n\bigl\|f(y_n) - f(p)\bigr\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|W_n y_n - p\|^2 + \delta_n\|J_{r_n} G y_n - p\|^2 \\
&\qquad + 2\alpha_n\bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \\
&\quad \leq \alpha_n\rho^2\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 + \delta_n\|G y_n - p\|^2 \\
&\qquad + 2\alpha_n\bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \\
&\quad \leq \alpha_n\rho\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 + \delta_n\|y_n - p\|^2 \\
&\qquad + 2\alpha_n\bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \\
&\quad = \bigl(1 - \beta_n - \alpha_n(1 - \rho)\bigr)\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + 2\alpha_n\bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \\
&\quad \leq \bigl(1 - \beta_n - \alpha_n(1 - \rho)\bigr)\bigl[\|x_n - p\|^2 - \sigma_n(1 - \sigma_n)g\bigl(\|x_n - J_{r_n} G x_n\|\bigr)\bigr] + \beta_n\|x_n - p\|^2 \\
&\qquad + 2\alpha_n\bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \\
&\quad = \bigl(1 - \alpha_n(1 - \rho)\bigr)\|x_n - p\|^2 - \bigl(1 - \beta_n - \alpha_n(1 - \rho)\bigr)\sigma_n(1 - \sigma_n)g\bigl(\|x_n - J_{r_n} G x_n\|\bigr) \\
&\qquad + 2\alpha_n\bigl\langle f(p) - p, J(x_{n+1} - p)\bigr\rangle \\
&\quad \leq \|x_n - p\|^2 - \bigl(1 - \beta_n - \alpha_n(1 - \rho)\bigr)\sigma_n(1 - \sigma_n)g\bigl(\|x_n - J_{r_n} G x_n\|\bigr) \\
&\qquad + 2\alpha_n\bigl\|f(p) - p\bigr\|\,\|x_{n+1} - p\|,
\end{aligned}$$

and thus

$$\begin{aligned}
&\bigl(1-\beta_n-\alpha_n(1-\rho)\bigr)\sigma_n(1-\sigma_n)g\bigl(\|x_n-J_{r_n}Gx_n\|\bigr)\\
&\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n\bigl\|f(p)-p\bigr\|\|x_{n+1}-p\|\\
&\quad\le\bigl(\|x_n-p\|+\|x_{n+1}-p\|\bigr)\|x_n-x_{n+1}\|+2\alpha_n\bigl\|f(p)-p\bigr\|\|x_{n+1}-p\|.
\end{aligned}$$

Since $\alpha_n\to 0$ and $\|x_{n+1}-x_n\|\to 0$, from condition (v) and the boundedness of $\{x_n\}$, it follows that

$$\lim_{n\to\infty}g\bigl(\|x_n-J_{r_n}Gx_n\|\bigr)=0.$$

Utilizing the properties of g, we have

$$\lim_{n\to\infty}\|x_n-J_{r_n}Gx_n\|=0, \tag{4.11}$$

and thus,

$$\lim_{n\to\infty}\|y_n-x_n\|=\lim_{n\to\infty}(1-\sigma_n)\|J_{r_n}Gx_n-x_n\|=0. \tag{4.12}$$

For simplicity, we put $q=\Pi_C(p-\mu_2B_2p)$, $u_n=\Pi_C(x_n-\mu_2B_2x_n)$ and $v_n=\Pi_C(u_n-\mu_1B_1u_n)$. Then $v_n=Gx_n$ for all $n\ge 0$. From Lemma 2.8, we have

$$\begin{aligned}
\|u_n-q\|^2&=\bigl\|\Pi_C(x_n-\mu_2B_2x_n)-\Pi_C(p-\mu_2B_2p)\bigr\|^2\\
&\le\bigl\|x_n-p-\mu_2(B_2x_n-B_2p)\bigr\|^2\\
&\le\|x_n-p\|^2-2\mu_2\bigl(\alpha_2-\kappa^2\mu_2\bigr)\|B_2x_n-B_2p\|^2,
\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}
\|v_n-p\|^2&=\bigl\|\Pi_C(u_n-\mu_1B_1u_n)-\Pi_C(q-\mu_1B_1q)\bigr\|^2\\
&\le\bigl\|u_n-q-\mu_1(B_1u_n-B_1q)\bigr\|^2\\
&\le\|u_n-q\|^2-2\mu_1\bigl(\alpha_1-\kappa^2\mu_1\bigr)\|B_1u_n-B_1q\|^2.
\end{aligned} \tag{4.14}$$

By combining (4.13) and (4.14), we obtain

$$\begin{aligned}
\|v_n-p\|^2\le{}&\|x_n-p\|^2-2\mu_2\bigl(\alpha_2-\kappa^2\mu_2\bigr)\|B_2x_n-B_2p\|^2\\
&-2\mu_1\bigl(\alpha_1-\kappa^2\mu_1\bigr)\|B_1u_n-B_1q\|^2.
\end{aligned} \tag{4.15}$$

By the convexity of $\|\cdot\|^2$, we have, from (4.1) and (4.15),

$$\begin{aligned}
&\|y_n-p\|^2\\
&\quad\le\sigma_n\|x_n-p\|^2+(1-\sigma_n)\|J_{r_n}Gx_n-p\|^2\\
&\quad\le\sigma_n\|x_n-p\|^2+(1-\sigma_n)\|v_n-p\|^2
\end{aligned}$$

$$\begin{aligned}
&\le \sigma_n\|x_n-p\|^2+(1-\sigma_n)\bigl[\|x_n-p\|^2-2\mu_2\bigl(\alpha_2-\kappa^2\mu_2\bigr)\|B_2x_n-B_2p\|^2\\
&\quad -2\mu_1\bigl(\alpha_1-\kappa^2\mu_1\bigr)\|B_1u_n-B_1q\|^2\bigr]\\
&= \|x_n-p\|^2-2(1-\sigma_n)\bigl[\mu_2\bigl(\alpha_2-\kappa^2\mu_2\bigr)\|B_2x_n-B_2p\|^2\\
&\quad +\mu_1\bigl(\alpha_1-\kappa^2\mu_1\bigr)\|B_1u_n-B_1q\|^2\bigr],
\end{aligned}$$

and thus

$$\begin{aligned}
&2(1-\sigma_n)\bigl[\mu_2\bigl(\alpha_2-\kappa^2\mu_2\bigr)\|B_2x_n-B_2p\|^2+\mu_1\bigl(\alpha_1-\kappa^2\mu_1\bigr)\|B_1u_n-B_1q\|^2\bigr]\\
&\quad\le \|x_n-p\|^2-\|y_n-p\|^2\\
&\quad\le \bigl(\|x_n-p\|+\|y_n-p\|\bigr)\|x_n-y_n\|.
\end{aligned}$$

Since $\|x_n-y_n\|\to 0$ and $0<\mu_i<\frac{\alpha_i}{\kappa^2}$ for $i=1,2$, and $\{x_n\}$ and $\{y_n\}$ are bounded, we obtain from condition (v) that

$$\lim_{n\to\infty}\|B_2x_n-B_2p\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|B_1u_n-B_1q\|=0. \tag{4.16}$$

Utilizing Proposition 2.2 and Lemma 2.10, we have

$$\begin{aligned}
\|u_n-q\|^2 &= \bigl\|\Pi_C(x_n-\mu_2B_2x_n)-\Pi_C(p-\mu_2B_2p)\bigr\|^2\\
&\le \bigl\langle x_n-\mu_2B_2x_n-(p-\mu_2B_2p),J(u_n-q)\bigr\rangle\\
&= \bigl\langle x_n-p,J(u_n-q)\bigr\rangle+\mu_2\bigl\langle B_2p-B_2x_n,J(u_n-q)\bigr\rangle\\
&\le \frac{1}{2}\bigl[\|x_n-p\|^2+\|u_n-q\|^2-g_1\bigl(\bigl\|x_n-u_n-(p-q)\bigr\|\bigr)\bigr]\\
&\quad +\mu_2\|B_2p-B_2x_n\|\|u_n-q\|,
\end{aligned}$$

which implies that

$$\|u_n-q\|^2\le\|x_n-p\|^2-g_1\bigl(\bigl\|x_n-u_n-(p-q)\bigr\|\bigr)+2\mu_2\|B_2p-B_2x_n\|\|u_n-q\|. \tag{4.17}$$

In the same way, we derive

$$\begin{aligned}
\|v_n-p\|^2 &= \bigl\|\Pi_C(u_n-\mu_1B_1u_n)-\Pi_C(q-\mu_1B_1q)\bigr\|^2\\
&\le \bigl\langle u_n-\mu_1B_1u_n-(q-\mu_1B_1q),J(v_n-p)\bigr\rangle\\
&= \bigl\langle u_n-q,J(v_n-p)\bigr\rangle+\mu_1\bigl\langle B_1q-B_1u_n,J(v_n-p)\bigr\rangle\\
&\le \frac{1}{2}\bigl[\|u_n-q\|^2+\|v_n-p\|^2-g_2\bigl(\bigl\|u_n-v_n+(p-q)\bigr\|\bigr)\bigr]\\
&\quad +\mu_1\|B_1q-B_1u_n\|\|v_n-p\|,
\end{aligned}$$

and we get

$$\|v_n-p\|^2\le\|u_n-q\|^2-g_2\bigl(\bigl\|u_n-v_n+(p-q)\bigr\|\bigr)+2\mu_1\|B_1q-B_1u_n\|\|v_n-p\|. \tag{4.18}$$

Combining (4.17) and (4.18), we get

$$\begin{aligned}\|v_n - p\|^2 \le{}& \|x_n - p\|^2 - g_1\bigl(\bigl\|x_n - u_n - (p-q)\bigr\|\bigr) - g_2\bigl(\bigl\|u_n - v_n + (p-q)\bigr\|\bigr)\\ &+ 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|.\end{aligned} \tag{4.19}$$

By the convexity of $\|\cdot\|^2$, we have, from (4.1) and (4.19),

$$\begin{aligned}&\|y_n - p\|^2\\ &\quad\le \sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|J_{r_n}Gx_n - p\|^2\\ &\quad\le \sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|v_n - p\|^2\\ &\quad\le \sigma_n\|x_n - p\|^2 + (1-\sigma_n)\bigl[\|x_n - p\|^2 - g_1\bigl(\bigl\|x_n - u_n - (p-q)\bigr\|\bigr)\\ &\qquad - g_2\bigl(\bigl\|u_n - v_n + (p-q)\bigr\|\bigr) + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\|\\ &\qquad + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|\bigr]\\ &\quad\le \|x_n - p\|^2 - (1-\sigma_n)\bigl[g_1\bigl(\bigl\|x_n - u_n - (p-q)\bigr\|\bigr) + g_2\bigl(\bigl\|u_n - v_n + (p-q)\bigr\|\bigr)\bigr]\\ &\qquad + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|,\end{aligned}$$

and hence

$$\begin{aligned}&(1-\sigma_n)\bigl[g_1\bigl(\bigl\|x_n - u_n - (p-q)\bigr\|\bigr) + g_2\bigl(\bigl\|u_n - v_n + (p-q)\bigr\|\bigr)\bigr]\\ &\quad\le \|x_n - p\|^2 - \|y_n - p\|^2 + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|\\ &\quad\le \bigl(\|x_n - p\| + \|y_n - p\|\bigr)\|x_n - y_n\| + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\|\\ &\qquad + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|.\end{aligned}$$

From (4.12), (4.16), condition (v), and the boundedness of $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, and $\{v_n\}$, we deduce

$$\lim_{n\to\infty} g_1\bigl(\bigl\|x_n - u_n - (p-q)\bigr\|\bigr) = 0 \quad\text{and}\quad \lim_{n\to\infty} g_2\bigl(\bigl\|u_n - v_n + (p-q)\bigr\|\bigr) = 0.$$

Utilizing the properties of g_1 and g_2, we obtain

$$\lim_{n\to\infty}\bigl\|x_n - u_n - (p-q)\bigr\| = 0 \quad\text{and}\quad \lim_{n\to\infty}\bigl\|u_n - v_n + (p-q)\bigr\| = 0. \tag{4.20}$$

Hence,

$$\|x_n - v_n\| \le \bigl\|x_n - u_n - (p-q)\bigr\| + \bigl\|u_n - v_n + (p-q)\bigr\| \to 0 \quad \text{as } n\to\infty,$$

that is,

$$\lim_{n\to\infty}\|x_n - Gx_n\| = 0. \tag{4.21}$$

Next, we show that

$$\lim_{n\to\infty}\|J_{r_n}x_n - x_n\| = 0 \quad\text{and}\quad \lim_{n\to\infty}\|W_nx_n - x_n\| = 0.$$

Indeed, observe that x_{n+1} can be rewritten as

$$\begin{aligned} x_{n+1} &= \alpha_n f(y_n) + \beta_n x_n + \gamma_n W_n y_n + \delta_n J_{r_n} G y_n \\ &= \alpha_n f(y_n) + \beta_n x_n + (\gamma_n + \delta_n) \frac{\gamma_n W_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n} \\ &= \alpha_n f(y_n) + \beta_n x_n + e_n \hat{z}_n, \end{aligned} \tag{4.22}$$

where $e_n = \gamma_n + \delta_n$ and $\hat{z}_n = \frac{\gamma_n W_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n}$. Utilizing Lemma 2.4 and (4.22), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \big\|\alpha_n\big(f(y_n) - p\big) + \beta_n(x_n - p) + e_n(\hat{z}_n - p)\big\|^2 \\ &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + e_n\|\hat{z}_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &= \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\quad + e_n \left\| \frac{\gamma_n W_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n} - p \right\|^2 \\ &= \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\quad + e_n \left\| \frac{\gamma_n}{\gamma_n + \delta_n}(W_n y_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G y_n - p) \right\|^2 \\ &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\quad + e_n \left[\frac{\gamma_n}{\gamma_n + \delta_n}\|W_n y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G y_n - p\|^2 \right] \\ &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\quad + e_n \left[\frac{\gamma_n}{\gamma_n + \delta_n}\|y_n - p\| + \frac{\delta_n}{\gamma_n + \delta_n}\|y_n - p\|^2 \right] \\ &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\quad + e_n \left[\frac{\gamma_n}{\gamma_n + \delta_n}\|x_n - p\| + \frac{\delta_n}{\gamma_n + \delta_n}\|x_n - p\|^2 \right] \\ &= \alpha_n \big\|f(y_n) - p\big\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big), \end{aligned}$$

which implies that

$$\begin{aligned} \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\|. \end{aligned}$$

Utilizing (4.4), conditions (i), (ii), (v), and the boundedness of $\{x_n\}$ and $\{f(y_n)\}$, we obtain

$$\lim_{n\to\infty} g_3\big(\|\hat{z}_n - x_n\|\big) = 0.$$

From the properties of g_3, we have

$$\lim_{n\to\infty} \|\hat{z}_n - x_n\| = 0.$$

Utilizing Lemma 2.3 and the definition of $\hat{z}_n$, we have

$$\begin{aligned}
\|\hat{z}_n - p\|^2 &= \left\| \frac{\gamma_n W_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n} - p \right\|^2 \\
&= \left\| \frac{\gamma_n}{\gamma_n + \delta_n}(W_n y_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G y_n - p) \right\|^2 \\
&\le \frac{\gamma_n}{\gamma_n + \delta_n}\|W_n y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G y_n - p\|^2 \\
&\quad - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_4\bigl(\|J_{r_n} G y_n - W_n y_n\|\bigr) \\
&\le \|y_n - p\|^2 - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_4\bigl(\|J_{r_n} G y_n - W_n y_n\|\bigr) \\
&\le \|x_n - p\|^2 - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_4\bigl(\|J_{r_n} G y_n - W_n y_n\|\bigr),
\end{aligned}$$

and thus

$$\begin{aligned}
\frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_4\bigl(\|J_{r_n} G y_n - W_n y_n\|\bigr) &\le \|x_n - p\|^2 - \|\hat{z}_n - p\|^2 \\
&\le \bigl(\|x_n - p\| + \|\hat{z}_n - p\|\bigr)\|x_n - \hat{z}_n\|.
\end{aligned}$$

Since $\{x_n\}$ and $\{\hat{z}_n\}$ are bounded and $\|\hat{z}_n - x_n\| \to 0$ as $n \to \infty$, we deduce from condition (ii) that

$$\lim_{n\to\infty} g_4\bigl(\|W_n y_n - J_{r_n} G y_n\|\bigr) = 0.$$

From the properties of g_4, we have

$$\lim_{n\to\infty} \|W_n y_n - J_{r_n} G y_n\| = 0. \tag{4.23}$$

On the other hand, x_{n+1} can also be rewritten as

$$\begin{aligned}
x_{n+1} &= \alpha_n f(y_n) + \beta_n x_n + \gamma_n W_n y_n + \delta_n J_{r_n} G y_n \\
&= \beta_n x_n + \gamma_n W_n y_n + (\alpha_n + \delta_n)\frac{\alpha_n f(y_n) + \delta_n J_{r_n} G y_n}{\alpha_n + \delta_n} \\
&= \beta_n x_n + \gamma_n W_n y_n + d_n \tilde{z}_n,
\end{aligned}$$

where $d_n = \alpha_n + \delta_n$ and $\tilde{z}_n = \frac{\alpha_n f(y_n) + \delta_n J_{r_n} G y_n}{\alpha_n + \delta_n}$. Utilizing Lemma 2.4 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \bigl\|\beta_n(x_n - p) + \gamma_n(W_n y_n - p) + d_n(\tilde{z}_n - p)\bigr\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + \gamma_n\|W_n y_n - p\|^2 + d_n\|\tilde{z}_n - p\|^2 - \beta_n \gamma_n g_5\bigl(\|x_n - W_n y_n\|\bigr) \\
&\quad = \beta_n\|x_n - p\|^2 + \gamma_n\|W_n y_n - p\|^2 + d_n\left\|\frac{\alpha_n f(y_n) + \delta_n J_{r_n} G y_n}{\alpha_n + \delta_n} - p\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - \beta_n\gamma_n g_5\bigl(\|x_n - W_n y_n\|\bigr) \\
&= \beta_n\|x_n - p\|^2 + \gamma_n\|W_n y_n - p\|^2 + d_n\left\|\frac{\alpha_n}{\alpha_n+\delta_n}\bigl(f(y_n)-p\bigr) + \frac{\delta_n}{\alpha_n+\delta_n}(J_{r_n}Gy_n - p)\right\|^2 \\
&\quad - \beta_n\gamma_n g_5\bigl(\|x_n - W_n y_n\|\bigr) \\
&\le \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 + d_n\left[\frac{\alpha_n}{\alpha_n+\delta_n}\bigl\|f(y_n)-p\bigr\|^2 + \frac{\delta_n}{\alpha_n+\delta_n}\|J_{r_n}Gy_n - p\|^2\right] \\
&\quad - \beta_n\gamma_n g_5\bigl(\|x_n - W_n y_n\|\bigr) \\
&\le \beta_n\|x_n - p\|^2 + \gamma_n\|y_n - p\|^2 + d_n\left[\frac{\alpha_n}{\alpha_n+\delta_n}\bigl\|f(y_n)-p\bigr\|^2 + \frac{\delta_n}{\alpha_n+\delta_n}\|y_n - p\|^2\right] \\
&\quad - \beta_n\gamma_n g_5\bigl(\|x_n - W_n y_n\|\bigr) \\
&\le \alpha_n\bigl\|f(y_n)-p\bigr\|^2 + (\beta_n+\gamma_n)\|x_n - p\|^2 + \delta_n\|x_n - p\|^2 - \beta_n\gamma_n g_5\bigl(\|x_n - W_n y_n\|\bigr) \\
&= \alpha_n\bigl\|f(y_n)-p\bigr\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \beta_n\gamma_n g_5\bigl(\|x_n - W_n y_n\|\bigr) \\
&\le \alpha_n\bigl\|f(y_n)-p\bigr\|^2 + \|x_n - p\|^2 - \beta_n\gamma_n g_5\bigl(\|x_n - W_n y_n\|\bigr),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\beta_n\gamma_n g_5\bigl(\|x_n - W_n y_n\|\bigr) &\le \alpha_n\bigl\|f(y_n)-p\bigr\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\le \alpha_n\bigl\|f(y_n)-p\bigr\|^2 + \bigl(\|x_n - p\| + \|x_{n+1} - p\|\bigr)\|x_n - x_{n+1}\|.
\end{aligned}
$$

From (4.4), conditions (i), (ii), (v), and the boundedness of $\{x_n\}$ and $\{f(y_n)\}$, we have

$$
\lim_{n\to\infty} g_5\bigl(\|x_n - W_n y_n\|\bigr) = 0.
$$

Utilizing the properties of g_5, we have

$$
\lim_{n\to\infty}\|x_n - W_n y_n\| = 0, \tag{4.24}
$$

which together with (4.12) and (4.24), implies that

$$
\begin{aligned}
\|x_n - W_n x_n\| &\le \|x_n - W_n y_n\| + \|W_n y_n - W_n x_n\| \\
&\le \|x_n - W_n y_n\| + \|y_n - x_n\| \to 0 \quad \text{as } n\to\infty,
\end{aligned}
$$

that is,

$$
\lim_{n\to\infty}\|x_n - W_n x_n\| = 0. \tag{4.25}
$$

We note that

$$
\begin{aligned}
&\|x_n - J_{r_n}x_n\| \\
&\quad\le \|x_n - W_n y_n\| + \|W_n y_n - J_{r_n}Gy_n\| + \|J_{r_n}Gy_n - J_{r_n}Gx_n\| + \|J_{r_n}Gx_n - J_{r_n}x_n\| \\
&\quad\le \|x_n - W_n y_n\| + \|W_n y_n - J_{r_n}Gy_n\| + \|y_n - x_n\| + \|Gx_n - x_n\|.
\end{aligned}
$$

Thus, from (4.12), (4.21), (4.23), and (4.24), it follows that

$$\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0. \tag{4.26}$$

Now, we claim that $\lim_{n\to\infty}\|x_n - J_r x_n\| = 0$ for a fixed number r such that $\varepsilon > r > 0$. In fact, using the resolvent identity in Proposition 2.2, we have

$$\begin{aligned}\|J_{r_n}x_n - J_r x_n\| &= \left\|J_r\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}x_n\right) - J_r x_n\right\| \\ &\le \left(1-\frac{r}{r_n}\right)\|x_n - J_{r_n}x_n\| \\ &\le \|x_n - J_{r_n}x_n\|.\end{aligned} \tag{4.27}$$

Thus, from (4.26) and (4.27), we get

$$\begin{aligned}\|x_n - J_r x_n\| &\le \|x_n - J_{r_n}x_n\| + \|J_{r_n}x_n - J_r x_n\| \\ &\le \|x_n - J_{r_n}x_n\| + \|x_n - J_{r_n}x_n\| \\ &= 2\|x_n - J_{r_n}x_n\| \to 0 \quad \text{as } n\to\infty,\end{aligned}$$

that is,

$$\lim_{n\to\infty}\|x_n - J_r x_n\| = 0. \tag{4.28}$$

Suppose that $\beta_n \equiv \beta$ for some fixed $\beta, \gamma \in (0,1)$ such that $\alpha_n + \beta + \gamma_n + \delta_n = 1$ for all $n \ge 0$. Define a mapping $Vx = (1-\theta_1-\theta_2)J_r x + \theta_1 Wx + \theta_2 Gx$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then, by Lemmas 2.5 and 2.13, we have $\mathrm{Fix}(V) = \mathrm{Fix}(J_r)\cap\mathrm{Fix}(W)\cap\mathrm{Fix}(G) = F$. For each $k \ge 1$, let $\{p_k\}$ be a unique element of C such that

$$p_k = \frac{1}{k}f(p_k) + \left(1-\frac{1}{k}\right)Vp_k.$$

From Lemma 2.9, we conclude that $p_k \to q \in \mathrm{Fix}(V) = F$ as $k\to\infty$. Observe that for every n, k

$$\begin{aligned}&\|x_{n+1} - Wp_k\| \\ &\quad = \bigl\|\alpha_n\bigl(f(y_n) - Wp_k\bigr) + \beta(x_n - Wp_k) + \gamma_n(W_n y_n - Wp_k) + \delta_n(J_{r_n}Gy_n - Wp_k)\bigr\| \\ &\quad \le \alpha_n\bigl\|f(y_n) - Wp_k\bigr\| + \beta\|x_n - Wp_k\| + \gamma_n\|W_n y_n - Wp_k\| \\ &\quad\quad + \delta_n\bigl(\|J_{r_n}Gy_n - W_n y_n\| + \|W_n y_n - Wp_k\|\bigr) \\ &\quad = \alpha_n\bigl\|f(y_n) - Wp_k\bigr\| + \beta\|x_n - Wp_k\| + (\gamma_n + \delta_n)\|W_n y_n - Wp_k\| + \delta_n\|J_{r_n}Gy_n - W_n y_n\| \\ &\quad = \alpha_n\bigl\|f(y_n) - Wp_k\bigr\| + \beta\|x_n - Wp_k\| + (1-\alpha_n-\beta)\|W_n y_n - Wp_k\| \\ &\quad\quad + \delta_n\|J_{r_n}Gy_n - W_n y_n\| \\ &\quad \le \alpha_n\bigl\|f(y_n) - Wp_k\bigr\| + \beta\|x_n - Wp_k\| \\ &\quad\quad + (1-\alpha_n-\beta)\bigl[\|W_n y_n - W_n p_k\| + \|W_n p_k - Wp_k\|\bigr] + \delta_n\|J_{r_n}Gy_n - W_n y_n\|\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n \big\|f(y_n) - Wp_k\big\| + \beta\|x_n - Wp_k\| + (1-\alpha_n-\beta)\big[\|y_n - p_k\| + \|W_np_k - Wp_k\|\big] \\
&\quad + \delta_n\|J_{r_n}Gy_n - W_ny_n\| \\
&\leq \alpha_n \big\|f(y_n) - Wp_k\big\| + \beta\|x_n - Wp_k\| \\
&\quad + (1-\beta)\big[\|x_n - p_k\| + \|y_n - x_n\| + \|W_np_k - Wp_k\|\big] + \delta_n\|J_{r_n}Gy_n - W_ny_n\| \\
&= \Delta_n + \beta\|x_n - Wp_k\| + (1-\beta)\|x_n - p_k\|,
\end{aligned}
\tag{4.29}
$$

where $\Delta_n = \alpha_n\|f(y_n) - Wp_k\| + (1-\beta)[\|y_n - x_n\| + \|W_np_k - Wp_k\|] + \delta_n\|J_{r_n}Gy_n - W_ny_n\|$. Since $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|W_np_k - Wp_k\| = \lim_{n\to\infty}\|J_{r_n}Gy_n - W_ny_n\| = 0$, we know that $\Delta_n \to 0$ as $n\to\infty$.

From (4.29), we obtain

$$
\begin{aligned}
&\|x_{n+1} - Wp_k\|^2 \\
&\quad\leq \big(\beta\|x_n - Wp_k\| + (1-\beta)\|x_n - p_k\|\big)^2 \\
&\qquad + \Delta_n\big[2\big(\beta\|x_n - Wp_k\| + (1-\beta)\|x_n - p_k\|\big) + \Delta_n\big] \\
&\quad= \beta^2\|x_n - Wp_k\|^2 + (1-\beta)^2\|x_n - p_k\|^2 + 2\beta(1-\beta)\|x_n - Wp_k\|\|x_n - p_k\| + \tau_n \\
&\quad\leq \beta^2\|x_n - Wp_k\|^2 + (1-\beta)^2\|x_n - p_k\|^2 \\
&\qquad + \beta(1-\beta)\big(\|x_n - Wp_k\|^2 + \|x_n - p_k\|^2\big) + \tau_n \\
&\quad= \beta\|x_n - Wp_k\|^2 + (1-\beta)\|x_n - p_k\|^2 + \tau_n,
\end{aligned}
\tag{4.30}
$$

where $\tau_n = \Delta_n[2(\beta\|x_n - Wp_k\| + (1-\beta)\|x_n - p_k\|) + \Delta_n] \to 0$ as $n\to\infty$.

For any Banach limit μ, from (4.30), we have

$$
\mu_n\|x_n - Wp_k\|^2 = \mu_n\|x_{n+1} - Wp_k\|^2 \leq \mu_n\|x_n - p_k\|^2. \tag{4.31}
$$

In addition, note that

$$
\begin{aligned}
\|x_n - Gp_k\|^2 &\leq \|x_n - Gx_n + Gx_n - Gp_k\|^2 \\
&\leq \big(\|x_n - Gx_n\| + \|x_n - p_k\|\big)^2 \\
&= \|x_n - p_k\|^2 + \|x_n - Gx_n\|\big(2\|x_n - p_k\| + \|x_n - Gx_n\|\big),
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_n - J_rp_k\|^2 &\leq \|x_n - J_rx_n + J_rx_n - J_rp_k\|^2 \\
&\leq \big(\|x_n - J_rx_n\| + \|x_n - p_k\|\big)^2 \\
&= \|x_n - p_k\|^2 + \|x_n - J_rx_n\|\big(2\|x_n - p_k\| + \|x_n - J_rx_n\|\big).
\end{aligned}
$$

It is easy to see from (4.21) and (4.28) that

$$
\mu_n\|x_n - Gp_k\|^2 \leq \mu_n\|x_n - p_k\|^2 \quad\text{and}\quad \mu_n\|x_n - J_rp_k\|^2 \leq \mu_n\|x_n - p_k\|^2. \tag{4.32}
$$

Utilizing (4.31) and (4.32), we have

$$\begin{aligned}\mu_n\|x_n - Vp_k\|^2 &= \mu_n\big\|(1-\theta_1-\theta_2)(x_n - J_r p_k) + \theta_1(x_n - Wp_k) + \theta_2(x_n - Gp_k)\big\|^2 \\ &\le (1-\theta_1-\theta_2)\mu_n\|x_n - J_r p_k\|^2 + \theta_1\mu_n\|x_n - Wp_k\|^2 + \theta_2\mu_n\|x_n - Gp_k\|^2 \\ &\le \mu_n\|x_n - p_k\|^2. \end{aligned} \tag{4.33}$$

Also, observe that

$$x_n - p_k = \frac{1}{k}\big(x_n - f(p_k)\big) + \left(1 - \frac{1}{k}\right)(x_n - Vp_k),$$

that is,

$$\left(1 - \frac{1}{k}\right)(x_n - Vp_k) = x_n - p_k - \frac{1}{k}\big(x_n - f(p_k)\big). \tag{4.34}$$

It follows from Lemma 2.2(ii) and (4.34) that

$$\begin{aligned}\left(1 - \frac{1}{k}\right)^2\|x_n - Vp_k\|^2 &\ge \|x_n - p_k\|^2 - \frac{2}{k}\big\langle x_n - p_k + p_k - f(p_k), J(x_n - p_k)\big\rangle \\ &= \left(1 - \frac{2}{k}\right)\|x_n - p_k\|^2 + \frac{2}{k}\big\langle f(p_k) - p_k, J(x_n - p_k)\big\rangle. \end{aligned} \tag{4.35}$$

So by (4.33) and (4.35), we have

$$\left(1 - \frac{1}{k}\right)^2\mu_n\|x_n - p_k\|^2 \ge \left(1 - \frac{2}{k}\right)\mu_n\|x_n - p_k\|^2 + \frac{2}{k}\mu_n\big\langle f(p_k) - p_k, J(x_n - p_k)\big\rangle,$$

and hence,

$$\frac{1}{k^2}\mu_n\|x_n - p_k\|^2 \ge \frac{2}{k}\mu_n\big\langle f(p_k) - p_k, J(x_n - p_k)\big\rangle.$$

This implies that

$$\frac{1}{2k}\mu_n\|x_n - p_k\|^2 \ge \mu_n\big\langle f(p_k) - p_k, J(x_n - p_k)\big\rangle. \tag{4.36}$$

Since $p_k \to q \in \mathrm{Fix}(V) = F$ as $k \to \infty$, by the uniform Fréchet differentiability of the norm of X, we have

$$\mu_n\big\langle f(q) - q, J(x_n - q)\big\rangle \le 0.$$

On the other hand, from (4.4) and the norm-to-norm uniform continuity of J on bounded subsets of X, we have

$$\lim_{n\to\infty}\big|\big\langle f(q) - q, J(x_{n+1} - q)\big\rangle - \big\langle f(q) - q, J(x_n - q)\big\rangle\big| = 0. \tag{4.37}$$

Utilizing Lemma 2.14, we deduce from (4.36) and (4.37) that

$$\limsup_{n\to\infty}\big\langle f(q) - q, J(x_n - q)\big\rangle \le 0.$$

Finally, we show that $x_n \to q$ as $n \to \infty$. It is easy to see from (4.1) that

$$\|y_n - q\|^2 \le \sigma_n \|x_n - q\|^2 + (1-\sigma_n)\|J_{r_n} G x_n - q\|^2 \le \|x_n - q\|^2.$$

Utilizing Lemma 2.2(a), from (4.1) and the convexity of $\|\cdot\|^2$ we get

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&\quad = \bigl\|\alpha_n\bigl(f(y_n) - f(q)\bigr) + \beta_n(x_n - q) + \gamma_n(W_n y_n - q) + \delta_n(J_{r_n} G y_n - q) + \alpha_n\bigl(f(q) - q\bigr)\bigr\|^2 \\
&\quad \le \bigl\|\alpha_n\bigl(f(y_n) - f(q)\bigr) + \beta_n(x_n - q) + \gamma_n(W_n y_n - q) + \delta_n(J_{r_n} G y_n - q)\bigr\|^2 \\
&\qquad + 2\alpha_n\bigl\langle f(q) - q, J(x_{n+1} - q)\bigr\rangle \\
&\quad \le \alpha_n\bigl\|f(y_n) - f(q)\bigr\|^2 + \beta_n\|x_n - q\|^2 + \gamma_n\|W_n y_n - q\|^2 + \delta_n\|J_{r_n} G y_n - q\|^2 \\
&\qquad + 2\alpha_n\bigl\langle f(q) - q, J(x_{n+1} - q)\bigr\rangle \\
&\quad \le \alpha_n\rho\|y_n - q\|^2 + \beta_n\|x_n - q\|^2 + \gamma_n\|y_n - q\|^2 + \delta_n\|y_n - q\|^2 \\
&\qquad + 2\alpha_n\bigl\langle f(q) - q, J(x_{n+1} - q)\bigr\rangle \\
&\quad \le \alpha_n\rho\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 + \gamma_n\|x_n - q\|^2 + \delta_n\|x_n - q\|^2 \\
&\qquad + 2\alpha_n\bigl\langle f(q) - q, J(x_{n+1} - q)\bigr\rangle \\
&\quad = \bigl(1 - \alpha_n(1-\rho)\bigr)\|x_n - q\|^2 + 2\alpha_n\bigl\langle f(q) - q, J(x_{n+1} - q)\bigr\rangle \\
&\quad = \bigl(1 - \alpha_n(1-\rho)\bigr)\|x_n - q\|^2 + \alpha_n(1-\rho)\frac{2\langle f(q) - q, J(x_{n+1} - q)\rangle}{1-\rho}.
\end{aligned} \tag{4.38}$$

Applying Lemma 2.7 to (4.38), we obtain $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 4.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X and Π_C be a sunny nonexpansive retraction from X onto C. Let $A \subset X \times X$ be an accretive operator such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $V: C \to C$ be α-strictly pseudocontractive mapping and $f: C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{T_i\}_{i=0}^\infty$ be an infinite family of nonexpansive mappings from C into itself such that $F = \bigcap_{i=0}^\infty \operatorname{Fix}(T_i) \cap \operatorname{Fix}(V) \cap A^{-1}0 \neq \emptyset$. Suppose that Assumption 4.1 holds. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = \sigma_n x_n + (1-\sigma_n) J_{r_n}((1-l)I + lV)x_n, \\ x_{n+1} = \alpha_n f(y_n) + \beta_n x_n + \gamma_n W_n y_n + \delta_n J_{r_n}((1-l)I + lV)y_n, \quad \forall n \ge 0, \end{cases} \tag{4.39}$$

where $0 < l < \frac{\alpha}{\kappa^2}$, W_n is the W-mapping generated by (2.1). Then

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;
(b) *the sequence $\{x_n\}_{n=0}^\infty$ converges strongly to some $q \in F$ which is a unique solution of the following variational inequality problem (VIP):*

$$\bigl\langle (I - f)q, J(q - p)\bigr\rangle \le 0, \quad \forall p \in F,$$

provided $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$.

Proof In Theorem 4.1, we put $B_1 = I - V$, $B_2 = 0$ and $\mu_1 = l$ where $0 < l < \frac{\alpha}{\kappa^2}$. Then SVI (3.1) is equivalent to the VIP of finding $x^* \in C$ such that

$$\langle B_1 x^*, J(x - x^*)\rangle \geq 0, \quad \forall x \in C.$$

In this case, $B_1 : C \to X$ is α-inverse strongly accretive. It is not hard to see that $\mathrm{Fix}(V) = \mathrm{VI}(C, B_1)$. Indeed, for $l > 0$, we have

$$\begin{aligned}
u \in \mathrm{VI}(C, B_1) \quad &\Leftrightarrow \quad \langle B_1 u, J(y - u)\rangle \geq 0 \quad \forall y \in C \\
&\Leftrightarrow \quad \langle u - lB_1 u - u, J(u - y)\rangle \geq 0 \quad \forall y \in C \\
&\Leftrightarrow \quad u = \Pi_C(u - lB_1 u) \\
&\Leftrightarrow \quad u = \Pi_C(u - lu + lVu) \\
&\Leftrightarrow \quad \langle u - lu + lVu - u, J(u - y)\rangle \geq 0 \quad \forall y \in C \\
&\Leftrightarrow \quad \langle u - Vu, J(u - y)\rangle \leq 0 \quad \forall y \in C \\
&\Leftrightarrow \quad u = Vu \\
&\Leftrightarrow \quad u \in \mathrm{Fix}(V).
\end{aligned}$$

Accordingly, we have $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \Omega \cap A^{-1}0 = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(V) \cap A^{-1}0$, and

$$\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x_n = \Pi_C(I - \mu_1 B_1)x_n = \Pi_C\big((1 - l)x_n + lVx_n\big) = \big((1 - l)I + lV\big)x_n.$$

Similarly, we get

$$\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)y_n = \big((1 - l)I + lV\big)y_n.$$

So, the scheme (4.1) reduces to (4.39), and therefore, the desired result follows from Theorem 4.1. □

We give the following important lemmas which will be used in our next result.

Lemma 4.1 *Let C be a nonempty closed convex subset of a smooth Banach space X and $B_i : C \to X$ be λ_i-strictly pseudocontractive mappings and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ for $i = 1, 2$. Then, for $\mu_i \in (0, 1]$,*

$$\big\|(I - \mu_i B_i)x - (I - \mu_i B_i)y\big\| \leq \left\{\sqrt{\frac{1 - \alpha_i}{\lambda_i}} + (1 - \mu_i)\left(1 + \frac{1}{\lambda_i}\right)\right\}\|x - y\|, \quad \forall x, y \in C,$$

for $i = 1, 2$. In particular, if $1 - \frac{\lambda_i}{1+\lambda_i}(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}) \leq \mu_i \leq 1$, then $I - \mu_i B_i$ is nonexpansive for $i = 1, 2$.

Proof Using the λ_i-strict pseudocontractivity of B_i, we derive for every $x, y \in C$

$$\begin{aligned}
\lambda_i \big\|(I - B_i)x - (I - B_i)y\big\|^2 &\leq \big\langle (I - B_i)x - (I - B_i)y, J(x - y)\big\rangle \\
&\leq \big\|(I - B_i)x - (I - B_i)y\big\| \|x - y\|,
\end{aligned}$$

which implies that

$$\|(I-B_i)x-(I-B_i)y\| \le \frac{1}{\lambda_i}\|x-y\|.$$

Hence,

$$\begin{aligned}\|B_ix-B_iy\| &\le \|(I-B_i)x-(I-B_i)y\| + \|x-y\| \\ &\le \left(1+\frac{1}{\lambda_i}\right)\|x-y\|.\end{aligned}$$

Utilizing the α_i-strong accretivity and λ_i-strict pseudocontractivity of B_i, we get

$$\begin{aligned}\lambda_i\|(I-B_i)x-(I-B_i)y\|^2 &\le \|x-y\|^2 - \langle B_ix-B_iy, J(x-y)\rangle \\ &\le (1-\alpha_i)\|x-y\|^2.\end{aligned}$$

So, we have

$$\|(I-B_i)x-(I-B_i)y\| \le \sqrt{\frac{1-\alpha_i}{\lambda_i}}\|x-y\|.$$

Therefore, for $\mu_i \in (0,1]$, we have

$$\begin{aligned}\|(I-\mu_iB_i)x-(I-\mu_iB_i)y\| &\le \|(I-B_i)x-(I-B_i)y\| + (1-\mu_i)\|B_ix-B_iy\| \\ &\le \sqrt{\frac{1-\alpha_i}{\lambda_i}}\|x-y\| + (1-\mu_i)\left(1+\frac{1}{\lambda_i}\right)\|x-y\| \\ &= \left\{\sqrt{\frac{1-\alpha_i}{\lambda_i}} + (1-\mu_i)\left(1+\frac{1}{\lambda_i}\right)\right\}\|x-y\|.\end{aligned}$$

Since $1-\frac{\lambda_i}{1+\lambda_i}(1-\sqrt{\frac{1-\alpha_i}{\lambda_i}}) \le \mu_i \le 1$, it follows that

$$\sqrt{\frac{1-\alpha_i}{\lambda_i}} + (1-\mu_i)\left(1+\frac{1}{\lambda_i}\right) \le 1.$$

This implies that $I-\mu_iB_i$ is nonexpansive for $i=1,2$. □

Lemma 4.2 *Let C be a nonempty closed convex subset of a smooth Banach space X and Π_C be a sunny nonexpansive retraction from X onto C. For each $i=1,2$, let $B_i: C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i+\lambda_i \ge 1$. Let $G: C \to C$ be the mapping defined by*

$$G(x) = \Pi_C\big[\Pi_C(x-\mu_2B_2x) - \mu_1B_1\Pi_C(x-\mu_2B_2x)\big], \quad \forall x \in C.$$

If $1-\frac{\lambda_i}{1+\lambda_i}(1-\sqrt{\frac{1-\alpha_i}{\lambda_i}}) \le \mu_i \le 1$, then $G: C \to C$ is nonexpansive.

Proof By Lemma 4.1, $I - \mu_i B_i$ is nonexpansive for $i = 1,2$. Therefore, for all $x, y \in C$, we have

$$\begin{aligned}\|G(x) - G(y)\| &= \|\Pi_C[\Pi_C(x - \mu_2 B_2 x) - \mu_1 B_1 \Pi_C(x - \mu_2 B_2 x)] \\ &\quad - \Pi_C[\Pi_C(y - \mu_2 B_2 y) - \mu_1 B_1 \Pi_C(y - \mu_2 B_2 y)]\| \\ &= \|\Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x - \Pi_C(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)y\| \\ &\leq \|(I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)x - (I - \mu_1 B_1)\Pi_C(I - \mu_2 B_2)y\| \\ &\leq \|\Pi_C(I - \mu_2 B_2)x - \Pi_C(I - \mu_2 B_2)y\| \\ &\leq \|(I - \mu_2 B_2)x - (I - \mu_2 B_2)y\| \\ &\leq \|x - y\|.\end{aligned}$$

This shows that $G : C \to C$ is nonexpansive. This completes the proof. □

Theorem 4.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C and $A \subset X \times X$ be an accretive operator in X such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. For each $i = 1,2$, let $B_i : C \to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i + \lambda_i \geq 1$ and $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{T_i\}_{i=0}^{\infty}$ be an infinite family of nonexpansive mappings from C into itself such that $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \Omega \cap A^{-1}0 \neq \emptyset$ with $1 - \frac{\lambda_i}{1+\lambda_i}(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}) \leq \mu_i \leq 1$ for $i = 1,2$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = \sigma_n G x_n + (1 - \sigma_n) J_{r_n} G x_n, \\ x_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n W_n y_n + \delta_n J_{r_n} G y_n, \quad \forall n \geq 0, \end{cases} \tag{4.40}$$

where W_n is the W-mapping generated by (2.1). *Assume that Assumption* 4.1 *holds except condition* (iii), *which is replaced by the following condition*:

(iii) $\sum_{n=1}^{\infty}(|\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$.

Then

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

(b) *the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $q \in F$ which is the unique solution of the following variational inequality problem* (*VIP*):

$$\langle (I - f)q, J(q - p)\rangle \leq 0, \quad \forall p \in F,$$

provided $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$.

Proof Take a fixed $p \in F$ arbitrarily. Then we obtain $p = Gp$, $p = W_n p$ and $J_{r_n} p = p$ for all $n \geq 0$. By using Lemma 4.2 and the same argument as in the proof beginning of the proof of Theorem 4.1, we have $\{x_n\}$, $\{y_n\}$, $\{Gx_n\}$, $\{Gy_n\}$, $\{f(y_n)\}$ are bounded sequences. Let us show that $\|x_{n+1}-x_n\| \to 0$ as $n \to \infty$. In fact, repeating the same argument as those in the proof of Theorem 4.1, we obtain

$$\begin{cases} \|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \le \|x_{n-1}-x_n\| + |r_{n-1}-r_n|M_0, \\ \|J_{r_n}Gy_n - J_{r_{n-1}}Gy_{n-1}\| \le \|y_{n-1}-y_n\| + |r_{n-1}-r_n|M_0, \quad \forall n \ge 1, \end{cases} \tag{4.41}$$

where

$$\sup_{n\ge1}\left\{\frac{1}{\varepsilon}\big(\|J_{r_n}Gx_n - Gx_{n-1}\| + \|J_{r_{n-1}}Gx_{n-1} - Gx_n\|\big)\right\} \le M_0$$

and

$$\sup_{n\ge1}\left\{\frac{1}{\varepsilon}\big(\|J_{r_n}Gy_n - Gy_{n-1}\| + \|J_{r_{n-1}}Gy_{n-1} - Gy_n\|\big)\right\} \le M_0,$$

for some $M_0 > 0$. By (4.40) and simple calculations, we have

$$\begin{aligned} y_n - y_{n-1} &= \sigma_n(Gx_n - Gx_{n-1}) + (\sigma_n - \sigma_{n-1})(Gx_{n-1} - J_{r_{n-1}}Gx_{n-1}) \\ &\quad + (1-\alpha_n)(J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}). \end{aligned}$$

It follows that

$$\begin{aligned} \|y_n - y_{n-1}\| &\le \sigma_n\|Gx_n - Gx_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - J_{r_{n-1}}Gx_{n-1}\| \\ &\quad + (1-\alpha_n)\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \\ &\le \sigma_n\|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - J_{r_{n-1}}Gx_{n-1}\| \\ &\quad + (1-\sigma_n)\big(\|x_{n-1}-x_n\| + |r_{n-1}-r_n|M_0\big) \\ &\le \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - J_{r_{n-1}}Gx_{n-1}\| + |r_n - r_{n-1}|M_0. \end{aligned} \tag{4.42}$$

Repeating the same argument as in (4.7) in the proof of Theorem 4.1, we get

$$\|W_ny_{n-1} - W_{n-1}y_{n-1}\| \le M\prod_{i=0}^{n-1}\lambda_i, \quad \text{for some constant } M > 0. \tag{4.43}$$

Considering condition (v), without loss of generality, we may assume that $\{\beta_n\} \subset [\hat{c}, \hat{d}]$ for some $\hat{c}, \hat{d} \in (0,1)$. From (4.40), it follows that x_{n+1} can be rewritten as

$$x_{n+1} = \beta_ny_n + (1-\beta_n)z_n, \tag{4.44}$$

where $z_n = \frac{\alpha_nf(y_n)+\gamma_nW_ny_n+\delta_nJ_{r_n}Gy_n}{1-\beta_n}$. Utilizing (4.3) and (4.42) we have

$$\begin{aligned} &\|z_n - z_{n-1}\| \\ &\quad = \left\|\frac{\alpha_nf(y_n)+\gamma_nW_ny_n+\delta_nJ_{r_n}Gy_n}{1-\beta_n} - \frac{\alpha_{n-1}f(y_{n-1})+\gamma_{n-1}W_{n-1}y_{n-1}+\delta_{n-1}J_{r_{n-1}}Gy_{n-1}}{1-\beta_{n-1}}\right\| \end{aligned}$$

$$
\begin{aligned}
&= \left\| \frac{x_{n+1} - \beta_n y_n}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_{n-1}} \right\| \\
&= \left\| \frac{x_{n+1} - \beta_n y_n}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_n} + \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_{n-1}} \right\| \\
&\le \left\| \frac{x_{n+1} - \beta_n y_n}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_n} \right\| + \left\| \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_{n-1}} \right\| \\
&= \frac{1}{1-\beta_n} \left\| x_{n+1} - \beta_n y_n - (x_n - \beta_{n-1} y_{n-1}) \right\| + \left| \frac{1}{1-\beta_n} - \frac{1}{1-\beta_{n-1}} \right| \|x_n - \beta_{n-1} y_{n-1}\| \\
&= \frac{1}{1-\beta_n} \left\| x_{n+1} - \beta_n y_n - (x_n - \beta_{n-1} y_{n-1}) \right\| + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&= \frac{1}{1-\beta_n} \\
&\quad \times \left\| \alpha_n f(y_n) + \gamma_n W_n y_n + \delta_n J_{r_n} G y_n - \alpha_{n-1} f(y_{n-1}) - \gamma_{n-1} W_{n-1} y_{n-1} - \delta_{n-1} J_{r_{n-1}} G y_{n-1} \right\| \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&\le \frac{1}{1-\beta_n} \big[\alpha_n \left\| f(y_n) - f(y_{n-1}) \right\| + \gamma_n \| W_n y_n - W_{n-1} y_{n-1} \| + \delta_n \| J_{r_n} G y_n - J_{r_{n-1}} G y_{n-1} \| \\
&\quad + |\alpha_n - \alpha_{n-1}| \left\| f(y_{n-1}) \right\| + |\gamma_n - \gamma_{n-1}| \| W_{n-1} y_{n-1} \| + |\delta_n - \delta_{n-1}| \| J_{r_{n-1}} G y_{n-1} \| \big] \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&\le \frac{1}{1-\beta_n} \big[\alpha_n \rho \| y_n - y_{n-1} \| + \gamma_n \| W_n y_n - W_n y_{n-1} \| + \delta_n \big[\| y_{n-1} - y_n \| + |r_{n-1} - r_n| M_0 \big] \\
&\quad + |\alpha_n - \alpha_{n-1}| \left\| f(y_{n-1}) \right\| + |\gamma_n - \gamma_{n-1}| \| W_{n-1} y_{n-1} \| + \gamma_n \| W_n y_{n-1} - W_{n-1} y_{n-1} \| \\
&\quad + |\delta_n - \delta_{n-1}| \| J_{r_{n-1}} G y_{n-1} \| \big] + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&\le \frac{1}{1-\beta_n} \Bigg[(\alpha_n \rho + \gamma_n + \delta_n) \| y_{n-1} - y_n \| + |r_{n-1} - r_n| M_0 + |\alpha_n - \alpha_{n-1}| \left\| f(y_{n-1}) \right\| \\
&\quad + |\gamma_n - \gamma_{n-1}| \| W_{n-1} y_{n-1} \| + \gamma_n M \prod_{i=0}^{n-1} \lambda_i + |\delta_n - \delta_{n-1}| \| J_{r_{n-1}} G y_{n-1} \| \Bigg] \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&\le \left(1 - \frac{(1-\rho)\alpha_n}{1-\beta_n} \right) \| y_n - y_{n-1} \| + \frac{1}{1-\beta_n} \big[|r_{n-1} - r_n| M_0 + |\alpha_n - \alpha_{n-1}| \left\| f(y_{n-1}) \right\| \\
&\quad + |\gamma_n - \gamma_{n-1}| \| W_{n-1} y_{n-1} \| + |\delta_n - \delta_{n-1}| \| J_{r_{n-1}} G y_{n-1} \| \big] + M \prod_{i=0}^{n-1} \lambda_i \\
&\quad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\|.
\end{aligned}
\tag{4.45}
$$

By simple calculations, (4.44) implies that

$$
x_{n+1} - x_n = \beta_n (y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(y_{n-1} - z_{n-1}) + (1-\beta_n)(z_n - z_{n-1}).
$$

This together with (4.42) and (4.45) we have

$$\begin{aligned}
&\|x_{n+1}-x_n\| \\
&\quad \le \beta_n\|y_n-y_{n-1}\| + |\beta_n-\beta_{n-1}|\|y_{n-1}-z_{n-1}\| + (1-\beta_n)\|z_n-z_{n-1}\| \\
&\quad \le \beta_n\|y_n-y_{n-1}\| + |\beta_n-\beta_{n-1}|\|y_{n-1}-z_{n-1}\| + (1-\beta_n)\biggl\{\biggl(1-\frac{(1-\rho)\alpha_n}{1-\beta_n}\biggr)\|y_n-y_{n-1}\| \\
&\qquad + \frac{1}{1-\beta_n}\bigl[|r_{n-1}-r_n|M_0 + |\alpha_n-\alpha_{n-1}|\bigl\|f(y_{n-1})\bigr\| + |\gamma_n-\gamma_{n-1}|\|W_{n-1}y_{n-1}\| \\
&\qquad + |\delta_n-\delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\|\bigr] + M\prod_{i=0}^{n-1}\lambda_i + \frac{|\beta_n-\beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|x_n-\beta_{n-1}y_{n-1}\|\biggr\} \\
&\quad \le \bigl(1-(1-\rho)\alpha_n\bigr)\|y_n-y_{n-1}\| + |\beta_n-\beta_{n-1}|\|y_{n-1}-z_{n-1}\| + |r_{n-1}-r_n|M_0 \\
&\qquad + |\alpha_n-\alpha_{n-1}|\bigl\|f(y_{n-1})\bigr\| + |\gamma_n-\gamma_{n-1}|\|W_{n-1}y_{n-1}\| + |\delta_n-\delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\| \\
&\qquad + M\prod_{i=0}^{n-1}\lambda_i + \frac{|\beta_n-\beta_{n-1}|}{1-\beta_{n-1}}\|x_n-\beta_{n-1}y_{n-1}\| \\
&\quad \le \bigl(1-(1-\rho)\alpha_n\bigr)\bigl[\|x_n-x_{n-1}\| + |\sigma_n-\sigma_{n-1}|\|Gx_{n-1}-J_{r_{n-1}}Gx_{n-1}\| + |r_n-r_{n-1}|M_0\bigr] \\
&\qquad + |\beta_n-\beta_{n-1}|\|y_{n-1}-z_{n-1}\| + |r_{n-1}-r_n|M_0 + |\alpha_n-\alpha_{n-1}|\bigl\|f(y_{n-1})\bigr\| \\
&\qquad + |\gamma_n-\gamma_{n-1}|\|W_{n-1}y_{n-1}\| + |\delta_n-\delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\| + M\prod_{i=0}^{n-1}\lambda_i \\
&\qquad + \frac{|\beta_n-\beta_{n-1}|}{1-\beta_{n-1}}\bigl\|\alpha_{n-1}f(y_{n-1}) + \gamma_{n-1}W_{n-1}y_{n-1} + \delta_{n-1}J_{r_{n-1}}Gy_{n-1}\bigr\| \\
&\quad \le \bigl(1-(1-\rho)\alpha_n\bigr)\|x_n-x_{n-1}\| + \bigl(|\sigma_n-\sigma_{n-1}| + |\alpha_n-\alpha_{n-1}| + |\beta_n-\beta_{n-1}| \\
&\qquad + |\gamma_n-\gamma_{n-1}| + |\delta_n-\delta_{n-1}| + |r_{n-1}-r_n|\bigr)\hat{M} + M\prod_{i=0}^{n-1}\lambda_i,
\end{aligned}$$

where $\frac{1}{1-\hat{d}}\sup_{n\ge 0}\{\|f(y_n)\| + \|W_ny_n\| + \|J_{r_n}Gy_n\| + \|Gx_n-J_{r_n}Gx_n\| + \|y_n-z_n\| + 2M_0\} \le \hat{M}$ for some $\hat{M}>0$. By Lemma 2.7 and conditions (i), (iii), and (iv), we conclude that (noting that $0<\lambda_i\le b<1$, $\forall i\ge 0$)

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{4.46}$$

Next we show that $\|x_n-Gx_n\|\to 0$ as $n\to\infty$. Indeed, utilizing Lemma 2.3, we get from (4.40)

$$\begin{aligned}
\|y_n-p\|^2 &= \bigl\|\sigma_n(Gx_n-p) + (1-\sigma_n)(J_{r_n}Gx_n-p)\bigr\|^2 \\
&\le \sigma_n\|Gx_n-p\|^2 + (1-\sigma_n)\|J_{r_n}Gx_n-p\|^2 - \sigma_n(1-\sigma_n)g\bigl(\|Gx_n-J_{r_n}Gx_n\|\bigr) \\
&\le \sigma_n\|x_n-p\|^2 + (1-\sigma_n)\|x_n-p\|^2 - \sigma_n(1-\sigma_n)g\bigl(\|Gx_n-J_{r_n}Gx_n\|\bigr) \\
&= \|x_n-p\|^2 - \sigma_n(1-\sigma_n)g\bigl(\|Gx_n-J_{r_n}Gx_n\|\bigr).
\end{aligned} \tag{4.47}$$

By Lemma 2.2 (a), (4.40), and (4.47), we have

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \big\|\alpha_n\big(f(y_n) - f(p)\big) + \beta_n(y_n - p) + \gamma_n(W_n y_n - p) + \delta_n(J_{r_n} G y_n - p) + \alpha_n\big(f(p) - p\big)\big\|^2 \\
&\quad \le \big\|\alpha_n\big(f(y_n) - f(p)\big) + \beta_n(y_n - p) + \gamma_n(W_n y_n - p) + \delta_n(J_{r_n} G y_n - p)\big\|^2 \\
&\qquad + 2\alpha_n\big\langle f(p) - p, J(x_{n+1} - p)\big\rangle \\
&\quad \le \alpha_n\big\|f(y_n) - f(p)\big\|^2 + \beta_n\|y_n - p\|^2 + \gamma_n\|W_n y_n - p\|^2 + \delta_n\|J_{r_n} G y_n - p\|^2 \\
&\qquad + 2\alpha_n\big\langle f(p) - p, J(x_{n+1} - p)\big\rangle \\
&\quad \le \alpha_n\rho\|y_n - p\|^2 + \beta_n\|y_n - p\|^2 + \gamma_n\|y_n - p\|^2 + \delta_n\|y_n - p\|^2 \\
&\qquad + 2\alpha_n\big\|f(p) - p\big\|\|x_{n+1} - p\| \\
&\quad = \big(1 - \alpha_n(1 - \rho)\big)\|y_n - p\|^2 + 2\alpha_n\big\|f(p) - p\big\|\|x_{n+1} - p\| \\
&\quad \le \|y_n - p\|^2 + 2\alpha_n\big\|f(p) - p\big\|\|x_{n+1} - p\| \\
&\quad \le \|x_n - p\|^2 - \sigma_n(1 - \sigma_n)g\big(\|Gx_n - J_{r_n} G x_n\|\big) + 2\alpha_n\big\|f(p) - p\big\|\|x_{n+1} - p\|,
\end{aligned}$$

which yields

$$\begin{aligned}
&\sigma_n(1 - \sigma_n)g\big(\|Gx_n - J_{r_n} G x_n\|\big) \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n\big\|f(p) - p\big\|\|x_{n+1} - p\| \\
&\quad \le \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\| + 2\alpha_n\big\|f(p) - p\big\|\|x_{n+1} - p\|.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, from condition (v) and the boundedness of $\{x_n\}$, it follows that

$$\lim_{n\to\infty} g\big(\|Gx_n - J_{r_n} G x_n\|\big) = 0.$$

Utilizing the properties of g, we have

$$\lim_{n\to\infty} \|Gx_n - J_{r_n} G x_n\| = 0. \tag{4.48}$$

On the other hand, observe that x_{n+1} can be rewritten as

$$\begin{aligned}
x_{n+1} &= \alpha_n f(y_n) + \beta_n y_n + \gamma_n W_n y_n + \delta_n J_{r_n} G y_n \\
&= \alpha_n f(y_n) + \beta_n y_n + (\gamma_n + \delta_n)\frac{\gamma_n W_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n} \\
&= \alpha_n f(y_n) + \beta_n y_n + e_n \hat{z}_n,
\end{aligned} \tag{4.49}$$

where $e_n = \gamma_n + \delta_n$ and $\hat{z}_n = \frac{\gamma_n W_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n}$. By Lemma 2.4, (4.3), and (4.49), we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \big\|\alpha_n\big(f(y_n) - p\big) + \beta_n(y_n - p) + e_n(\hat{z}_n - p)\big\|^2 \\
&\le \alpha_n\big\|f(y_n) - p\big\|^2 + \beta_n\|y_n - p\|^2 + e_n\|\hat{z}_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - y_n\|\big)
\end{aligned}$$

$$
\begin{aligned}
&= \alpha_n \left\|f(y_n) - p\right\|^2 + \beta_n\|y_n - p\|^2 - \beta_n e_n g_1\bigl(\|\hat{z}_n - y_n\|\bigr) \\
&\quad + e_n\left\|\frac{\gamma_n W_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n} - p\right\|^2 \\
&= \alpha_n \left\|f(y_n) - p\right\|^2 + \beta_n\|y_n - p\|^2 - \beta_n e_n g_1\bigl(\|\hat{z}_n - y_n\|\bigr) \\
&\quad + e_n\left\|\frac{\gamma_n}{\gamma_n + \delta_n}(W_n y_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G y_n - p)\right\|^2 \\
&\leq \alpha_n \left\|f(y_n) - p\right\|^2 + \beta_n\|y_n - p\|^2 - \beta_n e_n g_1\bigl(\|\hat{z}_n - y_n\|\bigr) \\
&\quad + e_n\left[\frac{\gamma_n}{\gamma_n + \delta_n}\|W_n y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G y_n - p\|^2\right] \\
&\leq \alpha_n \left\|f(y_n) - p\right\|^2 + \beta_n\|y_n - p\|^2 - \beta_n e_n g_1\bigl(\|\hat{z}_n - y_n\|\bigr) \\
&\quad + e_n\left[\frac{\gamma_n}{\gamma_n + \delta_n}\|y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|y_n - p\|^2\right] \\
&= \alpha_n \left\|f(y_n) - p\right\|^2 + (1-\alpha_n)\|y_n - p\|^2 - \beta_n e_n g_1\bigl(\|\hat{z}_n - y_n\|\bigr) \\
&\leq \alpha_n \left\|f(y_n) - p\right\|^2 + \|y_n - p\|^2 - \beta_n e_n g_1\bigl(\|\hat{z}_n - y_n\|\bigr) \\
&\leq \alpha_n \left\|f(y_n) - p\right\|^2 + \|x_n - p\|^2 - \beta_n e_n g_1\bigl(\|\hat{z}_n - y_n\|\bigr),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\beta_n e_n g_1\bigl(\|\hat{z}_n - y_n\|\bigr) &\leq \alpha_n \left\|f(y_n) - p\right\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\leq \alpha_n \left\|f(y_n) - p\right\|^2 + \bigl(\|x_n - p\| + \|x_{n+1} - p\|\bigr)\|x_n - x_{n+1}\|.
\end{aligned}
$$

Utilizing (4.46), conditions (i), (ii), (v), and the boundedness of $\{x_n\}$ and $\{f(y_n)\}$, we get

$$
\lim_{n\to\infty} g_1\bigl(\|\hat{z}_n - y_n\|\bigr) = 0.
$$

From the properties of g_1, we have

$$
\lim_{n\to\infty} \|\hat{z}_n - y_n\| = 0. \tag{4.50}
$$

Utilizing Lemma 2.3 and the definition of $\hat{z}_n$, we have

$$
\begin{aligned}
\|\hat{z}_n - p\|^2 &= \left\|\frac{\gamma_n W_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n} - p\right\|^2 \\
&= \left\|\frac{\gamma_n}{\gamma_n + \delta_n}(W_n y_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G y_n - p)\right\|^2 \\
&\leq \frac{\gamma_n}{\gamma_n + \delta_n}\|W_n y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G y_n - p\|^2 \\
&\quad - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2\bigl(\|J_{r_n} G y_n - W_n y_n\|\bigr) \\
&\leq \|y_n - p\|^2 - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2\bigl(\|J_{r_n} G y_n - W_n y_n\|\bigr),
\end{aligned}
$$

which leads to

$$\begin{aligned}\frac{\gamma_n\delta_n}{(\gamma_n+\delta_n)^2}g_2\bigl(\|J_{r_n}Gy_n-W_ny_n\|\bigr) &\le \|y_n-p\|^2-\|\hat{z}_n-p\|^2\\ &\le \bigl(\|y_n-p\|+\|\hat{z}_n-p\|\bigr)\|y_n-\hat{z}_n\|.\end{aligned}$$

Since $\{y_n\}$ and $\{\hat{z}_n\}$ are bounded, from (4.50) and condition (ii), we deduce

$$\lim_{n\to\infty} g_2\bigl(\|W_ny_n-J_{r_n}Gy_n\|\bigr)=0.$$

From the properties of g_2, we have

$$\lim_{n\to\infty}\|W_ny_n-J_{r_n}Gy_n\|=0. \tag{4.51}$$

Furthermore, x_{n+1} can also be rewritten as

$$\begin{aligned}x_{n+1} &= \alpha_nf(y_n)+\beta_ny_n+\gamma_nW_ny_n+\delta_nJ_{r_n}Gy_n\\ &= \beta_ny_n+\gamma_nW_ny_n+(\alpha_n+\delta_n)\frac{\alpha_nf(y_n)+\delta_nJ_{r_n}Gy_n}{\alpha_n+\delta_n}\\ &= \beta_ny_n+\gamma_nW_ny_n+d_n\tilde{z}_n,\end{aligned}$$

where $d_n=\alpha_n+\delta_n$ and $\tilde{z}_n=\frac{\alpha_nf(y_n)+\delta_nJ_{r_n}Gy_n}{\alpha_n+\delta_n}$. Utilizing Lemma 2.4, the convexity of $\|\cdot\|^2$, and (4.3), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\quad= \bigl\|\beta_n(y_n-p)+\gamma_n(W_ny_n-p)+d_n(\tilde{z}_n-p)\bigr\|^2\\ &\quad\le \beta_n\|y_n-p\|^2+\gamma_n\|W_ny_n-p\|^2+d_n\|\tilde{z}_n-p\|^2-\beta_n\gamma_ng_3\bigl(\|y_n-W_ny_n\|\bigr)\\ &\quad= \beta_n\|y_n-p\|^2+\gamma_n\|W_ny_n-p\|^2+d_n\left\|\frac{\alpha_nf(y_n)+\delta_nJ_{r_n}Gy_n}{\alpha_n+\delta_n}-p\right\|^2\\ &\qquad -\beta_n\gamma_ng_3\bigl(\|y_n-W_ny_n\|\bigr)\\ &\quad= \beta_n\|y_n-p\|^2+\gamma_n\|W_ny_n-p\|^2+d_n\left\|\frac{\alpha_n}{\alpha_n+\delta_n}\bigl(f(y_n)-p\bigr)+\frac{\delta_n}{\alpha_n+\delta_n}(J_{r_n}Gy_n-p)\right\|^2\\ &\qquad -\beta_n\gamma_ng_3\bigl(\|y_n-W_ny_n\|\bigr)\\ &\quad\le \beta_n\|y_n-p\|^2+\gamma_n\|y_n-p\|^2+d_n\left[\frac{\alpha_n}{\alpha_n+\delta_n}\bigl\|f(y_n)-p\bigr\|^2+\frac{\delta_n}{\alpha_n+\delta_n}\|J_{r_n}Gy_n-p\|^2\right]\\ &\qquad -\beta_n\gamma_ng_3\bigl(\|y_n-W_ny_n\|\bigr)\\ &\quad\le \alpha_n\bigl\|f(y_n)-p\bigr\|^2+(\beta_n+\gamma_n)\|y_n-p\|^2+\delta_n\|y_n-p\|^2-\beta_n\gamma_ng_3\bigl(\|y_n-W_ny_n\|\bigr)\\ &\quad= \alpha_n\bigl\|f(y_n)-p\bigr\|^2+(1-\alpha_n)\|y_n-p\|^2-\beta_n\gamma_ng_3\bigl(\|y_n-W_ny_n\|\bigr)\\ &\quad\le \alpha_n\bigl\|f(y_n)-p\bigr\|^2+\|y_n-p\|^2-\beta_n\gamma_ng_3\bigl(\|y_n-W_ny_n\|\bigr)\\ &\quad\le \alpha_n\bigl\|f(y_n)-p\bigr\|^2+\|x_n-p\|^2-\beta_n\gamma_ng_3\bigl(\|y_n-W_ny_n\|\bigr),\end{aligned}$$

which implies that

$$\begin{aligned}\beta_n\gamma_n g_3\bigl(\|y_n - W_n y_n\|\bigr) &\le \alpha_n \bigl\|f(y_n) - p\bigr\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n \bigl\|f(y_n) - p\bigr\|^2 + \bigl(\|x_n - p\| + \|x_{n+1} - p\|\bigr)\|x_n - x_{n+1}\|.\end{aligned}$$

From (4.46), conditions (i), (ii), (v), and the boundedness of $\{x_n\}$ and $\{f(y_n)\}$, we have

$$\lim_{n\to\infty} g_3\bigl(\|y_n - W_n y_n\|\bigr) = 0.$$

Utilizing the properties of g_3, we have

$$\lim_{n\to\infty} \|y_n - W_n y_n\| = 0. \tag{4.52}$$

Thus, from (4.51) and (4.52), we get

$$\|y_n - Jr_n G y_n\| \le \|y_n - W_n y_n\| + \|W_n y_n - Jr_n G y_n\| \to 0 \quad \text{as } n \to \infty,$$

that is,

$$\lim_{n\to\infty} \|y_n - Jr_n G y_n\| = 0. \tag{4.53}$$

Therefore, from (4.40), (4.46), (4.52), (4.53), and $\alpha_n \to 0$, we have

$$\begin{aligned}&\|x_n - y_n\| \\ &\quad\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\quad\le \|x_n - x_{n+1}\| + \alpha_n\bigl\|f(y_n) - y_n\bigr\| + \gamma_n\|W_n y_n - y_n\| + \delta_n\|Jr_n G y_n - y_n\| \\ &\quad\le \|x_n - x_{n+1}\| + \alpha_n\bigl\|f(y_n) - y_n\bigr\| + \|W_n y_n - y_n\| + \|Jr_n G y_n - y_n\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

that is,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.54}$$

Utilizing (4.40), (4.48), and (4.54), we obtain

$$\begin{aligned}\|x_n - Gx_n\| &\le \|x_n - y_n\| + \|y_n - Gx_n\| \\ &= \|x_n - y_n\| + (1 - \sigma_n)\|J_{r_n} G x_n - G x_n\| \\ &\le \|x_n - y_n\| + \|J_{r_n} G x_n - G x_n\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

that is,

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{4.55}$$

In addition, from (4.52) and (4.54), we have

$$\begin{aligned}\|x_n - W_n x_n\| &\le \|x_n - y_n\| + \|y_n - W_n y_n\| + \|W_n y_n - W_n x_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - W_n y_n\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

that is,

$$\lim_{n\to\infty} \|x_n - W_n x_n\| = 0. \tag{4.56}$$

Note that

$$\begin{aligned}\|x_n - J_{r_n}x_n\| &\le \|x_n - Gx_n\| + \|Gx_n - J_{r_n}Gx_n\| + \|J_{r_n}Gx_n - J_{r_n}x_n\| \\ &\le 2\|x_n - Gx_n\| + \|Gx_n - J_{r_n}Gx_n\|.\end{aligned}$$

So, from (4.48) and (4.55), it follows that

$$\lim_{n\to\infty} \|x_n - J_{r_n}x_n\| = 0. \tag{4.57}$$

Repeating the same argument as in (4.28) in the proof of Theorem 4.1, we get

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0, \tag{4.58}$$

for a fixed number r such that $\varepsilon > r > 0$.

Suppose that $\beta_n \equiv \beta$ for some fixed $\beta, \gamma \in (0,1)$ satisfying $\alpha_n + \beta + \gamma_n + \delta_n = 1$ for all $n \ge 0$. Define a mapping $Vx = (1 - \theta_1 - \theta_2)J_r x + \theta_1 Wx + \theta_2 Gx$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then, by Lemmas 2.5 and 2.13, we have $\mathrm{Fix}(V) = \mathrm{Fix}(J_r) \cap \mathrm{Fix}(W) \cap \mathrm{Fix}(G) = F$. For each $k \ge 1$, let $\{p_k\}$ be a unique element of C such that

$$p_k = \frac{1}{k}f(p_k) + \left(1 - \frac{1}{k}\right)Vp_k.$$

From Lemma 2.9, we conclude that $p_k \to q \in \mathrm{Fix}(V) = F$ as $k \to \infty$. Observe that for every n, k

$$\begin{aligned}
&\|x_{n+1} - Wp_k\| \\
&\quad\le \alpha_n \|f(y_n) - Wp_k\| + \beta\|y_n - Wp_k\| + \gamma_n\|W_n y_n - Wp_k\| \\
&\qquad + \delta_n\big(\|J_{r_n}Gy_n - W_n y_n\| + \|W_n y_n - Wp_k\|\big) \\
&\quad= \alpha_n \|f(y_n) - Wp_k\| + \beta\|y_n - Wp_k\| + (\gamma_n + \delta_n)\|W_n y_n - Wp_k\| + \delta_n\|J_{r_n}Gy_n - W_n y_n\| \\
&\quad= \alpha_n \|f(y_n) - Wp_k\| + \beta\|y_n - Wp_k\| + (1 - \alpha_n - \beta)\|W_n y_n - Wp_k\| \\
&\qquad + \delta_n\|J_{r_n}Gy_n - W_n y_n\| \\
&\quad\le \alpha_n \|f(y_n) - Wp_k\| + \beta\|y_n - Wp_k\| \\
&\qquad + (1 - \alpha_n - \beta)\big[\|W_n y_n - W_n p_k\| + \|W_n p_k - Wp_k\|\big] + \delta_n\|J_{r_n}Gy_n - W_n y_n\| \\
&\quad\le \alpha_n \|f(y_n) - Wp_k\| + \beta\|y_n - Wp_k\| + (1 - \alpha_n - \beta)\big(\|y_n - p_k\| + \|W_n p_k - Wp_k\|\big) \\
&\qquad + \delta_n\|J_{r_n}Gy_n - W_n y_n\| \\
&\quad\le \alpha_n \|f(y_n) - Wp_k\| + \beta\big(\|x_n - Wp_k\| + \|y_n - x_n\|\big) + (1 - \beta)\big[\|x_n - p_k\| + \|y_n - x_n\| \\
&\qquad + \|W_n p_k - Wp_k\|\big] + \delta_n\|J_{r_n}Gy_n - W_n y_n\| \\
&\quad= \Delta_n + \beta\|x_n - Wp_k\| + (1 - \beta)\|x_n - p_k\|,
\end{aligned} \tag{4.59}$$

where $\Delta_n = \alpha_n\|f(y_n) - Wp_k\| + (1-\beta)(\frac{1}{1-\beta}\|y_n - x_n\| + \|W_n p_k - Wp_k\|) + \delta_n\|J_{r_n}Gy_n - W_n y_n\|$. Since $\lim_{n\to\infty}\alpha_n = \lim_{n\to\infty}\|y_n - x_n\| = \lim_{n\to\infty}\|W_n p_k - Wp_k\| = \lim_{n\to\infty}\|J_{r_n}Gy_n - W_n y_n\| = 0$, we know that $\Delta_n \to 0$ as $n \to \infty$.

Repeating the same argument as in (4.31) and (4.32) in the proof of Theorem 4.1, we conclude that for any Banach limit μ,

$$\mu_n\|x_n - Wp_k\|^2 = \mu_n\|x_{n+1} - Wp_k\|^2 \le \mu_n\|x_n - p_k\|^2, \tag{4.60}$$

and

$$\mu_n\|x_n - Gp_k\|^2 \le \mu_n\|x_n - p_k\|^2 \quad \text{and} \quad \mu_n\|x_n - J_r p_k\|^2 \le \mu_n\|x_n - p_k\|^2. \tag{4.61}$$

Utilizing (4.60) and (4.61), we obtain

$$\begin{aligned} \mu_n\|x_n - Vp_k\|^2 &= \mu_n\big\|(1-\theta_1-\theta_2)(x_n - J_r p_k) + \theta_1(x_n - Wp_k) + \theta_2(x_n - Gp_k)\big\|^2 \\ &\le (1-\theta_1-\theta_2)\mu_n\|x_n - J_r p_k\|^2 + \theta_1\mu_n\|x_n - Wp_k\|^2 + \theta_2\mu_n\|x_n - Gp_k\|^2 \\ &\le \mu_n\|x_n - p_k\|^2. \end{aligned} \tag{4.62}$$

Repeating the same argument as in (4.36) in the proof of Theorem 4.1, we get

$$\frac{1}{2k}\mu_n\|x_n - p_k\|^2 \ge \mu_n\big\langle f(p_k) - p_k, J(x_n - p_k)\big\rangle. \tag{4.63}$$

Since $p_k \to q \in \operatorname{Fix}(V) = F$ as $k \to \infty$, by the uniform Gâteaux differentiability of the norm of X, we have

$$\mu_n\big\langle f(q) - q, J(x_n - q)\big\rangle \le 0.$$

On the other hand, from (4.4) and the norm-to-weak* uniform continuity of J on bounded subsets of X, it follows that

$$\lim_{n\to\infty}\big|\big\langle f(q) - q, J(x_{n+1} - q)\big\rangle - \big\langle f(q) - q, J(x_n - q)\big\rangle\big| = 0. \tag{4.64}$$

Using Lemma 2.14, we deduce from (4.63) and (4.64) that

$$\limsup_{n\to\infty}\big\langle f(q) - q, J(x_n - q)\big\rangle \le 0.$$

Finally, we show that $x_n \to q$ as $n \to \infty$. It is easy to see from (4.1) that

$$\|y_n - q\|^2 \le \sigma_n\|Gx_n - q\|^2 + (1-\sigma_n)\|J_{r_n}Gx_n - q\|^2 \le \|x_n - q\|^2.$$

Utilizing Lemma 2.2(a), from (4.1) and the convexity of $\|\cdot\|^2$ we get

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\quad = \big\|\alpha_n\big(f(y_n) - f(q)\big) + \beta_n(y_n - q) + \gamma_n(W_n y_n - q) + \delta_n(J_{r_n}Gy_n - q) + \alpha_n\big(f(q) - q\big)\big\|^2 \end{aligned}$$

$$
\begin{aligned}
&\le \alpha_n \|f(y_n) - f(q)\|^2 + \beta_n\|y_n - q\|^2 + \gamma_n\|W_n y_n - q\|^2 + \delta_n\|J_{r_n} G y_n - q\|^2 \\
&\quad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\le \alpha_n\rho\|y_n - q\|^2 + \beta_n\|y_n - q\|^2 + \gamma_n\|y_n - q\|^2 + \delta_n\|y_n - q\|^2 \\
&\quad + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&= \bigl(1 - \alpha_n(1-\rho)\bigr)\|y_n - q\|^2 + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\le \bigl(1 - \alpha_n(1-\rho)\bigr)\|x_n - q\|^2 + \alpha_n(1-\rho)\frac{2\langle f(q) - q, J(x_{n+1} - q)\rangle}{1-\rho}. \qquad (4.65)
\end{aligned}
$$

Applying Lemma 2.7 to (4.65), we obtain $x_n \to q$ as $n \to \infty$. This completes the proof. □

Corollary 4.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has an uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C and $A \subset X \times X$ be an accretive operator in X such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let $V : C \to C$ be a mapping such that $I - V : C \to X$ is ζ-strictly pseudocontractive and θ-strongly accretive with $\theta + \zeta \ge 1$. Let $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$ and $\{T_i\}_{i=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(V) \cap A^{-1}0 \neq \emptyset$. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$
\begin{cases}
y_n = \sigma_n((1-l)I + lV)x_n + (1-\sigma_n)J_{r_n}((1-l)I + lV)x_n, \\
x_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n W_n y_n + \delta_n J_{r_n}((1-l)I + lV)y_n, \quad \forall n \ge 0,
\end{cases} \qquad (4.66)
$$

where $1 - \frac{\zeta}{1+\zeta}(1 - \sqrt{\frac{1-\theta}{\zeta}}) \le l \le 1$, W_n is the W-mapping generated by (2.1). *Assume that Assumption* 4.1 *holds except condition* (iii), *which is replaced by the following condition*:

(iii) $\sum_{n=1}^{\infty}(|\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$.

Then

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;

(b) *the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $q \in F$ which is a unique solution of the following variational inequality problem* (*VIP*):

$$\langle (I - f)q, J(q - p)\rangle \le 0, \quad \forall p \in F,$$

provided $\beta_n \equiv \beta$ for some fixed $\beta \in (0,1)$.

Proof In Theorem 4.2, we put $B_1 = I - V$, $B_2 = 0$ and $\mu_1 = l$ where $1 - \frac{\zeta}{1+\zeta}(1 - \sqrt{\frac{1-\theta}{\zeta}}) \le l \le 1$. Then SVI (3.1) is equivalent to the VIP of finding $x^* \in C$ such that

$$\langle B_1 x^*, J(x - x^*)\rangle \ge 0, \quad \forall x \in C.$$

In this case, $B_1 : C \to X$ is ζ-strictly pseudocontractive and θ-strongly accretive. Repeating the same arguments as in the proof of Corollary 4.1, we can infer that $\mathrm{Fix}(V) = \mathrm{VI}(C, B_1)$. Accordingly, $F = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \Omega \cap A^{-1}0 = \bigcap_{i=0}^{\infty} \mathrm{Fix}(T_i) \cap \mathrm{Fix}(V) \cap A^{-1}0$,

$$G x_n = \bigl((1-l)I + lV\bigr)x_n \quad \text{and} \quad G y_n = \bigl((1-l)I + lV\bigr)y_n.$$

So, the scheme (4.40) reduces to (4.66). Therefore, the desired result follows from Theorem 4.2. □

Remark 4.1 Theorems 4.1 and 4.2 improve and extend [30, Theorem 3.2], [20, Theorem 3.1] and [29, Theorem 3.1] in the following aspects.

(a) The problem of finding a point $q \in \bigcap_n \mathrm{Fix}(T_n) \cap \Omega \cap A^{-1}0$ in Theorems 4.1 and 4.2 is more general and more subtle than the problem of finding a point $q \in \bigcap_n \mathrm{Fix}(T_n)$ in [30, Theorem 3.2], the problem of finding a point $q \in \bigcap_n \mathrm{Fix}(T_n) \cap \Omega$ in [20, Theorem 3.1] and the problem of finding a point $q \in A^{-1}0$ in [29, Theorem 3.1].

(b) Theorems 4.1 and 4.2 are proved without the assumption of the asymptotical regularity of $\{x_n\}$ in [29, Theorem 3.1] (that is, $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$).

(c) The iterative scheme in [20, Theorem 3.1] is extended to develop the iterative schemes (4.1) and (4.40) in Theorems 4.1 and 4.2 by virtue of the iterative schemes of [30, Theorem 3.2] and [29, Theorem 3.1]. The iterative schemes (4.1) and (4.40) in Theorems 4.1 and 4.2 are more advantageous and more flexible than the iterative scheme in [20, Theorem 3.1] because they involve several parameter sequences.

(d) The iterative schemes (4.1) and (4.40) in Theorems 4.1 and 4.2 are different from the iterative schemes in [30, Theorem 3.2], [20, Theorem 3.1] and [29, Theorem 3.1] because the mapping G in [20, Theorem 3.1] and the mapping J_{r_n} in [29, Theorem 3.1] are replaced by the composite mapping $J_{r_n}G$ in Theorems 4.1 and 4.2.

(e) The proof of [20, Theorem 3.1] depends on the argument techniques in [10], the inequality in 2-uniformly smooth Banach spaces, and the inequality in smooth and uniform convex Banach spaces. Because the composite mapping $J_{r_n}G$ appears in the iterative scheme (4.1) of Theorem 4.1, the proof of Theorem 4.1 depends on the argument techniques in [10], the inequality in 2-uniformly smooth Banach spaces, the inequality in smooth and uniform convex Banach spaces, the inequalities in uniform convex Banach spaces, and the properties of the W-mapping and the Banach limit. However, the proof of our Theorem 4.2 does not depend on the argument techniques in [10], the inequality in 2-uniformly smooth Banach spaces, and the inequality in smooth and uniform convex Banach spaces. It depends on only the inequalities in uniform convex Banach spaces and the properties of the W-mapping and the Banach limit.

(f) The assumption of the uniformly convex and 2-uniformly smooth Banach space X in [20, Theorem 3.1] is weakened to the uniformly convex Banach space X having a uniformly Gâteaux differentiable norm in Theorem 4.2.

5 Composite viscosity algorithms and convergence criteria

In this section, we introduce composite viscosity algorithms in real smooth and uniformly convex Banach spaces and study the strong convergence theorems. We first state the following important and useful lemma which will be used in the sequel.

Lemma 5.1 [27] *Let C be a nonempty closed convex subset of a Banach space X and $S_0, S_1, \ldots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^{\infty} \sup\{\|S_n x - S_{n-1}x\| : x \in C\} < \infty$. Then, for each $y \in C$, $\{S_n y\}$ converges strongly to some point in C. Moreover, let $S : C \to C$ be a mapping defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Sx - S_n x\| : x \in C\} = 0$.*

Assumption 5.1 Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$, $\{\sigma_n\}$ be the sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$ for all $n \geq 0$. Suppose that the following conditions hold:

(i) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\{\gamma_n\}, \{\delta_n\} \subset [c,d]$ for some $c, d \in (0,1)$;
(iii) $\sum_{n=1}^{\infty}(|\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}|) < \infty$;
(iv) $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$ and $r_n \geq \varepsilon > 0$ for all $n \geq 0$;
(v) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$ and $0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1$.

We now state and prove our first result on the composite implicit viscosity algorithm.

Theorem 5.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and $A \subset X \times X$ be an accretive operator on X such that $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. Let the mapping $B_i : C \to X$ be α_i-inverse strongly accretive for $i = 1, 2$, and $f : C \to C$ be a contraction with coefficient $\rho \in (0,1)$. Let $\{S_i\}_{i=0}^{\infty}$ be an infinite family of nonexpansive mappings of C into itself such that $F = \bigcap_{i=0}^{\infty} \operatorname{Fix}(S_i) \cap \Omega \cap A^{-1}0 \neq \emptyset$ with $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1, 2$. Suppose that Assumption 5.1 holds. For arbitrarily given $x_0 \in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n = \alpha_n f(y_n) + \beta_n x_n + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n, \\ x_{n+1} = \sigma_n y_n + (1 - \sigma_n) J_{r_n} G y_n, \quad \forall n \geq 0. \end{cases} \tag{5.1}$$

Assume that $\sum_{n=1}^{\infty} \sup_{x\in D} \|S_n x - S_{n-1} x\| < \infty$ for any bounded subset D of C, $S : C \to C$ be a mapping defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in C$, and $\operatorname{Fix}(S) = \bigcap_{n=0}^{\infty} \operatorname{Fix}(S_n)$. Then the sequence $\{x_n\}$ converges strongly to $q \in F$, which solves the following VIP:

$$\langle q - f(q), J(q - p)\rangle \leq 0, \quad \forall p \in F.$$

Proof First of all, let us show that the sequence $\{x_n\}$ is bounded. Indeed, take a fixed $p \in F$ arbitrarily. Then we get $p = Gp$, $p = S_n p$ and $p = J_{r_n} p$ for all $n \geq 0$. By Lemma 2.11, G is nonexpansive. Then, from (5.1), we have

$$\begin{aligned} \|y_n - p\| &\leq \alpha_n \|f(y_n) - p\| + \beta_n \|x_n - p\| + \gamma_n \|S_n x_n - p\| + \delta_n \|J_{r_n} G x_n - p\| \\ &\leq \alpha_n \big(\|f(y_n) - f(p)\| + \|f(p) - p\|\big) + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \delta_n \|G x_n - p\| \\ &\leq \alpha_n \big(\rho \|y_n - p\| + \|f(p) - p\|\big) + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \delta_n \|x_n - p\| \\ &= (1 - \alpha_n)\|x_n - p\| + \alpha_n \rho \|y_n - p\| + \alpha_n \|f(p) - p\|, \end{aligned}$$

which implies that

$$\|y_n - p\| \leq \left(1 - \frac{(1-\rho)\alpha_n}{1 - \alpha_n \rho}\right)\|x_n - p\| + \frac{\alpha_n}{1 - \alpha_n \rho}\|f(p) - p\|. \tag{5.2}$$

So, we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \sigma_n \|y_n - p\| + (1 - \sigma_n)\|J_{r_n} G y_n - p\| \\ &\leq \sigma_n \|y_n - p\| + (1 - \sigma_n)\|G y_n - p\| \end{aligned}$$

$$
\begin{aligned}
&\le \sigma_n\|y_n - p\| + (1-\sigma_n)\|y_n - p\| \\
&= \|y_n - p\| \\
&\le \left(1 - \frac{(1-\rho)\alpha_n}{1-\alpha_n\rho}\right)\|x_n - p\| + \frac{\alpha_n}{1-\alpha_n\rho}\big\|f(p) - p\big\| \\
&= \left(1 - \frac{(1-\rho)\alpha_n}{1-\alpha_n\rho}\right)\|x_n - p\| + \frac{(1-\rho)\alpha_n}{1-\alpha_n\rho}\frac{\|f(p) - p\|}{1-\rho} \\
&\le \max\left\{\|x_n - p\|, \frac{\|f(p) - p\|}{1-\rho}\right\}.
\end{aligned}
$$

By induction, we obtain

$$
\|x_n - p\| \le \max\left\{\|x_0 - p\|, \frac{\|f(p) - p\|}{1-\rho}\right\}, \quad \forall n \ge 0. \tag{5.3}
$$

Hence, $\{x_n\}$ is bounded, and so are the sequences $\{y_n\}$, $\{Gx_n\}$, $\{Gy_n\}$, and $\{f(y_n)\}$.

Let us show that

$$
\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{5.4}
$$

Observe that y_n can be rewritten as

$$
y_n = \beta_n x_n + (1-\beta_n)z_n,
$$

where $z_n = \frac{\alpha_n f(y_n)+\gamma_n S_n x_n+\delta_n J_{r_n} G x_n}{1-\beta_n}$. Note that

$$
\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\quad = \left\|\frac{\alpha_n f(y_n) + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{1-\beta_n} - \frac{\alpha_{n-1} f(y_{n-1}) + \gamma_{n-1} S_{n-1} x_{n-1} + \delta_{n-1} J_{r_{n-1}} G x_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad = \left\|\frac{y_n - \beta_n x_n}{1-\beta_n} - \frac{y_{n-1} - \beta_{n-1} x_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad = \left\|\frac{y_n - \beta_n x_n}{1-\beta_n} - \frac{y_{n-1} - \beta_{n-1} x_{n-1}}{1-\beta_n} + \frac{y_{n-1} - \beta_{n-1} x_{n-1}}{1-\beta_n} - \frac{y_{n-1} - \beta_{n-1} x_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad \le \left\|\frac{y_n - \beta_n x_n}{1-\beta_n} - \frac{y_{n-1} - \beta_{n-1} x_{n-1}}{1-\beta_n}\right\| + \left\|\frac{y_{n-1} - \beta_{n-1} x_{n-1}}{1-\beta_n} - \frac{y_{n-1} - \beta_{n-1} x_{n-1}}{1-\beta_{n-1}}\right\| \\
&\quad = \frac{1}{1-\beta_n}\big\|y_n - \beta_n x_n - (y_{n-1} - \beta_{n-1} x_{n-1})\big\| + \left|\frac{1}{1-\beta_n} - \frac{1}{1-\beta_{n-1}}\right|\|y_{n-1} - \beta_{n-1} x_{n-1}\| \\
&\quad = \frac{1}{1-\beta_n}\big\|y_n - \beta_n x_n - (y_{n-1} - \beta_{n-1} x_{n-1})\big\| + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|y_{n-1} - \beta_{n-1} x_{n-1}\| \\
&\quad = \frac{1}{1-\beta_n}\big\|\alpha_n f(y_n) + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n - \alpha_{n-1} f(y_{n-1}) - \gamma_{n-1} S_{n-1} x_{n-1} - \delta_{n-1} J_{r_{n-1}} G x_{n-1}\big\| \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|y_{n-1} - \beta_{n-1} x_{n-1}\| \\
&\quad \le \frac{1}{1-\beta_n}\big[\alpha_n\big\|f(y_n) - f(y_{n-1})\big\| + \gamma_n\|S_n x_n - S_{n-1} x_{n-1}\| + \delta_n\|J_{r_n} G x_n - J_{r_{n-1}} G x_{n-1}\|
\end{aligned}
$$

$$\begin{aligned}&+ |\alpha_n - \alpha_{n-1}| \|f(y_{n-1})\| + |\gamma_n - \gamma_{n-1}| \|S_{n-1}x_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} Gx_{n-1}\|\big] \\ &+ \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|y_{n-1} - \beta_{n-1}x_{n-1}\|. \end{aligned} \tag{5.5}$$

On the other hand, if $r_{n-1} \le r_n$, using the resolvent identity in Proposition 2.2,

$$J_{r_n}x_n = J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}x_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}x_n\right),$$

we get

$$\begin{aligned}\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| &= \left\|J_{r_{n-1}}\left(\frac{r_{n-1}}{r_n}Gx_n + \left(1 - \frac{r_{n-1}}{r_n}\right)J_{r_n}Gx_n\right) - J_{r_{n-1}}Gx_{n-1}\right\| \\ &\le \frac{r_{n-1}}{r_n}\|Gx_n - Gx_{n-1}\| + \left(1 - \frac{r_{n-1}}{r_n}\right)\|J_{r_n}Gx_n - Gx_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + \frac{r_n - r_{n-1}}{r_n}\|J_{r_n}Gx_n - Gx_{n-1}\| \\ &\le \|x_n - x_{n-1}\| + \frac{1}{\varepsilon}|r_n - r_{n-1}|\|J_{r_n}Gx_n - Gx_{n-1}\|.\end{aligned}$$

If $r_n \le r_{n-1}$, then it is easy to see that

$$\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \le \|x_{n-1} - x_n\| + \frac{1}{\varepsilon}|r_{n-1} - r_n|\|J_{r_{n-1}}Gx_{n-1} - Gx_n\|.$$

Thus, combining the above cases, we obtain

$$\begin{aligned}&\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \\ &\quad \le \|x_{n-1} - x_n\| + \frac{|r_{n-1} - r_n|}{\varepsilon}\sup_{n\ge1}\big\{\|J_{r_n}Gx_n - Gx_{n-1}\| + \|J_{r_{n-1}}Gx_{n-1} - Gx_n\|\big\}, \quad \forall n \ge 1.\end{aligned}$$

In a similar way, we derive

$$\begin{aligned}&\|J_{r_n}Gy_n - J_{r_{n-1}}Gy_{n-1}\| \\ &\quad \le \|y_{n-1} - y_n\| + \frac{|r_{n-1} - r_n|}{\varepsilon}\sup_{n\ge1}\big\{\|J_{r_n}Gy_n - Gy_{n-1}\| + \|J_{r_{n-1}}Gy_{n-1} - Gy_n\|\big\}, \quad \forall n \ge 1.\end{aligned}$$

Therefore, we have

$$\begin{cases}\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \le \|x_{n-1} - x_n\| + |r_{n-1} - r_n|M_0, \\ \|J_{r_n}Gy_n - J_{r_{n-1}}Gy_{n-1}\| \le \|y_{n-1} - y_n\| + |r_{n-1} - r_n|M_0,\end{cases} \tag{5.6}$$

for all $n \ge 1$, where

$$\sup_{n\ge1}\left\{\frac{1}{\varepsilon}\big(\|J_{r_n}Gx_n - Gx_{n-1}\| + \|J_{r_{n-1}}Gx_{n-1} - Gx_n\|\big)\right\} \le M_0,$$

and

$$\sup_{n\ge1}\left\{\frac{1}{\varepsilon}\big(\|J_{r_n}Gy_n - Gy_{n-1}\| + \|J_{r_{n-1}}Gy_{n-1} - Gy_n\|\big)\right\} \le M_0,$$

for some $M_0 > 0$. Combining (5.6) and (5.5), we have

$$\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\quad \le \frac{1}{1-\beta_n}\Big[\alpha_n \big\|f(y_n) - f(y_{n-1})\big\| + \gamma_n \|S_n x_n - S_{n-1}x_{n-1}\| \\
&\qquad + \delta_n\big(\|x_{n-1} - x_n\| + M_0|r_{n-1} - r_n|\big) + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| + |\gamma_n - \gamma_{n-1}|\|S_{n-1}x_{n-1}\| \\
&\qquad + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gx_{n-1}\|\Big] + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|y_{n-1} - \beta_{n-1}x_{n-1}\| \\
&\quad \le \frac{1}{1-\beta_n}\Big[\alpha_n\rho\|y_n - y_{n-1}\| + \gamma_n\|S_n x_n - S_n x_{n-1}\| + \delta_n\big(\|x_{n-1} - x_n\| + M_0|r_{n-1} - r_n|\big) \\
&\qquad + |\alpha_n - \alpha_{n-1}|\big\|f(y_{n-1})\big\| + |\gamma_n - \gamma_{n-1}|\|S_{n-1}x_{n-1}\| + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gx_{n-1}\| \\
&\qquad + \gamma_n\|S_n x_{n-1} - S_{n-1}x_{n-1}\|\Big] \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\big\|\alpha_{n-1}f(y_{n-1}) + \gamma_{n-1}S_{n-1}x_{n-1} + \delta_{n-1}J_{r_{n-1}}Gx_{n-1}\big\| \\
&\quad \le \frac{1}{1-\beta_n}\Big[\alpha_n\rho\|y_n - y_{n-1}\| + (\gamma_n + \delta_n)\|x_{n-1} - x_n\| + M\big(|\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}| \\
&\qquad + |\delta_n - \delta_{n-1}| + |r_{n-1} - r_n|\big) + \gamma_n\|S_n x_{n-1} - S_{n-1}x_{n-1}\|\Big] \\
&\qquad + \frac{1}{(1-\beta_{n-1})(1-\beta_n)}|\beta_n - \beta_{n-1}|M,
\end{aligned} \tag{5.7}$$

where $\sup_{n\ge 0}\{M_0 + \|f(y_n)\| + \|S_n x_n\| + \|J_{r_n}Gx_n\|\} \le M$ for some $M > 0$. By simple calculations, we have

$$y_n - y_{n-1} = \beta_n(x_n - x_{n-1}) + (1-\beta_n)(z_n - z_{n-1}) + (\beta_n - \beta_{n-1})(x_{n-1} - z_{n-1}). \tag{5.8}$$

Taking into account condition (v), without loss of generality, we may assume that $\{\beta_n\} \subset [a,b]$ for some $a, b \in (0,1)$. Hence, from (5.7) and (5.8), we deduce

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad \le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| + |\beta_n - \beta_{n-1}|\|x_{n-1} - z_{n-1}\| \\
&\quad \le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\bigg\{\frac{1}{1-\beta_n}\Big[\alpha_n\rho\|y_n - y_{n-1}\| + (\gamma_n + \delta_n)\|x_{n-1} - x_n\| \\
&\qquad + M\big(|\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}| + |r_{n-1} - r_n|\big) + \gamma_n\|S_n x_{n-1} - S_{n-1}x_{n-1}\|\Big] \\
&\qquad + \frac{1}{(1-\beta_{n-1})(1-\beta_n)}|\beta_n - \beta_{n-1}|M\bigg\} + |\beta_n - \beta_{n-1}|\|x_{n-1} - z_{n-1}\| \\
&\quad = (1-\alpha_n)\|x_{n-1} - x_n\| + \alpha_n\rho\|y_n - y_{n-1}\| + M\big(|\alpha_n - \alpha_{n-1}| + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}| \\
&\qquad + |r_{n-1} - r_n|\big) + \gamma_n\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\
&\qquad + \frac{1}{1-\beta_{n-1}}|\beta_n - \beta_{n-1}|M + |\beta_n - \beta_{n-1}|\|x_{n-1} - z_{n-1}\| \\
&\quad \le (1-\alpha_n)\|x_{n-1} - x_n\| + \alpha_n\rho\|y_n - y_{n-1}\| + M_1\big(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \\
&\qquad + |\delta_n - \delta_{n-1}| + |r_{n-1} - r_n|\big) + \|S_n x_{n-1} - S_{n-1}x_{n-1}\|,
\end{aligned}$$

where $\sup_{n\geq 0}\{\frac{M}{1-b} + \|x_n - z_n\|\} \leq M_1$ for some $M_1 > 0$. This leads to

$$\begin{aligned}
&\|y_n - y_{n-1}\| \\
&\quad\leq \left(1 - \frac{(1-\rho)\alpha_n}{1-\alpha_n\rho}\right)\|x_{n-1} - x_n\| + \frac{M_1}{1-\alpha_n\rho}\big(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}| \\
&\qquad + |\delta_n - \delta_{n-1}| + |r_{n-1} - r_n|\big) + \frac{1}{1-\alpha_n\rho}\|S_n x_{n-1} - S_{n-1}x_{n-1}\|.
\end{aligned} \tag{5.9}$$

Again by simple calculations, we have

$$\begin{aligned}
x_{n+1} - x_n &= \sigma_n(y_n - y_{n-1}) + (\sigma_n - \sigma_{n-1})(y_{n-1} - J_{r_{n-1}}Gy_{n-1}) \\
&\quad + (1-\sigma_n)(J_{r_n}Gy_n - J_{r_{n-1}}Gy_{n-1}).
\end{aligned}$$

This together with (5.6) and (5.9) implies that

$$\begin{aligned}
\|x_{n+1} - x_n\| &\leq \sigma_n\|y_n - y_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|y_{n-1} - J_{r_{n-1}}Gy_{n-1}\| \\
&\quad + (1-\sigma_n)\|J_{r_n}Gy_n - J_{r_{n-1}}Gy_{n-1}\| \\
&\leq \sigma_n\|y_n - y_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|y_{n-1} - J_{r_{n-1}}Gy_{n-1}\| \\
&\quad + (1-\sigma_n)\big(\|y_{n-1} - y_n\| + |r_{n-1} - r_n|M_0\big) \\
&\leq \|y_n - y_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|y_{n-1} - J_{r_{n-1}}Gy_{n-1}\| + |r_{n-1} - r_n|M_0 \\
&\leq \left(1 - \frac{(1-\rho)\alpha_n}{1-\alpha_n\rho}\right)\|x_{n-1} - x_n\| + \frac{M_1}{1-\alpha_n\rho}\big(|\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\
&\quad + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}| + |r_{n-1} - r_n|\big) + \frac{1}{1-\alpha_n\rho}\|S_n x_{n-1} - S_{n-1}x_{n-1}\| \\
&\quad + |\sigma_n - \sigma_{n-1}|\|y_{n-1} - J_{r_{n-1}}Gy_{n-1}\| + |r_{n-1} - r_n|M_0 \\
&\leq \left(1 - \frac{(1-\rho)\alpha_n}{1-\alpha_n\rho}\right)\|x_{n-1} - x_n\| + \widetilde{M}\big(|\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\
&\quad + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}| + |r_{n-1} - r_n| + \|S_n x_{n-1} - S_{n-1}x_{n-1}\|\big),
\end{aligned}$$

where $\sup_{n\geq 0}\{\frac{M_1+1}{1-\alpha_n\rho} + M_0 + \|y_n - J_{r_n}Gy_n\|\} \leq \widetilde{M}$ for some $\widetilde{M} > 0$. Noting that $\frac{(1-\rho)\alpha_n}{1-\alpha_n\rho} \geq (1-\rho)\alpha_n$ for all $n \geq 0$, from condition (i), we know that $\sum_{n=0}^{\infty}\frac{(1-\rho)\alpha_n}{1-\alpha_n\rho} = \infty$. Utilizing Lemma 2.7, we conclude from conditions (iii), (iv), and the assumption on $\{S_n\}$ that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Next we show that $\|x_n - Gx_n\| \to 0$ as $n \to \infty$.

Indeed, according to Lemma 2.2(a), we have from (5.1)

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad= \big\|\alpha_n\big(f(y_n) - f(p)\big) + \beta_n(x_n - p) + \gamma_n(S_n x_n - p) + \delta_n(J_{r_n}Gx_n - p) + \alpha_n\big(f(p) - p\big)\big\|^2 \\
&\quad\leq \big\|\alpha_n\big(f(y_n) - f(p)\big) + \beta_n(x_n - p) + \gamma_n(S_n x_n - p) + \delta_n(J_{r_n}Gx_n - p)\big\|^2 \\
&\qquad + 2\alpha_n\big\langle f(p) - p, J(y_n - p)\big\rangle
\end{aligned}$$

$$\begin{aligned}
&\le \alpha_n \|f(y_n) - f(p)\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|S_n x_n - p\|^2 + \delta_n \|J_{r_n} G x_n - p\|^2 \\
&\quad + 2\alpha_n \langle f(p) - p, J(y_n - p)\rangle \\
&\le \alpha_n \rho^2 \|y_n - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2 + \delta_n \|G x_n - p\|^2 \\
&\quad + 2\alpha_n \langle f(p) - p, J(y_n - p)\rangle \\
&\le \alpha_n \rho \|y_n - p\|^2 + \beta_n \|x_n - p\|^2 + \gamma_n \|x_n - p\|^2 + \delta_n \|x_n - p\|^2 \\
&\quad + 2\alpha_n \langle f(p) - p, J(y_n - p)\rangle \\
&= \alpha_n \rho \|y_n - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 + 2\alpha_n \langle f(p) - p, J(y_n - p)\rangle,
\end{aligned} \tag{5.10}$$

which implies that

$$\|y_n - p\|^2 \le \left(1 - \frac{(1-\rho)\alpha_n}{1 - \alpha_n \rho}\right)\|x_n - p\|^2 + \frac{2\alpha_n}{1 - \alpha_n \rho}\langle f(p) - p, J(y_n - p)\rangle.$$

Utilizing Lemma 2.3, we get from (5.1) and (5.10)

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\sigma_n (y_n - p) + (1 - \sigma_n)(J_{r_n} G y_n - p)\|^2 \\
&\le \sigma_n \|y_n - p\|^2 + (1 - \sigma_n)\|J_{r_n} G y_n - p\|^2 - \sigma_n (1 - \sigma_n) g(\|y_n - J_{r_n} G y_n\|) \\
&\le \sigma_n \|y_n - p\|^2 + (1 - \sigma_n)\|y_n - p\|^2 - \sigma_n (1 - \sigma_n) g(\|y_n - J_{r_n} G y_n\|) \\
&= \|y_n - p\|^2 - \sigma_n (1 - \sigma) g(\|y_n - J_{r_n} G y_n\|) \\
&\le \left(1 - \frac{(1-\rho)\alpha_n}{1 - \alpha_n \rho}\right)\|x_n - p\|^2 + \frac{2\alpha_n}{1 - \alpha_n \rho}\langle f(p) - p, J(y_n - p)\rangle \\
&\quad - \sigma_n (1 - \sigma_n) g(\|y_n - J_{r_n} G y_n\|) \\
&\le \|x_n - p\|^2 + \frac{2\alpha_n}{1 - \alpha_n \rho}\|f(p) - p\| \|y_n - p\| - \sigma_n (1 - \sigma_n) g(\|y_n - J_{r_n} G y_n\|),
\end{aligned}$$

and hence

$$\begin{aligned}
&\sigma_n (1 - \sigma_n) g(\|y_n - J_{r_n} G y_n\|) \\
&\quad \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \frac{2\alpha_n}{1 - \alpha_n \rho}\|f(p) - p\| \|y_n - p\| \\
&\quad \le (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + \frac{2\alpha_n}{1 - \alpha_n \rho}\|f(p) - p\| \|y_n - p\|.
\end{aligned}$$

Since $\alpha_n \to 0$ and $\|x_{n+1} - x_n\| \to 0$, from condition (v) and the boundedness of $\{x_n\}$ and $\{y_n\}$, it follows that

$$\lim_{n\to\infty} g(\|y_n - J_{r_n} G y_n\|) = 0.$$

Utilizing the properties of g, we have

$$\lim_{n\to\infty} \|y_n - J_{r_n} G y_n\| = 0. \tag{5.11}$$

Observe that

$$\begin{aligned}\|x_n - y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &= \|x_n - x_{n+1}\| + (1-\sigma_n)\|J_{r_n}Gy_n - y_n\| \\ &\le \|x_n - x_{n+1}\| + \|J_{r_n}Gy_n - y_n\|.\end{aligned}$$

From (5.4) and (5.11), we have

$$\lim_{n\to\infty}\|x_n - y_n\| = 0. \tag{5.12}$$

For simplicity, put $q = \Pi_C(p - \mu_2 B_2 p)$, $u_n = \Pi_C(x_n - \mu_2 B_2 x_n)$ and $v_n = \Pi_C(u_n - \mu_1 B_1 u_n)$. Then $v_n = Gx_n$ for all $n \ge 0$. From Lemma 2.8, we have

$$\begin{aligned}\|u_n - q\|^2 &= \big\|\Pi_C(x_n - \mu_2 B_2 x_n) - \Pi_C(p - \mu_2 B_2 p)\big\|^2 \\ &\le \big\|x_n - p - \mu_2(B_2 x_n - B_2 p)\big\|^2 \\ &\le \|x_n - p\|^2 - 2\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2,\end{aligned} \tag{5.13}$$

and

$$\begin{aligned}\|v_n - p\|^2 &= \big\|\Pi_C(u_n - \mu_1 B_1 u_n) - \Pi_C(q - \mu_1 B_1 q)\big\|^2 \\ &\le \big\|u_n - q - \mu_1(B_1 u_n - B_1 q)\big\|^2 \\ &\le \|u_n - q\|^2 - 2\mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2.\end{aligned} \tag{5.14}$$

Combining (5.13) and (5.14), we obtain

$$\begin{aligned}\|v_n - p\|^2 \le{}& \|x_n - p\|^2 - 2\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 \\ &- 2\mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2.\end{aligned} \tag{5.15}$$

By Lemma 2.2(a), (5.1), and (5.15), we have

$$\begin{aligned}&\|y_n - p\|^2 \\ &\quad= \big\|\alpha_n\big(f(y_n) - f(p)\big) + \beta_n(x_n - p) + \gamma_n(S_n x_n - p) + \delta_n(J_{r_n}Gx_n - p) + \alpha_n\big(f(p) - p\big)\big\|^2 \\ &\quad\le \alpha_n\big\|f(y_n) - f(p)\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + \delta_n\|J_{r_n}Gx_n - p\|^2 \\ &\qquad+ 2\alpha_n\big\langle f(p) - p, J(y_n - p)\big\rangle \\ &\quad\le \alpha_n\rho^2\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|v_n - p\|^2 \\ &\qquad+ 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\| \\ &\quad\le \alpha_n\rho\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\big[\|x_n - p\|^2 \\ &\qquad- 2\mu_2\big(\alpha_2 - \kappa^2\mu_2\big)\|B_2 x_n - B_2 p\|^2 - 2\mu_1\big(\alpha_1 - \kappa^2\mu_1\big)\|B_1 u_n - B_1 q\|^2\big] \\ &\qquad+ 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\|\end{aligned}$$

$$\leq \alpha_n\|y_n - p\|^2 + (1-\alpha_n)\|x_n - p\|^2 - 2\delta_n\big[\mu_2(\alpha_2 - \kappa^2\mu_2)\|B_2x_n - B_2p\|^2 + \mu_1(\alpha_1 - \kappa^2\mu_1)\|B_1u_n - B_1q\|^2\big] + 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\|.$$

Thus, we have

$$\begin{aligned}
&2\delta_n\big[\mu_2(\alpha_2 - \kappa^2\mu_2)\|B_2x_n - B_2p\|^2 + \mu_1(\alpha_1 - \kappa^2\mu_1)\|B_1u_n - B_1q\|^2\big] \\
&\quad\leq (1-\alpha_n)\|x_n - p\|^2 - (1-\alpha_n)\|y_n - p\|^2 + 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\| \\
&\quad\leq (1-\alpha_n)\big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\| + 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\|.
\end{aligned}$$

Since $0 < \mu_i < \frac{\alpha_i}{\kappa^2}$ for $i = 1,2$, from (5.12) and conditions (i), (ii), we obtain

$$\lim_{n\to\infty}\|B_2x_n - B_2p\| = 0 \quad\text{and}\quad \lim_{n\to\infty}\|B_1u_n - B_1q\| = 0. \tag{5.16}$$

Utilizing Proposition 2.2 and Lemma 2.10, we have

$$\begin{aligned}
\|u_n - q\|^2 &= \big\|\Pi_C(x_n - \mu_2B_2x_n) - \Pi_C(p - \mu_2B_2p)\big\|^2 \\
&\leq \big\langle x_n - \mu_2B_2x_n - (p - \mu_2B_2p), J(u_n - q)\big\rangle \\
&= \big\langle x_n - p, J(u_n - q)\big\rangle + \mu_2\big\langle B_2p - B_2x_n, J(u_n - q)\big\rangle \\
&\leq \frac{1}{2}\big[\|x_n - p\|^2 + \|u_n - q\|^2 - g_1\big(\big\|x_n - u_n - (p - q)\big\|\big)\big] \\
&\quad + \mu_2\|B_2p - B_2x_n\|\|u_n - q\|,
\end{aligned}$$

which implies that

$$\|u_n - q\|^2 \leq \|x_n - p\|^2 - g_1\big(\big\|x_n - u_n - (p - q)\big\|\big) + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\|. \tag{5.17}$$

In the same way, we derive

$$\begin{aligned}
\|v_n - p\|^2 &= \big\|\Pi_C(u_n - \mu_1B_1u_n) - \Pi_C(q - \mu_1B_1q)\big\|^2 \\
&\leq \big\langle u_n - \mu_1B_1u_n - (q - \mu_1B_1q), J(v_n - p)\big\rangle \\
&= \big\langle u_n - q, J(v_n - p)\big\rangle + \mu_1\big\langle B_1q - B_1u_n, J(v_n - p)\big\rangle \\
&\leq \frac{1}{2}\big[\|u_n - q\|^2 + \|v_n - p\|^2 - g_2\big(\big\|u_n - v_n + (p - q)\big\|\big)\big] \\
&\quad + \mu_1\|B_1q - B_1u_n\|\|v_n - p\|,
\end{aligned}$$

which implies that

$$\|v_n - p\|^2 \leq \|u_n - q\|^2 - g_2\big(\big\|u_n - v_n + (p - q)\big\|\big) + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|. \tag{5.18}$$

Combining (5.17) and (5.18), we get

$$\begin{aligned}
\|v_n - p\|^2 &\leq \|x_n - p\|^2 - g_1\big(\big\|x_n - u_n - (p - q)\big\|\big) - g_2\big(\big\|u_n - v_n + (p - q)\big\|\big) \\
&\quad + 2\mu_2\|B_2p - B_2x_n\|\|u_n - q\| + 2\mu_1\|B_1q - B_1u_n\|\|v_n - p\|.
\end{aligned} \tag{5.19}$$

By Lemma 2.2(a), (5.1), and (5.19), we have

$$
\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \big\|\alpha_n\big(f(y_n) - f(p)\big) + \beta_n(x_n - p) + \gamma_n(S_n x_n - p) + \delta_n(J_{r_n} G x_n - p) + \alpha_n\big(f(p) - p\big)\big\|^2 \\
&\quad \le \alpha_n\big\|f(y_n) - f(p)\big\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + \delta_n\|J_{r_n} G x_n - p\|^2 \\
&\qquad + 2\alpha_n\big\langle f(p) - p, J(y_n - p)\big\rangle \\
&\quad \le \alpha_n\rho\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\|v_n - p\|^2 + 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\| \\
&\quad \le \alpha_n\rho\|y_n - p\|^2 + \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + \delta_n\big[\|x_n - p\|^2 \\
&\qquad - g_1\big(\big\|x_n - u_n - (p - q)\big\|\big) - g_2\big(\big\|u_n - v_n + (p - q)\big\|\big) + 2\mu_2\|B_2 p - B_2 x_n\|\|u_n - q\| \\
&\qquad + 2\mu_1\|B_1 q - B_1 u_n\|\|v_n - p\|\big] + 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\| \\
&\quad \le \alpha_n\|y_n - p\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \delta_n\big[g_1\big(\big\|x_n - u_n - (p - q)\big\|\big) \\
&\qquad + g_2\big(\big\|u_n - v_n + (p - q)\big\|\big)\big] + 2\mu_2\|B_2 p - B_2 x_n\|\|u_n - q\| \\
&\qquad + 2\mu_1\|B_1 q - B_1 u_n\|\|v_n - p\| + 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\|,
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\delta_n\big[g_1\big(\big\|x_n - u_n - (p - q)\big\|\big) + g_2\big(\big\|u_n - v_n + (p - q)\big\|\big)\big] \\
&\quad \le (1 - \alpha_n)\|x_n - p\|^2 - (1 - \alpha_n)\|y_n - p\|^2 + 2\mu_2\|B_2 p - B_2 x_n\|\|u_n - q\| \\
&\qquad + 2\mu_1\|B_1 q - B_1 u_n\|\|v_n - p\| + 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\| \\
&\quad \le (1 - \alpha_n)\big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\| + 2\mu_2\|B_2 p - B_2 x_n\|\|u_n - q\| \\
&\qquad + 2\mu_1\|B_1 q - B_1 u_n\|\|v_n - p\| + 2\alpha_n\big\|f(p) - p\big\|\|y_n - p\|.
\end{aligned}
$$

Utilizing conditions (i), (ii), from (5.12) and (5.16), we have

$$
\lim_{n\to\infty} g_1\big(\big\|x_n - u_n - (p - q)\big\|\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} g_2\big(\big\|u_n - v_n + (p - q)\big\|\big) = 0. \tag{5.20}
$$

Utilizing the properties of g_1 and g_2, we have

$$
\lim_{n\to\infty} \big\|x_n - u_n - (p - q)\big\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \big\|u_n - v_n + (p - q)\big\| = 0. \tag{5.21}
$$

From (5.21), we get

$$
\|x_n - v_n\| \le \big\|x_n - u_n - (p - q)\big\| + \big\|u_n - v_n + (p - q)\big\| \to 0 \quad \text{as } n \to \infty,
$$

that is,

$$
\lim_{n\to\infty} \|x_n - G x_n\| = 0. \tag{5.22}
$$

Next, let us show that

$$
\lim_{n\to\infty} \|J_{r_n} x_n - x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|S_n x_n - x_n\| = 0.
$$

Indeed, observe that y_n can be rewritten as

$$\begin{aligned} y_n &= \alpha_n f(y_n) + \beta_n x_n + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n \\ &= \alpha_n f(y_n) + \beta_n x_n + (\gamma_n + \delta_n) \frac{\gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{\gamma_n + \delta_n} \\ &= \alpha_n f(y_n) + \beta_n x_n + e_n \hat{z}_n, \end{aligned} \tag{5.23}$$

where $e_n = \gamma_n + \delta_n$ and $\hat{z}_n = \frac{\gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{\gamma_n + \delta_n}$. Utilizing Lemma 2.4 and (5.23), we have

$$\begin{aligned} &\|y_n - p\|^2 \\ &\quad = \big\|\alpha_n\big(f(y_n) - p\big) + \beta_n(x_n - p) + e_n(\hat{z}_n - p)\big\|^2 \\ &\quad \le \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 + e_n\|\hat{z}_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\quad = \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\qquad + e_n \left\| \frac{\gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{\gamma_n + \delta_n} - p \right\|^2 \\ &\quad = \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\qquad + e_n \left\| \frac{\gamma_n}{\gamma_n + \delta_n}(S_n x_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G x_n - p) \right\|^2 \\ &\quad \le \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\qquad + e_n \left[\frac{\gamma_n}{\gamma_n + \delta_n}\|S_n x_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G x_n - p\|^2 \right] \\ &\quad \le \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\qquad + e_n \left[\frac{\gamma_n}{\gamma_n + \delta_n}\|x_n - p\| + \frac{\delta_n}{\gamma_n + \delta_n}\|x_n - p\|^2 \right] \\ &\quad = \alpha_n \big\|f(y_n) - p\big\|^2 + (1 - \alpha_n)\|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) \\ &\quad \le \alpha_n \big\|f(y_n) - p\big\|^2 + \|x_n - p\|^2 - \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big), \end{aligned}$$

which implies that

$$\begin{aligned} \beta_n e_n g_3\big(\|\hat{z}_n - x_n\|\big) &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\ &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\|. \end{aligned}$$

Utilizing (5.12), conditions (i), (ii), (v), and the boundedness of $\{x_n\}$, $\{y_n\}$, and $\{f(y_n)\}$, we get

$$\lim_{n\to\infty} g_3\big(\|\hat{z}_n - x_n\|\big) = 0.$$

From the properties of g_3, we have

$$\lim_{n\to\infty} \|\hat{z}_n - x_n\| = 0.$$

Utilizing Lemma 2.3 and the definition of $\hat{z}_n$, we have

$$\begin{aligned}
\|\hat{z}_n - p\|^2 &= \left\| \frac{\gamma_n S_n x_n + \delta_n J_{r_n} G x_n}{\gamma_n + \delta_n} - p \right\|^2 \\
&= \left\| \frac{\gamma_n}{\gamma_n + \delta_n}(S_n x_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G x_n - p) \right\|^2 \\
&\le \frac{\gamma_n}{\gamma_n + \delta_n}\|S_n x_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G x_n - p\|^2 \\
&\quad - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_4\bigl(\|J_{r_n} G x_n - S_n x_n\|\bigr) \\
&\le \|x_n - p\|^2 - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_4\bigl(\|J_{r_n} G x_n - S_n x_n\|\bigr),
\end{aligned}$$

which leads to

$$\begin{aligned}
\frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_4\bigl(\|J_{r_n} G x_n - S_n x_n\|\bigr) &\le \|x_n - p\|^2 - \|\hat{z}_n - p\|^2 \\
&\le \bigl(\|x_n - p\| + \|\hat{z}_n - p\|\bigr)\|x_n - \hat{z}_n\|.
\end{aligned}$$

Since $\{x_n\}$ and $\{\hat{z}_n\}$ are bounded and $\|\hat{z}_n - x_n\| \to 0$ as $n \to \infty$, we deduce from condition (ii) that

$$\lim_{n\to\infty} g_4\bigl(\|S_n x_n - J_{r_n} G x_n\|\bigr) = 0.$$

From the properties of g_4, we have

$$\lim_{n\to\infty} \|S_n x_n - J_{r_n} G x_n\| = 0. \tag{5.24}$$

On the other hand, y_n can also be rewritten as

$$\begin{aligned}
y_n &= \alpha_n f(y_n) + \beta_n x_n + \gamma_n S_n x_n + \delta_n J_{r_n} G x_n \\
&= \beta_n x_n + \gamma_n S_n x_n + (\alpha_n + \delta_n)\frac{\alpha_n f(y_n) + \delta_n J_{r_n} G x_n}{\alpha_n + \delta_n} = \beta_n x_n + \gamma_n S_n x_n + d_n \tilde{z}_n,
\end{aligned}$$

where $d_n = \alpha_n + \delta_n$ and $\tilde{z}_n = \frac{\alpha_n f(y_n) + \delta_n J_{r_n} G x_n}{\alpha_n + \delta_n}$. Utilizing Lemma 2.4 and the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}
&\|y_n - p\|^2 \\
&\quad = \bigl\|\beta_n(x_n - p) + \gamma_n(S_n x_n - p) + d_n(\tilde{z}_n - p)\bigr\|^2 \\
&\quad \le \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + d_n\|\tilde{z}_n - p\|^2 - \beta_n\gamma_n g_5\bigl(\|x_n - S_n x_n\|\bigr) \\
&\quad = \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + d_n\left\|\frac{\alpha_n f(y_n) + \delta_n J_{r_n} G x_n}{\alpha_n + \delta_n} - p\right\|^2 \\
&\qquad - \beta_n\gamma_n g_5\bigl(\|x_n - S_n x_n\|\bigr) \\
&\quad = \beta_n\|x_n - p\|^2 + \gamma_n\|S_n x_n - p\|^2 + d_n\left\|\frac{\alpha_n}{\alpha_n + \delta_n}\bigl(f(y_n) - p\bigr) + \frac{\delta_n}{\alpha_n + \delta_n}(J_{r_n} G x_n - p)\right\|^2
\end{aligned}$$

$$
\begin{aligned}
&\quad - \beta_n\gamma_n g_5\big(\|x_n - S_n x_n\|\big) \\
&\le \beta_n\|x_n - p\|^2 + \gamma_n\|x_n - p\|^2 + d_n\left[\frac{\alpha_n}{\alpha_n+\delta_n}\big\|f(y_n) - p\big\|^2 + \frac{\delta_n}{\alpha_n+\delta_n}\|J_{r_n}Gx_n - p\|^2\right] \\
&\quad - \beta_n\gamma_n g_5\big(\|x_n - S_n x_n\|\big) \\
&\le \alpha_n\big\|f(y_n) - p\big\|^2 + (\beta_n+\gamma_n)\|x_n - p\|^2 + \delta_n\|x_n - p\|^2 - \beta_n\gamma_n g_5\big(\|x_n - S_n x_n\|\big) \\
&= \alpha_n\big\|f(y_n) - p\big\|^2 + (1-\alpha_n)\|x_n - p\|^2 - \beta_n\gamma_n g_5\big(\|x_n - S_n x_n\|\big) \\
&\le \alpha_n\big\|f(y_n) - p\big\|^2 + \|x_n - p\|^2 - \beta_n\gamma_n g_5\big(\|x_n - S_n x_n\|\big),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\beta_n\gamma_n g_5\big(\|x_n - S_n x_n\|\big) &\le \alpha_n\big\|f(y_n) - p\big\|^2 + \|x_n - p\|^2 - \|y_n - p\|^2 \\
&\le \alpha_n\big\|f(y_n) - p\big\|^2 + \big(\|x_n - p\| + \|y_n - p\|\big)\|x_n - y_n\|.
\end{aligned}
$$

From (5.12), conditions (i), (ii), (v), and the boundedness of $\{x_n\}$, $\{y_n\}$, and $\{f(y_n)\}$, we have

$$\lim_{n\to\infty} g_5\big(\|x_n - S_n x_n\|\big) = 0.$$

Utilizing the properties of g_5, we have

$$\lim_{n\to\infty} \|x_n - S_n x_n\| = 0. \tag{5.25}$$

By Lemma 5.1, we get

$$\|x_n - Sx_n\| \le \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0 \quad \text{as } n\to\infty,$$

that is,

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{5.26}$$

We note that

$$
\begin{aligned}
\|x_n - J_{r_n}x_n\| &\le \|x_n - S_n x_n\| + \|S_n x_n - J_{r_n}Gx_n\| + \|J_{r_n}Gx_n - J_{r_n}x_n\| \\
&\le \|x_n - S_n x_n\| + \|S_n x_n - J_{r_n}Gx_n\| + \|Gx_n - x_n\|.
\end{aligned}
$$

So, from (5.22), (5.24), and (5.25), it follows that

$$\lim_{n\to\infty} \|x_n - J_{r_n}x_n\| = 0. \tag{5.27}$$

Furthermore, we claim that $\lim_{n\to\infty}\|x_n - J_r x_n\| = 0$ for a fixed number r such that $\varepsilon > r > 0$. In fact, taking into account the resolvent identity in Proposition 2.2, we have

$$
\begin{aligned}
\|J_{r_n}x_n - J_r x_n\| &= \left\|J_r\left(\frac{r}{r_n}x_n + \left(1-\frac{r}{r_n}\right)J_{r_n}x_n\right) - J_r x_n\right\| \\
&\le \left(1-\frac{r}{r_n}\right)\|x_n - J_{r_n}x_n\| \le \|x_n - J_{r_n}x_n\|.
\end{aligned}
\tag{5.28}
$$

From (5.27) and (5.8), we get

$$\begin{aligned}\|x_n - J_r x_n\| &\le \|x_n - J_{r_n} x_n\| + \|J_{r_n} x_n - J_r x_n\| \le \|x_n - J_{r_n} x_n\| + \|x_n - J_{r_n} x_n\| \\ &= 2\|x_n - J_{r_n} x_n\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

that is,

$$\lim_{n\to\infty} \|x_n - J_r x_n\| = 0. \tag{5.29}$$

Define a mapping $Wx = (1 - \theta_1 - \theta_2)J_r x + \theta_1 Sx + \theta_2 Gx$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then by Lemma 2.5, we have $\mathrm{Fix}(W) = \mathrm{Fix}(J_r) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = F$. We observe that

$$\begin{aligned}\|x_n - Wx_n\| &= \big\|(1 - \theta_1 - \theta_2)(x_n - J_r x_n) + \theta_1(x_n - Sx_n) + \theta_2(x_n - Gx_n)\big\| \\ &\le (1 - \theta_1 - \theta_2)\|x_n - J_r x_n\| + \theta_1\|x_n - Sx_n\| + \theta_2\|x_n - Gx_n\|.\end{aligned}$$

From (5.22), (5.26), and (5.29), we obtain

$$\lim_{n\to\infty} \|x_n - Wx_n\| = 0. \tag{5.30}$$

Now, we claim that

$$\limsup_{n\to\infty} \big\langle f(q) - q, J(x_n - q)\big\rangle \le 0, \tag{5.31}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Wx.$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Thus, we have

$$\|x_t - x_n\| = \big\|(1-t)(Wx_t - x_n) + t\big(f(x_t) - x_n\big)\big\|.$$

By Lemma 2.2(a), we obtain

$$\begin{aligned}&\|x_t - x_n\|^2 \\ &\quad = \big\|(1-t)(Wx_t - x_n) + t\big(f(x_t) - x_n\big)\big\|^2 \\ &\quad \le (1-t)^2\|Wx_t - x_n\|^2 + 2t\big\langle f(x_t) - x_n, J(x_t - x_n)\big\rangle \\ &\quad \le (1-t)^2\big(\|Wx_t - Wx_n\| + \|Wx_n - x_n\|\big)^2 + 2t\big\langle f(x_t) - x_n, J(x_t - x_n)\big\rangle \\ &\quad \le (1-t)^2\big(\|x_t - x_n\| + \|Wx_n - x_n\|\big)^2 + 2t\big\langle f(x_t) - x_n, J(x_t - x_n)\big\rangle \\ &\quad = (1-t)^2\big[\|x_t - x_n\|^2 + 2\|x_t - x_n\|\|Wx_n - x_n\| + \|Wx_n - x_n\|^2\big] \\ &\qquad + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\big\langle x_t - x_n, J(x_t - x_n)\big\rangle \\ &\quad = \big(1 - 2t + t^2\big)\|x_t - x_n\|^2 + f_n(t) + 2t\big\langle f(x_t) - x_t, J(x_t - x_n)\big\rangle + 2t\|x_t - x_n\|^2,\end{aligned} \tag{5.32}$$

where

$$f_n(t) = (1-t)^2\bigl(2\|x_t - x_n\| + \|x_n - Wx_n\|\bigr)\|x_n - Wx_n\| \to 0, \quad \text{as } n \to \infty. \tag{5.33}$$

It follows from (5.32) that

$$\bigl\langle x_t - f(x_t), J(x_t - x_n)\bigr\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}f_n(t). \tag{5.34}$$

Letting $n \to \infty$ in (5.34) and noticing (5.33), we derive

$$\limsup_{n\to\infty}\bigl\langle x_t - f(x_t), J(x_t - x_n)\bigr\rangle \le \frac{t}{2}M_2, \tag{5.35}$$

where $M_2 > 0$ is a constant such that $\|x_t - x_n\|^2 \le M_2$ for all $t \in (0,1)$ and $n \ge 0$. Taking $t \to 0$ in (5.35), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\bigl\langle x_t - f(x_t), J(x_t - x_n)\bigr\rangle \le 0. \tag{5.36}$$

On the other hand, we have

$$\begin{aligned}
&\bigl\langle f(q) - q, J(x_n - q)\bigr\rangle \\
&\quad = \bigl\langle f(q) - q, J(x_n - q)\bigr\rangle - \bigl\langle f(q) - q, J(x_n - x_t)\bigr\rangle + \bigl\langle f(q) - q, J(x_n - x_t)\bigr\rangle \\
&\qquad - \bigl\langle f(q) - x_t, J(x_n - x_t)\bigr\rangle + \bigl\langle f(q) - x_t, J(x_n - x_t)\bigr\rangle - \bigl\langle f(x_t) - x_t, J(x_n - x_t)\bigr\rangle \\
&\qquad + \bigl\langle f(x_t) - x_t, J(x_n - x_t)\bigr\rangle \\
&\quad = \bigl\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\bigr\rangle + \bigl\langle x_t - q, J(x_n - x_t)\bigr\rangle \\
&\qquad + \bigl\langle f(q) - f(x_t), J(x_n - x_t)\bigr\rangle + \bigl\langle f(x_t) - x_t, J(x_n - x_t)\bigr\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\limsup_{n\to\infty}\bigl\langle f(q) - q, J(x_n - q)\bigr\rangle \le{}& \limsup_{n\to\infty}\bigl\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\bigr\rangle \\
&+ \|x_t - q\| \limsup_{n\to\infty}\|x_n - x_t\| + \rho\|q - x_t\| \limsup_{n\to\infty}\|x_n - x_t\| \\
&+ \limsup_{n\to\infty}\bigl\langle f(x_t) - x_t, J(x_n - x_t)\bigr\rangle.
\end{aligned}$$

Taking into account that $x_t \to q$ as $t \to 0$, we have from (5.36)

$$\begin{aligned}
\limsup_{n\to\infty}\bigl\langle f(q) - q, J(x_n - q)\bigr\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\bigl\langle f(q) - q, J(x_n - q)\bigr\rangle \\
&\le \limsup_{t\to 0}\limsup_{n\to\infty}\bigl\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\bigr\rangle.
\end{aligned} \tag{5.37}$$

Since X has a uniformly Fréchet differentiable norm, the duality mapping J is norm-to-norm uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable and hence (5.31) holds. From (5.12) we get $(y_n - q) - (x_n - q) \to 0$. Noticing that J is norm-to-norm uniformly continuous on bounded subsets of X, we deduce from (5.31) that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q)-q,J(y_n-q)\rangle \\
&\quad=\limsup_{n\to\infty}\bigl(\langle f(q)-q,J(x_n-q)\rangle+\langle f(q)-q,J(y_n-q)-J(x_n-q)\rangle\bigr) \\
&\quad=\limsup_{n\to\infty}\langle f(q)-q,J(x_n-q)\rangle\le 0.
\end{aligned}$$

Finally, let us show that $x_n\to q$ as $n\to\infty$. We observe that

$$\begin{aligned}
&\|y_n-q\|^2 \\
&\quad=\bigl\|\alpha_n\bigl(f(y_n)-f(q)\bigr)+\beta_n(x_n-q)+\gamma_n(S_nx_n-q)+\delta_n(J_{r_n}Gx_n-q)+\alpha_n\bigl(f(q)-q\bigr)\bigr\|^2 \\
&\quad\le\bigl\|\alpha_n\bigl(f(y_n)-f(q)\bigr)+\beta_n(x_n-q)+\gamma_n(S_nx_n-q)+\delta_n(J_{r_n}Gx_n-q)\bigr\|^2 \\
&\qquad+2\alpha_n\langle f(q)-q,J(y_n-q)\rangle \\
&\quad\le\alpha_n\bigl\|f(y_n)-f(q)\bigr\|^2+\beta_n\|x_n-q\|^2+\gamma_n\|S_nx_n-q\|^2+\delta_n\|J_{r_n}Gx_n-q\|^2 \\
&\qquad+2\alpha_n\langle f(q)-q,J(y_n-q)\rangle \\
&\quad\le\alpha_n\rho\|y_n-q\|^2+(1-\alpha_n)\|x_n-q\|^2+2\alpha_n\langle f(q)-q,J(y_n-q)\rangle,
\end{aligned}$$

which implies that

$$\|y_n-q\|^2\le\left(1-\frac{\alpha_n(1-\rho)}{1-\alpha_n\rho}\right)\|x_n-q\|^2+\frac{\alpha_n(1-\rho)}{1-\alpha_n\rho}\cdot\frac{2\langle f(q)-q,J(y_n-q)\rangle}{1-\rho}. \tag{5.38}$$

From (5.1) and the convexity of $\|\cdot\|^2$, we get

$$\begin{aligned}
\|x_{n+1}-q\|^2&\le\sigma_n\|y_n-q\|^2+(1-\sigma_n)\|J_{r_n}Gy_n-q\|^2 \\
&\le\|y_n-q\|^2 \\
&\le\left(1-\frac{\alpha_n(1-\rho)}{1-\alpha_n\rho}\right)\|x_n-q\|^2+\frac{\alpha_n(1-\rho)}{1-\alpha_n\rho}\cdot\frac{2\langle f(q)-q,J(y_n-q)\rangle}{1-\rho}.
\end{aligned}\tag{5.39}$$

Applying Lemma 2.7 to (5.39), we obtain $x_n\to q$ as $n\to\infty$. This completes the proof. □

Corollary 5.1 *Let C be a nonempty closed convex subset of a uniformly convex and 2-uniformly smooth Banach space X. Let Π_C be a sunny nonexpansive retraction from X onto C and $A\subset X\times X$ be an accretive operator on X such that $\overline{D(A)}\subset C\subset\bigcap_{r>0}R(I+rA)$. Let the mapping $B_i:C\to X$ be α_i-inverse strongly accretive for $i=1,2$, and $f:C\to C$ be a contraction with coefficient $\rho\in(0,1)$. Let $S:C\to C$ be a nonexpansive mapping such that $F=\mathrm{Fix}(S)\cap\Omega\cap A^{-1}0\neq\emptyset$ with $0<\mu_i<\frac{\alpha_i}{\kappa^2}$ for $i=1,2$. For arbitrarily given $x_0\in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases}
y_n=\alpha_nf(y_n)+\beta_nx_n+\gamma_nSx_n+\delta_nJ_{r_n}Gx_n,\\
x_{n+1}=\sigma_ny_n+(1-\sigma_n)J_{r_n}Gy_n,\quad\forall n\ge 0.
\end{cases}$$

Suppose that Assumption 5.1 *holds. Assume that* $\sum_{n=1}^{\infty}\sup_{x\in D}\|S_nx - S_{n-1}x\| < \infty$ *for any bounded subset D of C*, $S : C \to C$ *is a mapping defined by* $Sx = \lim_{n\to\infty} S_nx$ *for all* $x \in C$, *and* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. *Then the sequence* $\{x_n\}$ *converges strongly to* $q \in F$, *which solves the following VIP*:

$$\langle q - f(q), J(q - p)\rangle \le 0, \quad \forall p \in F.$$

We now establish the following strong convergence result on the composite explicit viscosity algorithm.

Theorem 5.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let* Π_C *be a sunny nonexpansive retraction from X onto C and* $A \subset X \times X$ *be an accretive operator on X such that* $\overline{D(A)} \subset C \subset \bigcap_{r>0} R(I + rA)$. *For each* $i = 1, 2$, *let* $B_i : C \to X$ *be a* λ_i*-strictly pseudocontractive and* α_i*-strongly accretive mapping with* $\alpha_i + \lambda_i \ge 1$. *Let* $f : C \to C$ *be a contraction with coefficient* $\rho \in (0, 1)$ *and* $\{S_i\}_{i=0}^{\infty}$ *be an infinite family of nonexpansive mappings* $S_i : C \to C$ *such that* $F = \bigcap_{i=0}^{\infty}\mathrm{Fix}(S_i) \cap \Omega \cap A^{-1}0 \ne \emptyset$ *with* $1 - \frac{\lambda_i}{1+\lambda_i}(1 - \sqrt{\frac{1-\alpha_i}{\lambda_i}}) \le \mu_i \le 1$ *for* $i = 1, 2$. *Suppose that Assumption* 5.1 *holds. For arbitrarily given* $x_0 \in C$, *let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} y_n = \sigma_n Gx_n + (1 - \sigma_n)J_{r_n}Gx_n, \\ x_{n+1} = \alpha_n f(y_n) + \beta_n y_n + \gamma_n S_n y_n + \delta_n J_{r_n}Gy_n, \quad \forall n \ge 0. \end{cases} \tag{5.40}$$

Assume that $\sum_{n=1}^{\infty}\sup_{x\in D}\|S_nx - S_{n-1}x\| < \infty$ *for any bounded subset D of C*, $S : C \to C$ *is a mapping defined by* $Sx = \lim_{n\to\infty} S_nx$ *for all* $x \in C$, *and* $\mathrm{Fix}(S) = \bigcap_{n=0}^{\infty}\mathrm{Fix}(S_n)$. *Then* $\{x_n\}$ *converges strongly to* $q \in F$, *which solves the following VIP*:

$$\langle q - f(q), J(q - p)\rangle \le 0, \quad \forall p \in F.$$

Proof Take a fixed $p \in F$ arbitrarily. Then we obtain $p = Gp$, $p = S_np$ and $J_{r_n}p = p$ for all $n \ge 0$. Moreover, by Lemma 4.2, we have

$$\begin{aligned} \|y_n - p\| &\le \sigma_n\|Gx_n - p\| + (1 - \sigma_n)\|J_{r_n}Gx_n - p\| \\ &\le \sigma_n\|x_n - p\| + (1 - \sigma_n)\|x_n - p\| \\ &= \|x_n - p\|, \end{aligned} \tag{5.41}$$

and therefore

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n\|f(y_n) - p\| + \beta_n\|y_n - p\| + \gamma_n\|S_ny_n - p\| + \delta_n\|J_{r_n}Gy_n - p\| \\ &\le \alpha_n\big(\|f(y_n) - f(p)\| + \|f(p) - p\|\big) + \beta_n\|y_n - p\| \\ &\quad + \gamma_n\|y_n - p\| + \delta_n\|y_n - p\| \\ &\le \alpha_n\rho\|y_n - p\| + \alpha_n\|f(p) - p\| + (\beta_n + \gamma_n + \delta_n)\|y_n - p\| \\ &= \big(1 - \alpha_n(1 - \rho)\big)\|y_n - p\| + \alpha_n\|f(p) - p\| \end{aligned}$$

$$
\begin{aligned}
&\le \bigl(1-\alpha_n(1-\rho)\bigr)\|x_n-p\| + \alpha_n\bigl\|f(p)-p\bigr\| \\
&= \bigl(1-\alpha_n(1-\rho)\bigr)\|x_n-p\| + \alpha_n(1-\rho)\cdot\frac{\|f(p)-p\|}{1-\rho} \\
&\le \max\biggl\{\|x_n-p\|, \frac{\|f(p)-p\|}{1-\rho}\biggr\}.
\end{aligned}
$$

By induction, we get

$$
\|x_n-p\| \le \max\biggl\{\|x_0-p\|, \frac{\|f(p)-p\|}{1-\rho}\biggr\}, \quad \forall n\ge 0,
$$

which implies that $\{x_n\}$ is bounded and so are the sequences $\{y_n\}$, $\{Gx_n\}$, $\{Gy_n\}$, $\{f(y_n)\}$.

Let us show that $\|x_{n+1}-x_n\| \to 0$ as $n\to\infty$. As a matter of fact, repeating the same arguments as those in the proof of Theorem 4.1, we obtain

$$
\begin{cases}
\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \le \|x_{n-1}-x_n\| + |r_{n-1}-r_n|M_0, \\
\|J_{r_n}Gy_n - J_{r_{n-1}}Gy_{n-1}\| \le \|y_{n-1}-y_n\| + |r_{n-1}-r_n|M_0, \quad \forall n\ge 1,
\end{cases}
\tag{5.42}
$$

where

$$
\sup_{n\ge 1}\biggl\{\frac{1}{\varepsilon}\bigl(\|J_{r_n}Gx_n - Gx_{n-1}\| + \|J_{r_{n-1}}Gx_{n-1} - Gx_n\|\bigr)\biggr\} \le M_0,
$$

and

$$
\sup_{n\ge 1}\biggl\{\frac{1}{\varepsilon}\bigl(\|J_{r_n}Gy_n - Gy_{n-1}\| + \|J_{r_{n-1}}Gy_{n-1} - Gy_n\|\bigr)\biggr\} \le M_0,
$$

for some $M_0 > 0$. By (5.40) and simple calculations, we have

$$
\begin{aligned}
y_n - y_{n-1} = {}& \sigma_n(Gx_n - Gx_{n-1}) + (\sigma_n - \sigma_{n-1})(Gx_{n-1} - J_{r_{n-1}}Gx_{n-1}) \\
&+ (1-\alpha_n)(J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|y_n - y_{n-1}\| &\le \sigma_n\|Gx_n - Gx_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - J_{r_{n-1}}Gx_{n-1}\| \\
&\quad + (1-\alpha_n)\|J_{r_n}Gx_n - J_{r_{n-1}}Gx_{n-1}\| \\
&\le \sigma_n\|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - J_{r_{n-1}}Gx_{n-1}\| \\
&\quad + (1-\sigma_n)\bigl(\|x_{n-1}-x_n\| + |r_{n-1}-r_n|M_0\bigr) \\
&\le \|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - J_{r_{n-1}}Gx_{n-1}\| + |r_n - r_{n-1}|M_0.
\end{aligned}
\tag{5.43}
$$

Taking into account condition (v), without loss of generality we may assume that $\{\beta_n\} \subset [a,b]$ for some $a,b\in(0,1)$. From (5.40), x_{n+1} can be rewritten as

$$
x_{n+1} = \beta_n y_n + (1-\beta_n)z_n, \tag{5.44}
$$

where $z_n = \frac{\alpha_n f(y_n)+\gamma_n S_n y_n+\delta_n J_{r_n} G y_n}{1-\beta_n}$. Utilizing (5.42) and (5.43), we have

$$\begin{aligned}
&\|z_n - z_{n-1}\| \\
&\quad = \left\| \frac{\alpha_n f(y_n) + \gamma_n S_n y_n + \delta_n J_{r_n} G y_n}{1-\beta_n} - \frac{\alpha_{n-1} f(y_{n-1}) + \gamma_{n-1} S_{n-1} y_{n-1} + \delta_{n-1} J_{r_{n-1}} G y_{n-1}}{1-\beta_{n-1}} \right\| \\
&\quad = \left\| \frac{x_{n+1} - \beta_n y_n}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_{n-1}} \right\| \\
&\quad = \left\| \frac{x_{n+1} - \beta_n y_n}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_n} + \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_{n-1}} \right\| \\
&\quad \le \left\| \frac{x_{n+1} - \beta_n y_n}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_n} \right\| + \left\| \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_n} - \frac{x_n - \beta_{n-1} y_{n-1}}{1-\beta_{n-1}} \right\| \\
&\quad = \frac{1}{1-\beta_n} \left\| x_{n+1} - \beta_n y_n - (x_n - \beta_{n-1} y_{n-1}) \right\| + \left| \frac{1}{1-\beta_n} - \frac{1}{1-\beta_{n-1}} \right| \|x_n - \beta_{n-1} y_{n-1}\| \\
&\quad = \frac{1}{1-\beta_n} \left\| x_{n+1} - \beta_n y_n - (x_n - \beta_{n-1} y_{n-1}) \right\| + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&\quad = \frac{1}{1-\beta_n} \left\| \alpha_n f(y_n) + \gamma_n S_n y_n + \delta_n J_{r_n} G y_n - \alpha_{n-1} f(y_{n-1}) - \gamma_{n-1} S_{n-1} y_{n-1} - \delta_{n-1} J_{r_{n-1}} G y_{n-1} \right\| \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&\quad \le \frac{1}{1-\beta_n} \big[\alpha_n \left\| f(y_n) - f(y_{n-1}) \right\| + \gamma_n \|S_n y_n - S_{n-1} y_{n-1}\| + \delta_n \|J_{r_n} G y_n - J_{r_{n-1}} G y_{n-1}\| \\
&\qquad + |\alpha_n - \alpha_{n-1}| \left\| f(y_{n-1}) \right\| + |\gamma_n - \gamma_{n-1}| \|S_{n-1} y_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} G y_{n-1}\| \big] \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&\quad \le \frac{1}{1-\beta_n} \big[\alpha_n \rho \|y_n - y_{n-1}\| + \gamma_n \|S_n y_n - S_n y_{n-1}\| + \delta_n \big[\|y_{n-1} - y_n\| + |r_{n-1} - r_n| M_0 \big] \\
&\qquad + |\alpha_n - \alpha_{n-1}| \left\| f(y_{n-1}) \right\| + |\gamma_n - \gamma_{n-1}| \|S_{n-1} y_{n-1}\| + \gamma_n \|S_n y_{n-1} - S_{n-1} y_{n-1}\| \\
&\qquad + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} G y_{n-1}\| \big] + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&\quad \le \frac{1}{1-\beta_n} \big[(\alpha_n \rho + \gamma_n + \delta_n) \|y_{n-1} - y_n\| + |r_{n-1} - r_n| M_0 + |\alpha_n - \alpha_{n-1}| \left\| f(y_{n-1}) \right\| \\
&\qquad + |\gamma_n - \gamma_{n-1}| \|S_{n-1} y_{n-1}\| + \gamma_n \|S_n y_{n-1} - S_{n-1} y_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} G y_{n-1}\| \big] \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\| \\
&\quad \le \left(1 - \frac{(1-\rho)\alpha_n}{1-\beta_n} \right) \|y_n - y_{n-1}\| + \frac{1}{1-\beta_n} \big[|r_{n-1} - r_n| M_0 + |\alpha_n - \alpha_{n-1}| \left\| f(y_{n-1}) \right\| \\
&\qquad + |\gamma_n - \gamma_{n-1}| \|S_{n-1} y_{n-1}\| + |\delta_n - \delta_{n-1}| \|J_{r_{n-1}} G y_{n-1}\| \big] + \|S_n y_{n-1} - S_{n-1} y_{n-1}\| \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)} \|x_n - \beta_{n-1} y_{n-1}\|. \qquad (5.45)
\end{aligned}$$

By simple calculations and (5.44), we get

$$x_{n+1} - x_n = \beta_n (y_n - y_{n-1}) + (\beta_n - \beta_{n-1})(y_{n-1} - z_{n-1}) + (1-\beta_n)(z_n - z_{n-1}).$$

This together with (5.43) and (5.45) implies that

$$
\begin{aligned}
&\|x_{n+1} - x_n\| \\
&\quad \le \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_{n-1} - z_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| \\
&\quad \le \beta_n\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_{n-1} - z_{n-1}\| + (1-\beta_n)\left\{\left(1 - \frac{(1-\rho)\alpha_n}{1-\beta_n}\right)\|y_n - y_{n-1}\|\right. \\
&\qquad + \frac{1}{1-\beta_n}\bigl[|r_{n-1} - r_n|M_0 + |\alpha_n - \alpha_{n-1}|\bigl\|f(y_{n-1})\bigr\| + |\gamma_n - \gamma_{n-1}|\|S_{n-1}y_{n-1}\| \\
&\qquad + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\|\bigr] + \|S_n y_{n-1} - S_{n-1}y_{n-1}\| \\
&\qquad \left. + \frac{|\beta_n - \beta_{n-1}|}{(1-\beta_{n-1})(1-\beta_n)}\|x_n - \beta_{n-1}y_{n-1}\|\right\} \\
&\quad \le \bigl(1 - (1-\rho)\alpha_n\bigr)\|y_n - y_{n-1}\| + |\beta_n - \beta_{n-1}|\|y_{n-1} - z_{n-1}\| + |r_{n-1} - r_n|M_0 \\
&\qquad + |\alpha_n - \alpha_{n-1}|\bigl\|f(y_{n-1})\bigr\| + |\gamma_n - \gamma_{n-1}|\|S_{n-1}y_{n-1}\| + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\| \\
&\qquad + \|S_n y_{n-1} - S_{n-1}y_{n-1}\| + \frac{|\beta_n - \beta_{n-1}|}{1-\beta_{n-1}}\|x_n - \beta_{n-1}y_{n-1}\| \\
&\quad \le \bigl(1 - (1-\rho)\alpha_n\bigr)\bigl[\|x_n - x_{n-1}\| + |\sigma_n - \sigma_{n-1}|\|Gx_{n-1} - J_{r_{n-1}}Gx_{n-1}\| + |r_n - r_{n-1}|M_0\bigr] \\
&\qquad + |\beta_n - \beta_{n-1}|\|y_{n-1} - z_{n-1}\| + |r_{n-1} - r_n|M_0 + |\alpha_n - \alpha_{n-1}|\bigl\|f(y_{n-1})\bigr\| \\
&\qquad + |\gamma_n - \gamma_{n-1}|\|S_{n-1}y_{n-1}\| + |\delta_n - \delta_{n-1}|\|J_{r_{n-1}}Gy_{n-1}\| + \|S_n y_{n-1} - S_{n-1}y_{n-1}\| \\
&\qquad + \frac{|\beta_n - \beta_{n-1}|}{1-\beta_{n-1}}\bigl\|\alpha_{n-1}f(y_{n-1}) + \gamma_{n-1}S_{n-1}y_{n-1} + \delta_{n-1}J_{r_{n-1}}Gy_{n-1}\bigr\| \\
&\quad \le \bigl(1 - (1-\rho)\alpha_n\bigr)\|x_n - x_{n-1}\| + \bigl(|\sigma_n - \sigma_{n-1}| + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| \\
&\qquad + |\gamma_n - \gamma_{n-1}| + |\delta_n - \delta_{n-1}| + |r_{n-1} - r_n|\bigr)M + \|S_n y_{n-1} - S_{n-1}y_{n-1}\|,
\end{aligned}
$$

where $\frac{1}{1-b}\sup_{n\ge 0}\{\|f(y_n)\| + \|S_n y_n\| + \|J_{r_n}Gy_n\| + \|Gx_n - J_{r_n}Gx_n\| + \|y_n - z_n\| + 2M_0\} \le M$ for some $M > 0$. So, in terms of Lemma 2.7 and conditions (i), (iii), and (iv), we conclude that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{5.46}$$

Next we show that $\|x_n - Gx_n\| \to 0$ as $n \to \infty$.

Indeed, utilizing Lemma 2.3 and (5.40), we get

$$
\begin{aligned}
\|y_n - p\|^2 &= \bigl\|\sigma_n(Gx_n - p) + (1-\sigma_n)(J_{r_n}Gx_n - p)\bigr\|^2 \\
&\le \sigma_n\|Gx_n - p\|^2 + (1-\sigma_n)\|J_{r_n}Gx_n - p\|^2 - \sigma_n(1-\sigma_n)g\bigl(\|Gx_n - J_{r_n}Gx_n\|\bigr) \\
&\le \sigma_n\|x_n - p\|^2 + (1-\sigma_n)\|x_n - p\|^2 - \sigma_n(1-\sigma_n)g\bigl(\|Gx_n - J_{r_n}Gx_n\|\bigr) \\
&= \|x_n - p\|^2 - \sigma_n(1-\sigma_n)g\bigl(\|Gx_n - J_{r_n}Gx_n\|\bigr).
\end{aligned}
\tag{5.47}
$$

According to Lemma 2.2, we have from (5.40) and (5.47)

$$
\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\quad = \bigl\|\alpha_n\bigl(f(y_n) - f(p)\bigr) + \beta_n(y_n - p) + \gamma_n(S_n y_n - p) + \delta_n(J_{r_n}Gy_n - p) + \alpha_n\bigl(f(p) - p\bigr)\bigr\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\le \bigl\|\alpha_n\bigl(f(y_n)-f(p)\bigr)+\beta_n(y_n-p)+\gamma_n(S_ny_n-p)+\delta_n(J_{r_n}Gy_n-p)\bigr\|^2 \\
&\quad + 2\alpha_n\bigl\langle f(p)-p, J(x_{n+1}-p)\bigr\rangle \\
&\le \alpha_n\bigl\|f(y_n)-f(p)\bigr\|^2+\beta_n\|y_n-p\|^2+\gamma_n\|S_ny_n-p\|^2+\delta_n\|J_{r_n}Gy_n-p\|^2 \\
&\quad + 2\alpha_n\bigl\langle f(p)-p, J(x_{n+1}-p)\bigr\rangle \\
&\le \alpha_n\rho^2\|y_n-p\|^2+\beta_n\|y_n-p\|^2+\gamma_n\|y_n-p\|^2+\delta_n\|Gy_n-p\|^2 \\
&\quad + 2\alpha_n\bigl\langle f(p)-p, J(x_{n+1}-p)\bigr\rangle \\
&\le \alpha_n\rho\|y_n-p\|^2+\beta_n\|y_n-p\|^2+\gamma_n\|y_n-p\|^2+\delta_n\|y_n-p\|^2 \\
&\quad + 2\alpha_n\bigl\|f(p)-p\bigr\|\|x_{n+1}-p\| \\
&= \bigl(1-\alpha_n(1-\rho)\bigr)\|y_n-p\|^2+2\alpha_n\bigl\|f(p)-p\bigr\|\|x_{n+1}-p\| \\
&\le \|y_n-p\|^2+2\alpha_n\bigl\|f(p)-p\bigr\|\|x_{n+1}-p\| \\
&\le \|x_n-p\|^2-\sigma_n(1-\sigma_n)g\bigl(\|Gx_n-J_{r_n}Gx_n\|\bigr)+2\alpha_n\bigl\|f(p)-p\bigr\|\|x_{n+1}-p\|,
\end{aligned}
$$

which hence yields

$$
\begin{aligned}
&\sigma_n(1-\sigma_n)g\bigl(\|Gx_n-J_{r_n}Gx_n\|\bigr) \\
&\quad \le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n\bigl\|f(p)-p\bigr\|\|x_{n+1}-p\| \\
&\quad \le \bigl(\|x_n-p\|+\|x_{n+1}-p\|\bigr)\|x_n-x_{n+1}\|+2\alpha_n\bigl\|f(p)-p\bigr\|\|x_{n+1}-p\|.
\end{aligned}
$$

Since $\alpha_n \to 0$ and $\|x_{n+1}-x_n\| \to 0$, from condition (v) and the boundedness of $\{x_n\}$, it follows that

$$\lim_{n\to\infty} g\bigl(\|Gx_n-J_{r_n}Gx_n\|\bigr)=0.$$

Utilizing the properties of g, we have

$$\lim_{n\to\infty}\|Gx_n-J_{r_n}Gx_n\|=0. \tag{5.48}$$

On the other hand, x_{n+1} can be rewritten as

$$
\begin{aligned}
x_{n+1} &= \alpha_nf(y_n)+\beta_ny_n+\gamma_nS_ny_n+\delta_nJ_{r_n}Gy_n \\
&= \alpha_nf(y_n)+\beta_ny_n+(\gamma_n+\delta_n)\frac{\gamma_nS_ny_n+\delta_nJ_{r_n}Gy_n}{\gamma_n+\delta_n} \\
&= \alpha_nf(y_n)+\beta_ny_n+e_n\hat{z}_n,
\end{aligned} \tag{5.49}
$$

where $e_n=\gamma_n+\delta_n$ and $\hat{z}_n=\frac{\gamma_nS_ny_n+\delta_nJ_{r_n}Gy_n}{\gamma_n+\delta_n}$. Utilizing Lemma 2.4, from (5.41) and (5.49), we have

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \bigl\|\alpha_n\bigl(f(y_n)-p\bigr)+\beta_n(y_n-p)+e_n(\hat{z}_n-p)\bigr\|^2 \\
&\le \alpha_n\bigl\|f(y_n)-p\bigr\|^2+\beta_n\|y_n-p\|^2+e_n\|\hat{z}_n-p\|^2-\beta_ne_ng_1\bigl(\|\hat{z}_n-y_n\|\bigr)
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|y_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - y_n\|\big) \\
&\quad + e_n\left\|\frac{\gamma_n S_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n} - p\right\|^2 \\
&= \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|y_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - y_n\|\big) \\
&\quad + e_n\left\|\frac{\gamma_n}{\gamma_n + \delta_n}(S_n y_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G y_n - p)\right\|^2 \\
&\le \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|y_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - y_n\|\big) \\
&\quad + e_n\left[\frac{\gamma_n}{\gamma_n + \delta_n}\|S_n y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G y_n - p\|^2\right] \\
&\le \alpha_n \big\|f(y_n) - p\big\|^2 + \beta_n\|y_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - y_n\|\big) \\
&\quad + e_n\left[\frac{\gamma_n}{\gamma_n + \delta_n}\|y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|y_n - p\|^2\right] \\
&= \alpha_n \big\|f(y_n) - p\big\|^2 + (1 - \alpha_n)\|y_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - y_n\|\big) \\
&\le \alpha_n \big\|f(y_n) - p\big\|^2 + \|y_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - y_n\|\big) \\
&\le \alpha_n \big\|f(y_n) - p\big\|^2 + \|x_n - p\|^2 - \beta_n e_n g_1\big(\|\hat{z}_n - y_n\|\big),
\end{aligned}
$$

which hence implies that

$$
\begin{aligned}
\beta_n e_n g_1\big(\|\hat{z}_n - y_n\|\big) &\le \alpha_n \big\|f(y_n) - p\big\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\le \alpha_n \big\|f(y_n) - p\big\|^2 + \big(\|x_n - p\| + \|x_{n+1} - p\|\big)\|x_n - x_{n+1}\|.
\end{aligned}
$$

Utilizing (5.46), conditions (i), (ii), (v), and the boundedness of $\{x_n\}$ and $\{f(y_n)\}$, we get

$$
\lim_{n\to\infty} g_1\big(\|\hat{z}_n - y_n\|\big) = 0.
$$

From the properties of g_1, we have

$$
\lim_{n\to\infty} \|\hat{z}_n - y_n\| = 0. \tag{5.50}
$$

Utilizing Lemma 2.3 and the definition of $\hat{z}_n$, we have

$$
\begin{aligned}
\|\hat{z}_n - p\|^2 &= \left\|\frac{\gamma_n S_n y_n + \delta_n J_{r_n} G y_n}{\gamma_n + \delta_n} - p\right\|^2 \\
&= \left\|\frac{\gamma_n}{\gamma_n + \delta_n}(S_n y_n - p) + \frac{\delta_n}{\gamma_n + \delta_n}(J_{r_n} G y_n - p)\right\|^2 \\
&\le \frac{\gamma_n}{\gamma_n + \delta_n}\|S_n y_n - p\|^2 + \frac{\delta_n}{\gamma_n + \delta_n}\|J_{r_n} G y_n - p\|^2 \\
&\quad - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2\big(\|J_{r_n} G y_n - S_n y_n\|\big) \\
&\le \|y_n - p\|^2 - \frac{\gamma_n \delta_n}{(\gamma_n + \delta_n)^2} g_2\big(\|J_{r_n} G y_n - S_n y_n\|\big),
\end{aligned}
$$

which leads to

$$\begin{aligned}\frac{\gamma_n\delta_n}{(\gamma_n+\delta_n)^2}g_2\big(\|J_{r_n}Gy_n-S_ny_n\|\big) &\le \|y_n-p\|^2-\|\hat{z}_n-p\|^2\\ &\le \big(\|y_n-p\|+\|\hat{z}_n-p\|\big)\|y_n-\hat{z}_n\|.\end{aligned}$$

Since $\{y_n\}$ and $\{\hat{z}_n\}$ are bounded, we deduce from (5.50) and condition (ii) that

$$\lim_{n\to\infty} g_2\big(\|S_ny_n-J_{r_n}Gy_n\|\big)=0.$$

From the properties of g_2, we have

$$\lim_{n\to\infty}\|S_ny_n-J_{r_n}Gy_n\|=0. \tag{5.51}$$

Furthermore, x_{n+1} can also be rewritten as

$$\begin{aligned}x_{n+1}&=\alpha_nf(y_n)+\beta_ny_n+\gamma_nS_ny_n+\delta_nJ_{r_n}Gy_n\\ &=\beta_ny_n+\gamma_nS_ny_n+(\alpha_n+\delta_n)\frac{\alpha_nf(y_n)+\delta_nJ_{r_n}Gy_n}{\alpha_n+\delta_n}\\ &=\beta_ny_n+\gamma_nS_ny_n+d_n\tilde{z}_n,\end{aligned}$$

where $d_n=\alpha_n+\delta_n$ and $\tilde{z}_n=\frac{\alpha_nf(y_n)+\delta_nJ_{r_n}Gy_n}{\alpha_n+\delta_n}$. Utilizing Lemma 2.4 and the convexity of $\|\cdot\|^2$, we have from (5.41)

$$\begin{aligned}&\|x_{n+1}-p\|^2\\ &\quad=\big\|\beta_n(y_n-p)+\gamma_n(S_ny_n-p)+d_n(\tilde{z}_n-p)\big\|^2\\ &\quad\le\beta_n\|y_n-p\|^2+\gamma_n\|S_ny_n-p\|^2+d_n\|\tilde{z}_n-p\|^2-\beta_n\gamma_ng_3\big(\|y_n-S_ny_n\|\big)\\ &\quad=\beta_n\|y_n-p\|^2+\gamma_n\|S_ny_n-p\|^2+d_n\left\|\frac{\alpha_nf(y_n)+\delta_nJ_{r_n}Gy_n}{\alpha_n+\delta_n}-p\right\|^2\\ &\qquad-\beta_n\gamma_ng_3\big(\|y_n-S_ny_n\|\big)\\ &\quad=\beta_n\|y_n-p\|^2+\gamma_n\|S_ny_n-p\|^2+d_n\left\|\frac{\alpha_n}{\alpha_n+\delta_n}\big(f(y_n)-p\big)+\frac{\delta_n}{\alpha_n+\delta_n}(J_{r_n}Gy_n-p)\right\|^2\\ &\qquad-\beta_n\gamma_ng_3\big(\|y_n-S_ny_n\|\big)\\ &\quad\le\beta_n\|y_n-p\|^2+\gamma_n\|y_n-p\|^2+d_n\left[\frac{\alpha_n}{\alpha_n+\delta_n}\big\|f(y_n)-p\big\|^2+\frac{\delta_n}{\alpha_n+\delta_n}\|J_{r_n}Gy_n-p\|^2\right]\\ &\qquad-\beta_n\gamma_ng_3\big(\|y_n-S_ny_n\|\big)\\ &\quad\le\alpha_n\big\|f(y_n)-p\big\|^2+(\beta_n+\gamma_n)\|y_n-p\|^2+\delta_n\|y_n-p\|^2-\beta_n\gamma_ng_3\big(\|y_n-S_ny_n\|\big)\\ &\quad=\alpha_n\big\|f(y_n)-p\big\|^2+(1-\alpha_n)\|y_n-p\|^2-\beta_n\gamma_ng_3\big(\|y_n-S_ny_n\|\big)\\ &\quad\le\alpha_n\big\|f(y_n)-p\big\|^2+\|y_n-p\|^2-\beta_n\gamma_ng_3\big(\|y_n-S_ny_n\|\big)\\ &\quad\le\alpha_n\big\|f(y_n)-p\big\|^2+\|x_n-p\|^2-\beta_n\gamma_ng_3\big(\|y_n-S_ny_n\|\big),\end{aligned}$$

which implies that

$$\begin{aligned}\beta_n\gamma_n g_3\bigl(\|y_n - S_ny_n\|\bigr) &\le \alpha_n\bigl\|f(y_n) - p\bigr\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\ &\le \alpha_n\bigl\|f(y_n) - p\bigr\|^2 + \bigl(\|x_n - p\| + \|x_{n+1} - p\|\bigr)\|x_n - x_{n+1}\|.\end{aligned}$$

From (5.46), conditions (i), (ii), (v), and the boundedness of $\{x_n\}$ and $\{f(y_n)\}$, we have

$$\lim_{n\to\infty} g_3\bigl(\|y_n - S_ny_n\|\bigr) = 0.$$

Utilizing the properties of g_3, we have

$$\lim_{n\to\infty} \|y_n - S_ny_n\| = 0. \tag{5.52}$$

Thus, from (5.51) and (5.52), we get

$$\|y_n - Jr_nGy_n\| \le \|y_n - S_ny_n\| + \|S_ny_n - Jr_nGy_n\| \to 0 \quad \text{as } n \to \infty,$$

that is,

$$\lim_{n\to\infty} \|y_n - Jr_nGy_n\| = 0. \tag{5.53}$$

Therefore, from (5.40), (5.46), (5.52), (5.53), and $\alpha_n \to 0$, it follows that

$$\begin{aligned}&\|x_n - y_n\| \\ &\quad\le \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\| \\ &\quad\le \|x_n - x_{n+1}\| + \alpha_n\bigl\|f(y_n) - y_n\bigr\| + \gamma_n\|S_ny_n - y_n\| + \delta_n\|Jr_nGy_n - y_n\| \\ &\quad\le \|x_n - x_{n+1}\| + \alpha_n\bigl\|f(y_n) - y_n\bigr\| + \|S_ny_n - y_n\| + \|Jr_nGy_n - y_n\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

that is,

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{5.54}$$

Utilizing (5.40), (5.48), and (5.54), we obtain

$$\begin{aligned}\|x_n - Gx_n\| &\le \|x_n - y_n\| + \|y_n - Gx_n\| = \|x_n - y_n\| + (1 - \sigma_n)\|J_{r_n}Gx_n - Gx_n\| \\ &\le \|x_n - y_n\| + \|J_{r_n}Gx_n - Gx_n\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

that is,

$$\lim_{n\to\infty} \|x_n - Gx_n\| = 0. \tag{5.55}$$

In addition, from (5.52) and (5.54), we have

$$\begin{aligned}\|x_n - S_nx_n\| &\le \|x_n - y_n\| + \|y_n - S_ny_n\| + \|S_ny_n - S_nx_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - S_ny_n\| \to 0 \quad \text{as } n \to \infty,\end{aligned}$$

that is,

$$\lim_{n\to\infty}\|x_n - S_n x_n\| = 0. \tag{5.56}$$

In terms of (5.56) and Lemma 2.6, we have

$$\|x_n - Sx_n\| \le \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0 \quad \text{as } n \to \infty,$$

that is,

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0. \tag{5.57}$$

We note that

$$\begin{aligned}\|x_n - J_{r_n}x_n\| &\le \|x_n - y_n\| + \|y_n - J_{r_n}Gy_n\| + \|J_{r_n}Gy_n - J_{r_n}Gx_n\| + \|J_{r_n}Gx_n - J_{r_n}x_n\| \\ &\le 2\|x_n - y_n\| + \|y_n - J_{r_n}Gy_n\| + \|Gx_n - x_n\|.\end{aligned}$$

So, from (5.53), (5.54), and (5.55), we obtain

$$\lim_{n\to\infty}\|x_n - J_{r_n}x_n\| = 0. \tag{5.58}$$

Furthermore, repeating the same arguments as those of (5.29) in the proof of Theorem 4.1, we can derive

$$\lim_{n\to\infty}\|x_n - J_r x_n\| = 0, \tag{5.59}$$

for a fixed number $r \in (0,\varepsilon)$. Define a mapping $Wx = (1-\theta_1-\theta_2)J_r x + \theta_1 Sx + \theta_2 Gx$, where $\theta_1, \theta_2 \in (0,1)$ are two constants with $\theta_1 + \theta_2 < 1$. Then by Lemma 2.5, we have $\mathrm{Fix}(W) = \mathrm{Fix}(J_r) \cap \mathrm{Fix}(S) \cap \mathrm{Fix}(G) = F$. We observe that

$$\begin{aligned}\|x_n - Wx_n\| &= \big\|(1-\theta_1-\theta_2)(x_n - J_r x_n) + \theta_1(x_n - Sx_n) + \theta_2(x_n - Gx_n)\big\| \\ &\le (1-\theta_1-\theta_2)\|x_n - J_r x_n\| + \theta_1\|x_n - Sx_n\| + \theta_2\|x_n - Gx_n\|.\end{aligned}$$

From (5.55), (5.57), and (5.59), we obtain

$$\lim_{n\to\infty}\|x_n - Wx_n\| = 0. \tag{5.60}$$

Now, we claim that

$$\limsup_{n\to\infty}\big\langle f(q) - q, J(x_n - q)\big\rangle \le 0, \tag{5.61}$$

where $q = s - \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction

$$x \mapsto tf(x) + (1-t)Wx.$$

Then x_t solves the fixed point equation $x_t = tf(x_t) + (1-t)Wx_t$. Repeating the same arguments as those of (5.36) in the proof of Theorem 4.1, we derive

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0. \tag{5.62}$$

Repeating the same arguments as those of (5.37) in the proof of Theorem 4.1, we obtain

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \\
&\quad = \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \\
&\quad \le \limsup_{t\to 0}\limsup_{n\to\infty}\langle f(q) - q, J(x_n - q) - J(x_n - x_t)\rangle.
\end{aligned} \tag{5.63}$$

Since X has a uniformly Gâteaux differentiable norm, the duality mapping J is norm-to-weak* uniformly continuous on bounded subsets of X. Consequently, the two limits are interchangeable, and hence (5.61) holds. From (5.46), we get $(x_{n+1} - q) - (x_n - q) \to 0$. Noticing the norm-to-weak* uniform continuity of J on bounded subsets of X, we deduce from (5.61) that

$$\begin{aligned}
&\limsup_{n\to\infty}\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad = \limsup_{n\to\infty}\bigl(\langle f(q) - q, J(x_{n+1} - q) - J(x_n - q)\rangle + \langle f(q) - q, J(x_n - q)\rangle\bigr) \\
&\quad = \limsup_{n\to\infty}\langle f(q) - q, J(x_n - q)\rangle \le 0.
\end{aligned}$$

Finally, let us show that $x_n \to q$ as $n \to \infty$. We observe that

$$\begin{aligned}
\|y_n - q\| &= \bigl\|\alpha_n\bigl(G(x_n) - q\bigr) + (1-\alpha_n)\bigl(J_{r_n}G(x_n) - q\bigr)\bigr\| \\
&\le \alpha_n\|x_n - q\| + (1-\alpha_n)\|x_n - q\| = \|x_n - q\|,
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \alpha_n\langle f(y_n) - f(q) + f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad + \bigl\langle \beta_n(y_n - q) + \gamma_n(S_n y_n - q) + \delta_n\bigl(J_{r_n}G(y_n) - q\bigr), J(x_{n+1} - q)\bigr\rangle \\
&\le \alpha_n\bigl\|f(y_n) - f(q)\bigr\|\,\|x_{n+1} - q\| + \alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad + \bigl\|\beta_n(y_n - q) + \gamma_n(S_n y_n - q) + \delta_n\bigl(J_{r_n}G(y_n) - q\bigr)\bigr\|\,\|x_{n+1} - q\| \\
&\le \alpha_n\rho\|y_n - q\|\,\|x_{n+1} - q\| + \alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad + \bigl(\beta_n\|y_n - q\| + \gamma_n\|y_n - q\| + \delta_n\|y_n - q\|\bigr)\|x_{n+1} - q\| \\
&= \alpha_n\rho\|y_n - q\|\,\|x_{n+1} - q\| + \alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\quad + (1-\alpha_n)\|y_n - q\|\,\|x_{n+1} - q\| \\
&\le \bigl(1 - \alpha_n(1-\rho)\bigr)\|y_n - q\|\,\|x_{n+1} - q\| + \alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle \\
&\le \bigl(1 - \alpha_n(1-\rho)\bigr)\|x_n - q\|\,\|x_{n+1} - q\| + \alpha_n\langle f(q) - q, J(x_{n+1} - q)\rangle
\end{aligned}$$

$$\begin{aligned} &= \frac{1-\alpha_n(1-\rho)}{2}\big(\|x_n-q\|^2+\|x_{n+1}-q\|^2\big)+\alpha_n\big\langle f(q)-q,J(x_{n+1}-q)\big\rangle \\ &\le \frac{1-\alpha_n(1-\rho)}{2}\|x_n-q\|^2+\frac{1}{2}\|x_{n+1}-q\|^2+\alpha_n\big\langle f(q)-q,J(x_{n+1}-q)\big\rangle. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|x_{n+1}-q\|^2 &\le \big(1-\alpha_n(1-\rho)\big)\|x_n-q\|^2+2\alpha_n\big\langle f(q)-q,J(x_{n+1}-q)\big\rangle \\ &= \big(1-\alpha_n(1-\rho)\big)\|x_n-q\|^2+\alpha_n(1-\rho)\frac{2\langle f(q)-q,J(x_{n+1}-q)\rangle}{1-\rho}. \end{aligned} \tag{5.64}$$

Since $\sum_{n=0}^{\infty}\alpha_n=\infty$ and $\limsup_{n\to\infty}\langle f(q)-q,J(x_{n+1}-q)\rangle\le 0$, by Lemma 2.7, we conclude from (5.64) that $x_n\to q$ as $n\to\infty$. This completes the proof. □

Corollary 5.2 *Let C be a nonempty closed convex subset of a uniformly convex Banach space X which has a uniformly Gâteaux differentiable norm. Let Π_C be a sunny nonexpansive retraction from X onto C and $A\subset X\times X$ be an accretive operator on X such that $\overline{D(A)}\subset C\subset\bigcap_{r>0}R(I+rA)$. Let the mapping $B_i:C\to X$ be λ_i-strictly pseudocontractive and α_i-strongly accretive with $\alpha_i+\lambda_i\ge 1$ for $i=1,2$. Let $f:C\to C$ be a contraction with coefficient $\rho\in(0,1)$ and $S:C\to C$ be a nonexpansive mapping such that $F=\mathrm{Fix}(S)\cap\Omega\cap A^{-1}0\neq\emptyset$ with $1-\frac{\lambda_i}{1+\lambda_i}(1-\sqrt{\frac{1-\alpha_i}{\lambda_i}})\le\mu_i\le 1$ for $i=1,2$. Suppose that Assumption* 5.1 *holds. For arbitrarily given $x_0\in C$, let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} y_n=\sigma_n Gx_n+(1-\sigma_n)J_{r_n}Gx_n, \\ x_{n+1}=\alpha_n f(y_n)+\beta_n y_n+\gamma_n Sy_n+\delta_n J_{r_n}Gy_n, \quad \forall n\ge 0. \end{cases}$$

Then the sequence $\{x_n\}$ converges strongly to $q\in F$, which solves the following VIP:

$$\big\langle q-f(q),J(q-p)\big\rangle\le 0,\quad \forall p\in F.$$

Remark 5.1 Our Theorems 5.1 and 5.2 improve and extend [30, Theorem 3.2], [20, Theorem 3.1] and [29, Theorem 3.1] in the following aspects.

(a) The problem of finding a point $q\in\bigcap_n\mathrm{Fix}(S_n)\cap\Omega\cap A^{-1}0$ in Theorems 5.1 and 5.2 is more general and more subtle than the problem of finding $q\in\bigcap_n\mathrm{Fix}(T_n)$ in [30, Theorem 3.2], the problem of finding $q\in\bigcap_n\mathrm{Fix}(T_n)\cap\Omega$ in [20, Theorem 3.1] and the problem of finding $q\in A^{-1}0$ in [29, Theorem 3.1].

(b) Theorems 5.1 and 5.2 are proved without the asymptotical regularity assumption of $\{x_n\}$ in [29, Theorem 3.1] (that is, $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$).

(c) The iterative scheme in [20, Theorem 3.1] is extended to develop the iterative schemes (5.1) and (5.40) in Theorems 5.1 and 5.2 by virtue of the iterative schemes of [30, Theorem 3.2] and [29, Theorem 3.1]. The iterative schemes (5.1) and (5.40) in Theorems 5.1 and 5.2 are more advantageous and more flexible than the iterative scheme in [20, Theorem 3.1] because they involves several parameter sequences.

(d) The iterative schemes (5.1) and (5.40) in Theorems 5.1 and 5.2 are different from the one given in [30, Theorem 3.2], [20, Theorem 3.1] and [29, Theorem 3.1] because the first iteration step in (5.1) is implicit and because the mapping G in [20, Theorem 3.1] and the mapping J_{r_n} in [29, Theorem 3.1] are replaced by the same composite mapping $J_{r_n}G$ in Theorems 5.1 and 5.2.

(e) The proof of [20, Theorem 3.1] depends on the argument techniques in [10], the inequality in 2-uniformly smooth Banach spaces and the inequality in smooth and uniform convex Banach spaces. Because the composite mapping $J_{r_n}G$ appears in the iterative scheme (5.1) in Theorem 5.1, the proof of Theorem 5.1 depends on the argument techniques in [10], the inequality in 2-uniformly smooth Banach spaces, the inequality in smooth and uniform convex Banach spaces, and the inequalities in uniform convex Banach spaces. However, the proof of our Theorem 5.1 does not depend on the argument techniques in [10], the inequality in 2-uniformly smooth Banach spaces, and the inequality in smooth and uniform convex Banach spaces. It depends on only the inequalities in uniform convex Banach spaces.

(f) The assumption of the uniformly convex and 2-uniformly smooth Banach space X in [20, Theorem 3.1] is weakened to the one of the uniformly convex Banach space X having a uniformly Gâteaux differentiable norm in Theorem 5.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors participated in the design of this work and performed equally. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Shanghai Normal University, and Scientific Computing Key Laboratory of Shanghai Universities, Shanghai, 200234, China. [2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21589, Saudi Arabia. [3]Department of Mathematics, Aligarh Muslim University, Aligarh, India. [4]Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran, Saudi Arabia.

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