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A note on spherical maxima sharing the same Lagrange multiplier

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Dedicated to Professor Wataru Takahashi, with esteem and friendship, on the occasion of his 70th birthday

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Abstract
In this paper, we establish a general result on spherical maxima sharing the same Lagrange multiplier of which the following is a particular consequence: Let X be a real Hilbert space. For each $r > 0$, let $S_r = \{x \in X : \|x\|^2 = r\}$. Let $J : X \to \mathbf{R}$ be a sequentially weakly upper semicontinuous functional which is Gâteaux differentiable in $X \setminus \{0\}$. Assume that $\limsup_{x\to 0} \frac{J(x)}{\|x\|^2} = +\infty$. Then, for each $\rho > 0$, there exists an open interval $I \subseteq]0, +\infty[$ and an increasing function $\varphi : I \to]0, \rho[$ such that, for each $\lambda \in I$, one has $\emptyset \neq \{x \in S_{\varphi(\lambda)} : J(x) = \sup_{S_{\varphi(\lambda)}} J\} \subseteq \{x \in X : x = \lambda J'(x)\}$.

Here and in what follows, X is a real Hilbert space and $J : X \to \mathbf{R}$ is a functional, with $J(0) = 0$. For each $r > 0$, set

$$S_r = \{x \in X : \|x\|^2 = r\},$$
$$B_r = \{x \in X : \|x\|^2 \le r\}.$$

A point $\hat{x} \in S_r$ such that

$$J(\hat{x}) = \sup_{S_r} J$$

is called a spherical maximum of J. Assuming that J is C^1, spherical maxima are important in connection with the eigenvalue problem

$$J'(x) = \mu x. \tag{1}$$

Actually, if $\hat{x}$ is a spherical maximum of J, by the classical Lagrange multiplier theorem, there exists $\mu_{\hat{x}} \in \mathbf{R}$ such that

$$J'(\hat{x}) = \mu_{\hat{x}}\hat{x}.$$

More specifically, one could be interested in the multiplicity of solutions for (1), in the sense of finding some $\mu \in \mathbf{R}$ for which there are more points x satisfying (1). In this connection, however, just because of dependence of $\mu_{\hat{x}}$ on $\hat{x}$, the existence of more spherical maxima in S_r does not imply automatically the existence of some $\mu \in \mathbf{R}$ for which (1) has more solutions. So, in order to the multiplicity of solutions of (1), it is important to know when, at least for some $r > 0$, the spherical maxima in S_r share the same Lagrange multiplier.

The aim of the present note is to give a contribution along such a direction.

Here is our basic result.

Theorem 1 *For some $\rho > 0$, assume that J is Gâteaux differentiable in $\mathrm{int}(B_\rho) \setminus \{0\}$ and that*

$$\frac{\beta_\rho}{\rho} < \delta_\rho, \tag{2}$$

where

$$\beta_\rho = \sup_{B_\rho} J$$

and

$$\delta_\rho = \sup_{x \in B_\rho \setminus \{0\}} \frac{J(x)}{\|x\|^2}.$$

Assume also that, for some $a > 0$, with

$$a > \frac{\rho}{\rho\delta_\rho - \beta_\rho}$$

if $\delta_\rho < +\infty$, the restriction of the functional $\|\cdot\|^2 - aJ(\cdot)$ to B_ρ is sequentially weakly lower semicontinuous.

For each $r \in]\beta_\rho, +\infty[$, put

$$\eta(r) = \sup_{y \in B_\rho} \frac{\rho - \|y\|^2}{r - J(y)}$$

and

$$\Gamma(r) = \left\{ x \in B_\rho : \frac{\rho - \|x\|^2}{r - J(x)} = \eta(r) \right\}.$$

Then the following assertions hold:

(i) *the function η is convex and decreasing in $]\beta_\rho, +\infty[$, with $\lim_{r \to +\infty} \eta(r) = 0$;*

(ii) *for each $r \in]\beta_\rho + \frac{\rho}{a}, \rho\delta_\rho[$, the set $\Gamma(r)$ is non-empty and, for every $\hat{x} \in \Gamma(r)$, one has*

$$0 < \|\hat{x}\|^2 < \rho$$

and

$$\begin{aligned} \hat{x} &\in \left\{ x \in S_{\|\hat{x}\|^2} : J(x) = \sup_{S_{\|\hat{x}\|^2}} J \right\} \\ &\subseteq \left\{ x \in \mathrm{int}(B_\rho) : \|x\|^2 - \eta(r)J(x) = \inf_{y \in B_\rho} \left(\|y\|^2 - \eta(r)J(y) \right) \right\} \\ &\subseteq \left\{ x \in X : x = \frac{\eta(r)}{2} J'(x) \right\}; \end{aligned}$$

(iii) *for each* $r_1, r_2 \in]\beta_\rho + \frac{\rho}{a}, \rho\delta_\rho[$, *with* $r_1 < r_2$, *and each* $\hat{x} \in \Gamma(r_1)$, $\hat{y} \in \Gamma(r_2)$, *one has*

$$\|\hat{y}\| < \|\hat{x}\|;$$

(iv) *if* A *denotes the set of all* $r \in]\beta_\rho + \frac{\rho}{a}, \rho\delta_\rho[$ *such that* $\Gamma(r)$ *is a singleton, then the function* $r \to \Gamma(r)$ $(r \in A)$ *is continuous with respect to the weak topology*; *if, in addition,* J *is sequentially weakly upper semicontinuous in* B_ρ, *then* $\Gamma_{|A}$ *is continuous with respect to the strong topology.*

Before proving Theorem 1, let us recall a proposition from [1] that will be used in the proof.

Proposition 1 *Let* Y *be a non-empty set,* $f, g : Y \to \mathbf{R}$ *two functions, and* a, b *two real numbers, with* $a < b$. *Let* y_a *be a global minimum of the function* $f + ag$ *and* y_b *a global minimum of the function* $f + bg$.

Then one has $g(y_b) \leq g(y_a)$.

Proof of Theorem 1 By definition, the function η is the upper envelope of a family of functions which are decreasing and convex in $]\beta_\rho, +\infty[$. So, η is convex and non-increasing. We also have

$$\eta(r) \leq \frac{\rho}{r - \beta_\rho} \tag{3}$$

for all $r > \beta_\rho$ and so

$$\lim_{r \to +\infty} \eta(r) = 0.$$

In turn, this implies that η is decreasing as it never vanishes. Now, fix $r \in]\beta_\rho + \frac{\rho}{a}, \rho\delta_\rho[$. So, we have

$$\frac{\rho}{r - \beta_\rho} < a.$$

Consequently, by (3),

$$\eta(r) < a.$$

Observe that, for each $\lambda \in]0, a[$, the restriction to B_ρ of the functional $\|\cdot\|^2 - \lambda J(\cdot)$ is sequentially weakly lower semicontinuous. In this connection, it is enough to notice that

$$\frac{a}{a - \lambda}\left(\|x\|^2 - \lambda J(x)\right) = \|x\|^2 + \frac{\lambda}{a - \lambda}\left(\|x\|^2 - aJ(x)\right).$$

Fix a sequence $\{x_n\}$ in B_ρ such that

$$\lim_{n \to \infty} \frac{\rho - \|x_n\|^2}{r - J(x_n)} = \eta(r).$$

Up to a subsequence, we can suppose that $\{x_n\}$ converges weakly to some $\hat{x}_r \in B_\rho$. Fix $\epsilon \in]0, \eta(r)[$. For each $n \in \mathbf{N}$ large enough, we have

$$\frac{\rho - \|x_n\|^2}{r - J(x_n)} > \eta(r) - \epsilon$$

and so

$$\|x_n\|^2 + \big(\eta(r) - \epsilon\big)\big(r - J(x_n)\big) < \rho.$$

But then, by sequential weak lower semicontinuity, we have

$$\|\hat{x}_r\|^2 + \big(\eta(r) - \epsilon\big)\big(r - J(\hat{x}_r)\big) \le \liminf_{n\to\infty}\big(\|x_n\|^2 + \big(\eta(r) - \epsilon\big)\big(r - J(x_n)\big)\big) \le \rho.$$

Hence, since ϵ is arbitrary, we have

$$\|\hat{x}_r\|^2 + \eta(r)\big(r - J(\hat{x}_r)\big) \le \rho$$

and so

$$\frac{\rho - \|\hat{x}_r\|^2}{r - J(\hat{x}_r)} = \eta(r),$$

that is, $\hat{x}_r \in \Gamma(r)$. Now, let $\hat{x}$ be any point of $\Gamma(r)$. Let us show that $\hat{x} \neq 0$. Indeed, since $\frac{r}{\rho} < \delta_\rho$, there exists $\tilde{x} \in B_\rho \setminus \{0\}$ such that

$$\frac{J(\tilde{x})}{\|\tilde{x}\|^2} > \frac{r}{\rho}.$$

Clearly, this is equivalent to

$$\frac{\rho}{r} < \frac{\rho - \|\tilde{x}\|^2}{r - J(\tilde{x})}.$$

So

$$\frac{\rho}{r} < \frac{\rho - \|\hat{x}\|^2}{r - J(\hat{x})}$$

and hence, since $J(0) = 0$, we have $\hat{x} \neq 0$, as claimed. Clearly, $\|\hat{x}\|^2 < \rho$ as $\eta(r) > 0$. Moreover, if $x \in S_{\|\hat{x}\|^2}$, we have

$$\frac{1}{r - J(x)} \le \frac{1}{r - J(\hat{x})}$$

from which we get

$$J(\hat{x}) = \sup_{S_{\|\hat{x}\|^2}} J.$$

Now, let u be any global maximum of $J_{|S_{\|\hat{x}\|^2}}$. Then we have

$$\frac{\rho - \|u\|^2}{r - J(u)} = \eta(r)$$

and so

$$\|u\|^2 - \eta(r)J(u) = \rho - r\eta(r) \leq \|x\|^2 - \eta(r)J(x)$$

for all $x \in B_\rho$. Hence, as $\|u\|^2 < \rho$, the point u is a local minimum of the functional $\|\cdot\|^2 - \eta(r)J(\cdot)$. Consequently, we have

$$u = \frac{\eta(r)}{2}J'(u),$$

and the proof of (ii) is complete. To prove (iii), observe that

$$\frac{1}{\eta(r)} = \inf_{\|x\|^2<\rho} \frac{r - J(x)}{\rho - \|x\|^2}.$$

As a consequence, for each $r_1, r_2 \in]\beta_\rho + \frac{\rho}{a}, \rho\delta_\rho[$, with $r_1 < r_2$, and for each $\hat{x} \in \Gamma(r_1)$, $\hat{y} \in \Gamma(r_2)$, we have

$$\frac{r_1 - J(\hat{x})}{\rho - \|\hat{x}\|^2} = \inf_{\|x\|^2<\rho} \frac{r_1 - J(x)}{\rho - \|x\|^2}$$

and

$$\frac{r_2 - J(\hat{y})}{\rho - \|\hat{y}\|^2} = \inf_{\|x\|^2<\rho} \frac{r_2 - J(x)}{\rho - \|x\|^2}.$$

Therefore, in view of Proposition 1, we have

$$\frac{1}{\rho - \|\hat{y}\|^2} \leq \frac{1}{\rho - \|\hat{x}\|^2}$$

and so

$$\|\hat{y}\| \leq \|\hat{x}\|.$$

We claim that

$$\|\hat{y}\| < \|\hat{x}\|.$$

Arguing by contradiction, assume that $\|\hat{y}\| = \|\hat{x}\|$. In view of (ii), this would imply that $J(\hat{y}) = J(\hat{x})$ and so, at the same time,

$$\hat{y} = \frac{\eta(r_2)}{2}J'(\hat{y})$$

and

$$\hat{y} = \frac{\eta(r_1)}{2}J'(\hat{y}).$$

In turn, this would imply $\eta(r_1) = \eta(r_2)$ and hence $r_1 = r_2$, a contradiction. So, (iii) holds. Finally, let us prove (iv). For each $r \in A$, continue to denote by $\Gamma(r)$ the unique point of $\Gamma(r)$. Let $r \in A$ and let $\{r_k\}$ be any sequence in A converging to r. Up to a subsequence, $\{\Gamma(r_k)\}$ converges weakly to some $\tilde{x} \in B_\rho$. Moreover, for each $k \in \mathbf{N}$, $x \in B_\rho$, one has

$$\frac{\rho - \|x\|^2}{r_k - J(x)} \le \frac{\rho - \|\Gamma(r_k)\|^2}{r_k - J(\Gamma(r_k))}.$$

From this, after easy manipulations, we get

$$\begin{aligned} &\big\|\Gamma(r_k)\big\|^2 - \frac{\rho - \|x\|^2}{r - J(x)} J\big(\Gamma(r_k)\big) - \left(\frac{\rho - \|x\|^2}{r_k - J(x)} - \frac{\rho - \|x\|^2}{r - J(x)}\right) J\big(\Gamma(r_k)\big) \\ &\quad \le \rho - \frac{\rho - \|x\|^2}{r_k - J(x)} r_k. \end{aligned} \tag{4}$$

Since the sequence $\{J(\Gamma(r_k))\}$ is bounded above, we have

$$\limsup_{k\to\infty} \left(\frac{\rho - \|x\|^2}{r_k - J(x)} - \frac{\rho - \|x\|^2}{r - J(x)}\right) J\big(\Gamma(r_k)\big) \le 0. \tag{5}$$

On the other hand, by sequential weak semicontinuity, we also have

$$\|\tilde{x}\|^2 - \frac{\rho - \|x\|^2}{r - J(x)} J(\tilde{x}) \le \liminf_{k\to\infty} \left(\big\|\Gamma(r_k)\big\|^2 - \frac{\rho - \|x\|^2}{r - J(x)} J\big(\Gamma(r_k)\big)\right). \tag{6}$$

Now, passing in (4) to the $\liminf$, in view of (5) and (6), we obtain

$$\|\tilde{x}\|^2 - \frac{\rho - \|x\|^2}{r - J(x)} J(\tilde{x}) \le \rho - \frac{\rho - \|x\|^2}{r - J(x)} r,$$

which is equivalent to

$$\frac{\rho - \|x\|^2}{r - J(x)} \le \frac{\rho - \|\tilde{x}\|^2}{r - J(\tilde{x})}.$$

Since this holds for all $x \in B_\rho$, we have $\tilde{x} = \Gamma(r)$. So, $\Gamma_{|A}$ is continuous at r with respect to the weak topology. Now, assuming also that J is sequentially weakly upper semicontinuous, in view of the continuity of η in $]\beta_\rho, +\infty[$, we have

$$\lim_{k\to\infty} \frac{\rho - \|\Gamma(r_k)\|^2}{r_k - J(\Gamma(r_k))} = \frac{\rho - \|\Gamma(r)\|^2}{r - J(\Gamma(r))},$$

and hence

$$\begin{aligned} \liminf_{k\to\infty} \big(\rho - \big\|\Gamma(r_k)\big\|^2\big) &= \frac{\rho - \|\Gamma(r)\|^2}{r - J(\Gamma(r))} \liminf_{k\to\infty} \big(r_k - J\big(\Gamma(r_k)\big)\big) \\ &= \frac{\rho - \|\Gamma(r)\|^2}{r - J(\Gamma(r))} \Big(r - \limsup_{k\to\infty} J\big(\Gamma(r_k)\big)\Big) \\ &\ge \frac{\rho - \|\Gamma(r)\|^2}{r - J(\Gamma(r))} \big(r - J\big(\Gamma(r)\big)\big) = \rho - \big\|\Gamma(r)\big\|^2 \end{aligned}$$

from which

$$\limsup_{k\to\infty}\|\Gamma(r_k)\| \le \|\Gamma(r)\|.$$

Since X is a Hilbert space and $\{\Gamma(r_k)\}$ converges weakly to $\Gamma(r)$, this implies that

$$\lim_{k\to\infty}\|\Gamma(r_k) - \Gamma(r)\| = 0,$$

which shows the continuity of $\Gamma_{|A}$ at r in the strong topology. □

Remark 1 Clearly, when J is sequentially weakly upper semicontinuous in B_ρ, the assertions of Theorem 1 hold in the whole interval $]\beta_\rho, \rho\delta_\rho[$, since a can be any positive number.

Remark 2 The simplest way to satisfy condition (2) is, of course, to assume that

$$\limsup_{x\to 0}\frac{J(x)}{\|x\|^2} = +\infty.$$

Another reasonable way is provided by the following proposition.

Proposition 2 *For some $s > 0$, assume that J is Gâteaux differentiable in $B_s \setminus \{0\}$ and that there exists a global maximum $\hat{x}$ of $J_{|B_s}$ such that*

$$\langle J'(\hat{x}), \hat{x}\rangle < 2J(\hat{x}).$$

Then (2) *holds with $\rho = \|\hat{x}\|^2$.*

Proof For each $t \in]0,1]$, set

$$\omega(t) = \frac{J(t\hat{x})}{\|t\hat{x}\|^2}.$$

Clearly, ω is derivable in $]0,1]$. In particular, one has

$$\omega'(1) = \frac{\langle J'(\hat{x}), \hat{x}\rangle - 2J(\hat{x})}{\|\hat{x}\|^2}.$$

So, by assumption, $\omega'(1) < 0$ and hence, in a left neighborhood of 1, we have

$$\omega(t) > \omega(1),$$

which implies the validity of (2) with $\rho = \|\hat{x}\|^2$. □

Also, notice the following consequence of Theorem 1.

Theorem 2 *For some $\rho > 0$, let the assumptions of Theorem* 1 *be satisfied.*

Then there exists an open interval $I \subseteq]0,+\infty[$ and an increasing function $\varphi : I \to]0,\rho[$ such that, for each $\lambda \in I$, one has

$$\emptyset \neq \Big\{x \in S_{\varphi(\lambda)} : J(x) = \sup_{S_{\varphi(\lambda)}} J\Big\} \subseteq \big\{x \in X : x = \lambda J'(x)\big\}.$$

Proof Take

$$I = \frac{1}{2}\eta\left(\left]\beta_\rho + \frac{\rho}{a}, \rho\delta_\rho\right[\right).$$

Clearly, I is an open interval since η is continuous and decreasing. Now, for each $r \in]\beta_\rho + \frac{\rho}{a}, \rho\delta_\rho[$, pick $v_r \in \Gamma(r)$. Finally, set

$$\varphi(\lambda) = \|v_{\eta^{-1}(2\lambda)}\|^2$$

for all $\lambda \in I$. Taking (iii) into account, we then realize that the function φ (whose range is contained in $]0, \rho[$) is the composition of two decreasing functions, and so it is increasing. Clearly, the conclusion follows directly from (ii). □

We conclude deriving from Theorem 1 the following multiplicity result.

Theorem 3 *For some* $\rho > 0$, *assume that* J *is sequentially weakly upper semicontinuous in* B_ρ, *Gâteaux differentiable in* $\operatorname{int}(B_\rho) \setminus \{0\}$ *and satisfies* (2). *Moreover, assume that there exists* $\tilde{\rho}$ *satisfying*

$$\inf_{x \in D} \|x\|^2 < \tilde{\rho} < \sup_{x \in D} \|x\|^2, \tag{7}$$

where

$$D = \bigcup_{r \in]\beta_\rho, \rho\delta_\rho[} \Gamma(r),$$

such that $J_{|S_{\tilde{\rho}}}$ *has either two global maxima or a global maximum at which* J' *vanishes.*

Then there exists $\tilde{\lambda} > 0$ *such that the equation*

$$x = \tilde{\lambda} J'(x)$$

has at least two non-zero solutions which are global minima of the restriction of the functional $\frac{1}{2}\|\cdot\|^2 - \tilde{\lambda}J(\cdot)$ *to* $\operatorname{int}(B_\rho)$.

Proof For each $r \in]\beta_\rho, \rho\delta_\rho[$, in view of (7), we can pick $v_r \in \Gamma(r)$ (recall Remark 1), so that

$$\inf_{]\beta_\rho, \rho\delta_\rho[} \psi < \tilde{\rho} < \sup_{]\beta_\rho, \rho\delta_\rho[} \psi, \tag{8}$$

where

$$\psi(r) = \|v_r\|^2.$$

Two cases can occur. First, assume that $\tilde{\rho} \in \psi(]\beta_\rho, \rho\delta_\rho[)$. So, $\psi(\tilde{r}) = \tilde{\rho}$ for some $\tilde{r} \in]\beta_\rho, \rho\delta_\rho[$. So, by (ii), for each global maximum u of $J_{|S_{\tilde{\rho}}}$, we have $J'(u) \neq 0$. As a consequence, in this case, $J_{|S_{\tilde{\rho}}}$ has at least two global maxima which, by (ii) again, satisfies the conclusion with $\tilde{\lambda} = \frac{1}{2}\eta(\tilde{r})$. Now, suppose that $\tilde{\rho} \notin \psi(]\beta_\rho, \rho\delta_\rho[)$. In this case, in view of (8), the function ψ is discontinuous and hence, in view of (iv), there exists some $r^* \in]\beta_\rho, \rho\delta_\rho[$ such that $\Gamma(r^*)$ has at least two elements which, by (ii), satisfy the conclusion with $\tilde{\lambda} = \frac{1}{2}\eta(r^*)$. □

Competing interests
The author declares that he has no competing interests.

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