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Contraction mapping principle with generalized altering distance function in ordered metric spaces and applications to ordinary differential equations

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Abstract

The aim of this paper is to present the definition of a generalized altering distance function and to extend the results of Yan *et al.* (Fixed Point Theory Appl. 2012:152, 2012) and some others, and to prove a new fixed point theorem of generalized contraction mappings in a complete metric space endowed with a partial order by using generalized altering distance functions. The results of this paper can be used to investigate a large class of nonlinear problems. As an application, we discuss the existence of a solution for a periodic boundary value problem.

Keywords: contraction mapping principle; partially ordered metric spaces; fixed point; generalized altering distance function; ordinary differential equations

1 Introduction

The Banach contraction mapping principle is a classical and powerful tool in nonlinear analysis. Weak contractions are generalizations of the Banach contraction mapping, which have been studied by several authors. In [1–8], the authors prove some types of weak contractions in complete metric spaces, respectively. In particular, the existence of a fixed point for weak contractions and generalized contractions was extended to partially ordered metric spaces in [2, 9–22]. Among them, some involve altering distance functions. Such functions were introduced by Khan *et al.* in [1], where they present some fixed point theorems with the help of such functions. First, we recall the definition of an altering distance function.

Definition 1.1 An altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ which satisfies:

(a) ψ is continuous and non-decreasing.
(b) $\psi = 0$ if and only if $t = 0$.

Recently, Harjani and Sadarangani proved some fixed point theorems for weak contractions and generalized contractions in partially ordered metric spaces by using the altering distance function in [11, 23], respectively. Their results improve the theorems of [2, 3].

Theorem 1.2 ([11]) *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric $d \in X$ such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous and non-decreasing mapping such that*

$$d\bigl(f(x), f(y)\bigr) \leq d(x, y) - \psi\bigl(d(x, y)\bigr), \quad \text{for } x \geq y,$$

where $\psi : [0, \infty) \to [0, \infty)$ is continuous and non-decreasing function such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Theorem 1.3 ([23]) *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric $d \in X$ such that (X, d) is a complete metric space. Let $f : X \to X$ be a continuous and non-decreasing mapping such that*

$$\psi d\bigl(f(x), f(y)\bigr) \leq \psi\bigl(d(x, y)\bigr) - \phi\bigl(d(x, y)\bigr), \quad \text{for } x \geq y,$$

where ψ and ϕ are altering distance functions. If there exists $x_0 \in X$ with $x_0 \leq f(x_0)$, then f has a fixed point.

Subsequently, Amini-Harandi and Emami proved another fixed point theorem for contraction type maps in partially ordered metric spaces in [10]. The following class of functions is used in [10].

Let $\Re$ denote the class of those functions $\beta : [0, \infty) \to [0, 1)$ which satisfy the condition: $\beta(t_n) \to 1 \Rightarrow t_n \to 0$.

Theorem 1.4 ([10]) *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d such that (X, d) is a complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq f(x_0)$. Suppose that there exists $\beta \in \Re$ such that*

$$d\bigl(f(x), f(y)\bigr) \leq \beta\bigl(d(x, y)\bigr)d(x, y) \quad \text{for each } x, y \in X \text{ with } x \geq y.$$

Assume that either f is continuous or M is such that if an increasing sequence $x_n \to x \in X$, then $x_n \leq x$, $\forall n$. Besides, if for each $x, y \in X$ there exists $z \in m$ which is comparable to x and y, then f has a unique fixed point.

In 2012, Yan *et al.* proved the following result.

Theorem 1.5 ([24]) *Let X be a partially ordered set and suppose that there exists a metric d in x such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that*

$$\psi\bigl(d(Tx, Ty)\bigr) \leq \phi\bigl(d(x, y)\bigr), \quad \forall x \geq y,$$

where ψ is an altering distance function and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

The aim of this paper is to present the definition of generalized altering distance function and to extend the results of Yan *et al.* [24] and some others, and to prove a new fixed point theorem of generalized contraction mappings in a complete metric space endowed with a partial order by using generalized altering distance functions. The results of this paper can be used to investigate a large class of nonlinear problems. As an application, we discuss the existence of a solution for a periodic boundary value problem.

2 Main results

We first give the definition of generalized altering distance function as follows.

Definition 2.1 A generalized altering distance function is a function $\psi : [0,\infty) \to [0,\infty)$ which satisfies:

(a) ψ is non-decreasing;
(b) $\psi = 0$ if and only if $t = 0$.

We first recall the following notion of a monotone non-decreasing function in a partially ordered set.

Definition 2.2 If $(X,\leq)$ is a partially ordered set and $T : X \to X$, we say that T is monotone non-decreasing if $x, y \in X$, $x \leq y \Rightarrow T(x) \leq T(y)$.

This definition coincides with the notion of a non-decreasing function in the case where $X = R$ and $\leq$ represents the usual total order in R.

In what follows, we prove the following theorem, which is the generalized type of Theorems 1.2-1.5.

Theorem 2.3 *Let X be a partially ordered set and suppose that there exists a metric d in x such that (X,d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that*

$$\psi\big(d(Tx,Ty)\big) \leq \phi\big(d(x,y)\big), \quad \forall x \geq y,$$

where ψ is a generalized altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a right upper semi-continuous function with the condition: $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.

Proof Since T is a non-decreasing function, we obtain by induction that

$$x_0 \leq Tx_0 \leq T^2x_0 \leq T^3x_0 \leq \cdots \leq T^nx_0 \leq T^{n+1}x_0 \leq \cdots. \tag{1}$$

Put $x_{n+1} = Tx_n$. Then, for each integer $n \geq 1$, from (1) and, as the elements x_{n+1} and x_n are comparable, we get

$$\psi\big(d(x_{n+1},x_n)\big) = \psi\big(d(Tx_n,Tx_{n-1})\big) \leq \phi\big(d(x_n,x_{n-1})\big). \tag{2}$$

Using the condition of Theorem 2.3 we have

$$d(x_{n+1},x_n) < d(x_n,x_{n-1}). \tag{3}$$

Hence the sequence $\{d(x_{n+1}, x_n)\}$ is decreasing and, consequently, there exists $r \geq 0$ such that

$$d(x_{n+1}, x_n) \to r^+,$$

as $n \to \infty$. Consider the properties of ψ and ϕ, letting $n \to \infty$ in (2) we get

$$\psi(r) \leq \lim_{n\to\infty} \psi\bigl(d(x_{n+1}, x_n)\bigr) \leq \lim_{n\to\infty} \phi\bigl(d(x_n, x_{n-1})\bigr) \leq \phi(r).$$

By using the condition: $\psi(t) > \phi(t)$ for all $t > 0$, we have $r = 0$, and hence

$$d(x_{n+1}, x_n) \to 0, \tag{4}$$

as $n \to \infty$. In what follows, we will show that $\{x_n\}$ is a Cauchy sequence. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find subsequences $\{x_{n_k}\}$ with $n_k > m_k > k$ such that

$$d(x_{n_k}, x_{m_k}) \geq \varepsilon \tag{5}$$

for all $k \geq 1$. Further, corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (5). Then

$$d(x_{n_{k-1}}, x_{m_{k-1}}) < \varepsilon. \tag{6}$$

From (5) and (6), we have

$$\varepsilon \leq d(x_{n_k}, x_{m_k}) \leq \bigl(d(x_{n_k}), x_{n_{k-1}}\bigr) + d(x_{n_{k-1}}, x_{m_k}) < d(x_{n_k}, x_{n_{k-1}}) + \varepsilon.$$

Letting $k \to \infty$ and using (4), we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k}) = \varepsilon. \tag{7}$$

By using the triangular inequality we have

$$d(x_{n_k}, x_{m_k}) \leq d(x_{n_k}, x_{n_{k-1}}) + d(x_{n_{k-1}}, x_{m_{k-1}}) + d(x_{m_{k-1}}, x_{m_k}),$$
$$d(x_{n_{k-1}}, x_{m_{k-1}}) \leq d(x_{n_{k-1}}, x_{n_k}) + d(x_{n_k}, x_{m_k}) + d(x_{m_k}, x_{m_{k-1}}).$$

Letting $k \to \infty$ in the above two inequalities and using (4) and (7), we have

$$\lim_{k\to\infty} d(x_{n_{k-1}}, x_{m_{k-1}}) = \varepsilon. \tag{8}$$

As $n_k > m_k$ and $x_{n_{k-1}}$ and $x_{m_{k-1}}$ are comparable, using (1) we have

$$\psi\bigl(d(x_{n_k}, x_{m_k})\bigr) \leq \phi\bigl(d(x_{n_{k-1}}, x_{m_{k-1}})\bigr).$$

Consider the properties of ψ and ϕ, letting $k \to \infty$ and taking into account (7) and (8), we have

$$\psi(\varepsilon) \leq \phi(\varepsilon).$$

From the condition $\psi(t) > \phi(t)$ for all $t > 0$, we get $\varepsilon = 0$, which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence and, since X is a complete metric space, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$. Moreover, the continuity of T implies that

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz,$$

and this proves that z is a fixed point. This completes the proof. □

In what follows, we prove that Theorem 2.3 is still valid for T being not necessarily continuous, assuming the following hypothesis in X:

$$\begin{aligned} &\text{If } (x_n) \text{ is a non-decreasing sequence in } X \text{ such that } x_n \to x \\ &\text{then } x_n \leq x \text{ for all } n \in N. \end{aligned} \tag{9}$$

Theorem 2.4 *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Assume that X satisfies* (9). *Let $T : X \to X$ be a non-decreasing mapping such that*

$$\psi\big(d(Tx, Ty)\big) \leq \phi\big(d(x,y)\big), \quad \forall x \geq y,$$

where ψ is a generalized altering distance functions and ϕ: $[0,\infty) \to [0,\infty)$ *is a right upper semi-continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point.*

Proof Following the proof of Theorem 2.3 we only have to check that $T(z) = z$. As (x_n) is a non-decreasing sequence in X and $\lim_{n\to\infty} x_n = z$ the condition (9) gives us that $x_n \leq z$ for every $n \in N$ and consequently,

$$\psi\big(d\big(x_{n+1}, T(z)\big)\big) = \psi\big(d\big(T(x_n), T(z)\big)\big) \leq \phi\big(d(x_n, z)\big).$$

Letting $n \to \infty$ and taking into account that ψ is an altering distance function, we have

$$\psi\big(d\big(z, T(z)\big)\big) \leq \phi(0).$$

Using condition of theorem we have $\phi(0) = 0$, this implies $\Psi(d(z, T(z))) = 0$. Thus, $d(z, T(z)) = 0$ or equivalently, $T(z) = z$. □

Now, we present an example where it can be appreciated that the hypotheses in Theorems 2.3 and Theorems 2.4 do not guarantee uniqueness of the fixed point. An example appears in [12].

Let $X = \{(1,0),(0,1)\} \subset R^2$ and consider the usual order $(x,y) \leq (z,t) \Leftrightarrow x \leq z$, $y \leq t$. Thus, (x,y) is a partially ordered set whose different elements are not comparable. Besides

(X, d_2) is a complete metric space considering d_2 the Euclidean distance. The identity map $T(x, y) = (x, y)$ is trivially continuous and non-decreasing and condition (1) of Theorem 2.4 is satisfied, since the elements in X are only comparable to themselves. Moreover, $(1, 0) \leq T(1, 0) = (1, 0)$ and T has two fixed points in X.

In what follows, we give a sufficient condition for the uniqueness of the point in Theorems 2.3 and 2.4. This condition is:

$$\text{for } x, y \in X \text{ there exists a lower bound or an upper bound.} \tag{10}$$

In [12] it is proved that condition (10) is equivalent to:

$$\text{for } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{11}$$

Theorem 2.5 *Adding condition* (11) *to the hypotheses of Theorem* 2.3 (*resp. Theorem* 2.4) *we obtain the uniqueness of the fixed point of* T.

Proof Suppose that there exist $z, y \in X$ which are fixed points. We distinguish two cases.

Case 1. If y is comparable to z then $T^n(y) = y$ is comparable to $T^n(z) = z$ for $n = 0, 1, 2, \dots$ and

$$\begin{aligned} \psi\bigl(d(z, y)\bigr) &= \psi\bigl(d\bigl(T^n(z), T^n(y)\bigr)\bigr) \\ &\leq \phi\bigl(d\bigl(T^{n-1}(z), T^{n-1}(y)\bigr)\bigr) \\ &\leq \phi\bigl(d(z, y)\bigr). \end{aligned}$$

As we have the condition $\psi(t) > \phi(t)$ for $t > 0$ we obtain $d(z, y) = 0$ and this implies $z = y$.

Case 2. If y is not comparable to z then there exists $x \in X$ comparable to y and z. Monotonicity of T implies that $T^n(x)$ is comparable to $T^n(y)$ and to $T^n(z) = z$, for $n = 0, 1, 2, \dots$ Moreover,

$$\begin{aligned} \psi\bigl(d\bigl(z, T^n(x)\bigr)\bigr) &= \psi\bigl(d\bigl(T^n(z), T^n(x)\bigr)\bigr) \\ &\leq \phi\bigl(d\bigl(T^{n-1}(z), T^{n-1}(x)\bigr)\bigr) \\ &= \phi\bigl(d\bigl(z, T^{n-1}(x)\bigr)\bigr). \end{aligned} \tag{12}$$

Hence, ψ is a generalized altering distance function and we have the condition $\psi(t) > \phi(t)$ for $t > 0$, this gives us that $\{d(z, f^n(x))\}$ is a non-negative decreasing sequence and, consequently, there exists γ such that

$$\lim_{n \to \infty} d\bigl(z, T^n(x)\bigr) = \gamma.$$

Letting $n \to \infty$ in (12) and, taking into account the properties of ψ and ϕ, we obtain

$$\psi(\gamma) \leq \phi(\gamma).$$

This and the condition $\psi(t) > \phi(t)$ for $t > 0$ imply $\gamma = 0$. Analogously, it can be proved that

$$\lim_{n \to \infty} d\bigl(y, T^n(x)\bigr) = 0.$$

Finally, as

$$\lim_{n\to\infty} d\bigl(z, T^n(x)\bigr) = \lim_{n\to\infty} d\bigl(y, T^n(x)\bigr) = 0$$

the uniqueness of the limit gives us $y = z$. This finishes the proof. □

Remark 2.6 Under the assumption of Theorem 2.3, it can be proved that for every $x \in X$, $\lim_{n\to\infty} T^n(x) = z$, where z is the fixed point (*i.e.* the operator f is Picard).

Remark 2.7 Theorem 1.2 is a particular case of Theorem 2.3 for ψ being the identity function, and $\phi(t) = t - \psi(t)$. Theorem 1.3 is a particular case of our Theorem 2.3 for $\phi(t)$ being replaced by $\psi(t) - \phi(t)$. Theorem 1.4 is a particular case of Theorem 2.3 for ψ being the identity function, and $\phi(t) = \beta(t)t$. Theorem 1.5 is also a particular case of Theorem 2.3 for ψ and ϕ being continuous.

Example 2.8 The following are some generalized altering distance functions:

$$\psi_1(t) = \begin{cases} 0, & t = 0, \\ [t] + 1, & t > 0, \end{cases}$$

$$\psi_2(t) = \begin{cases} 0, & t = 0, \\ \lambda([t] + 1), & t > 0, \end{cases}$$

where $\alpha > 0$ is a constant.

$$\psi_3(t) = \begin{cases} t, & 0 \le t < 1, \\ \alpha t^2, & t \ge 1, \end{cases}$$

where $\alpha \ge 1$ is a constant.

We choose $\psi(t) = \psi_3(t)$ and

$$\phi(t) = \begin{cases} t^2, & 0 \le t < 1, \\ \beta t, & t \ge 1, \end{cases}$$

where $0 < \beta < \alpha$ is a constant. By using Theorem 2.3, we can get the following result.

Theorem 2.9 *Let X be a partially ordered set and suppose that there exists a metric d in x such that (X, d) is a complete metric space. Let $T : X \to X$ be a continuous and non-decreasing mapping such that*

$$\begin{aligned} &0 \le d(Tx, Ty) < 1 \quad \Rightarrow \quad d(Tx, Ty) \le \bigl(d(x, y)\bigr)^2, \\ &d(Tx, Ty) \ge 1 \quad \Rightarrow \quad \alpha\bigl(d(Tx, Ty)\bigr)^2 \le \beta d(x, y) \end{aligned}$$

for any $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \le Tx_0$, then T has a fixed point.

3 Application to ordinary differential equations

In this section we present two examples where our Theorems 2.3 and 2.4 can be applied. The first example is inspired by [17]. We study the existence of a solution for the following first-order periodic problem:

$$\begin{cases} u'(t) = f(t, u(t)), & t \in [0, T], \\ u(0) = u(T), \end{cases} \tag{13}$$

where $T > 0$ and $f : I \times R \to R$ is a continuous function. Previously, we considered the space $C(I)$ ($I = [0, T]$) of continuous functions defined on I. Obviously, this space with the metric given by

$$d(x, y) = \sup\{|x(t) - y(t)| : t \in I\}, \quad \text{for } x, y \in C(I),$$

is a complete metric space. $C(I)$ can also be equipped with a partial order given by

$$x, y \in C(I), \quad x \leq y \quad \Leftrightarrow \quad x(t) \leq y(t)m, \quad \text{for } t \in I.$$

Clearly, $(C(I), \leq)$ satisfies condition (10), since for $x, y \in C(I)$ the functions $\max\{x, y\}$ and $\min\{x, y\}$ are least upper and greatest lower bounds of x and y, respectively. Moreover, in [17] it is proved that $(C(I), \leq)$ with the above mentioned metric satisfies condition (9).

Now we give the following definition.

Definition 3.1 A lower solution for (13) is a function $\alpha \in C^{(1)}(I)$ such that

$$\begin{cases} \alpha'(t) \leq f(t, \alpha(t)), & \text{for } t \in I, \\ \alpha(0) \leq \alpha(T). \end{cases}$$

Theorem 3.2 *Consider problem* (13) *with $f : I \times R \to R$ continuous and suppose that there exist $\lambda, \alpha > 0$ with*

$$\alpha \leq \left(\frac{2\lambda(e^{\lambda T} - 1)}{T(e^{\lambda T} + 1)} \right)^{\frac{1}{2}}$$

such that for $x, y \in R$ with $x \geq y$

$$0 \leq f(t, x) + \lambda x - [f(t, y) + \lambda y] \leq \alpha \sqrt{g(x - y)},$$

where $g(t) : [0, +\infty) \to [0, +\infty)$ is a light upper semi-continuous function with $g(0) = 0$, $g(t) < t^2$, $\forall t > 0$. Then the existence of a lower solution for (13) *provides the existence of an unique solution of* (13).

Proof Problem (13) can be written as

$$\begin{cases} u'(t) + \lambda u(t) = f(t, u(t)) + \lambda u(t), & \text{for } t \in I = [0, T], \\ u(0) = u(T). \end{cases}$$

This problem is equivalent to the integral equation

$$u(t) = \int_0^T G(t,s)\big[f\big(s,u(s)\big) + \lambda u(s)\big]\,ds,$$

where $G(t,s)$ is the Green function given by

$$G(t,s) = \begin{cases} \frac{e^{\lambda(T+s-t)}}{e^{\lambda T}-1}, & 0 \le s < t \le T, \\ \frac{e^{\lambda(s-t)}}{e^{\lambda T}-1}, & 0 \le t < s \le T. \end{cases}$$

Define $F : C(I) \to C(I)$ by

$$(Fu)(t) = \int_0^T G(t,s)\big[f\big(s,u(s)\big) + \lambda u(s)\big]\,ds.$$

Note that if $u \in C(I)$ is a fixed point of F then $u \in C^1(I)$ is a solution of (13). In what follows, we check that the hypotheses in Theorems 2.3 and 2.4 are satisfied. The mapping F is non-decreasing, since we have $u \ge v$, and using our assumption. We can obtain

$$f(t,u) + \lambda u \ge f(t,v) + \lambda v$$

which implies, since $G(t,s) > 0$, that for $t \in I$

$$\begin{aligned} (Fu)(t) &= \int_0^T G(t,s)\big[f\big(s,u(s)\big) + \lambda u(s)\big]\,ds \\ &\ge \int_0^T G(t,s)\big[f\big(s,v(s)\big) + \lambda v(s)\big]\,ds = (Fv)(t). \end{aligned}$$

Besides, for $u \ge v$, we have

$$\begin{aligned} d(Fu,Fv) &= \sup_{t\in I}\big|(Fu)(t) - (Fv)(t)\big| \\ &= \sup_{t\in I}\big((Fu)(t) - (Fv)(t)\big) \\ &= \sup_{t\in I}\int_0^T G(t,s)\big[f\big(s,u(s)\big) + \lambda u(s) - f\big(s,v(s)\big) - \lambda v(s)\big]\,ds \\ &\le \sup_{t\in I}\int_0^T G(t,s)\alpha\sqrt{g\big(u(s)-v(s)\big)}\,ds. \end{aligned} \tag{14}$$

Using the Cauchy-Schwarz inequality in the last integral we get

$$\begin{aligned} &\int_0^T G(t,s)\alpha\sqrt{g\big(u(s)-v(s)\big)}\,ds \\ &\quad\le \left(\int_0^T G(t,s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^T \alpha^2 g\big(u(s)-v(s)\big)\,ds\right)^{\frac{1}{2}}. \end{aligned} \tag{15}$$

The first integral gives us

$$\begin{aligned}\int_0^T G(t,s)^2\,ds &= \int_0^t G(t,s)^2\,ds + \int_t^T G(t,s)^2\,ds\\ &= \int_0^t \frac{e^{2\lambda(T+s-t)}}{(e^{\lambda T}-1)^2}\,ds + \int_t^T \frac{e^{2\lambda(s-t)}}{(e^{\lambda T}-1)^2}\,ds\\ &= \frac{1}{2\lambda(e^{\lambda T}-1)^2}e^{(2\lambda T-1)}\\ &= \frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}.\end{aligned}\tag{16}$$

The second integral in (15) gives the following estimate:

$$\begin{aligned}\int_0^T \alpha^2 g\bigl(u(s)-v(s)\bigr)\,ds &\le \alpha^2 g\bigl(\|u-v\|\bigr)\cdot T\\ &= \alpha^2 g\bigl(d(u,v)\bigr)\cdot T.\end{aligned}\tag{17}$$

Taking into account (14)-(17) we have

$$\begin{aligned}d(Fu,Fv) &\le \sup_{t\in I}\left(\frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}\right)^{\frac{1}{2}}\cdot\bigl(\alpha^2 g\bigl(d(u,v)\bigr)\cdot T\bigr)^{\frac{1}{2}}\\ &= \left(\frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}\right)^{\frac{1}{2}}\cdot\alpha\cdot\sqrt{T}\cdot\bigl(g\bigl(d(u,v)\bigr)\bigr)^{\frac{1}{2}}\end{aligned}$$

and from the last inequality we obtain

$$d(Fu,Fv)^2 \le \frac{e^{\lambda T}+1}{2\lambda(e^{\lambda T}-1)}\cdot\alpha^2\cdot T\cdot g\bigl(d(u,v)\bigr)$$

or, equivalently.

$$2\lambda\bigl(e^{\lambda T}-1\bigr)d(Fu,Fv)^2 \le \bigl(e^{\lambda T}+1\bigr)\cdot\alpha^2\cdot T\cdot g\bigl(d(u,v)\bigr).$$

By our assumption, as

$$\alpha \le \left(\frac{2\lambda(e^{\lambda T}-1)}{T(e^{\lambda T}+1)}\right)^{\frac{1}{2}},$$

the last inequality gives us

$$2\lambda\bigl(e^{\lambda T}-1\bigr)d(Fu,Fv)^2 \le 2\lambda\bigl(e^{\lambda T}-1\bigr)\cdot g\bigl(d(u,v)\bigr),$$

and, hence,

$$d(Fu,Fv)^2 \le g\bigl(d(u,v)\bigr).\tag{18}$$

Put $\psi(t) = t^2$ and $\phi(t) = g(t)$. Obviously, ψ is a generalized altering distance function, $\psi(t)$ and $\phi(t)$ satisfy the condition of $\psi(t) > \phi(t)$ for $t > 0$. From (18), we obtain for $u \geq v$

$$\psi\big(d(Fu, Fv)\big) \leq \phi\big(d(u, v)\big).$$

Finally, let $\alpha(t)$ be a lower solution for (13); we claim that $\alpha \leq F(\alpha)$. In fact

$$\alpha'(t) + \lambda\alpha(t) \leq f\big(t, \alpha(t)\big) + \lambda\alpha(t), \quad \text{for } t \in I.$$

We multiply by $e^{\lambda t}$,

$$\big(\alpha(t)e^{\lambda t}\big)' \leq \big[f\big(t, \alpha(t)\big) + \lambda\alpha(t)\big]e^{\lambda t}, \quad \text{for } t \in I,$$

and this gives us

$$\alpha(t)e^{\lambda t} \leq \alpha(0) + \int_0^t \big[f\big(s, \alpha(s)\big) + \lambda\alpha(s)\big]e^{\lambda s}\,ds, \quad \text{for } t \in I. \tag{19}$$

As $\alpha(0) \leq \alpha(T)$, the last inequality gives us

$$\alpha(0)e^{\lambda t} \leq \alpha(T)e^{\lambda T} \leq \alpha(0) + \int_0^T \big[f\big(s, \alpha(s)\big) + \lambda\alpha(s)\big]e^{\lambda s}\,ds,$$

and so

$$\alpha(0) \leq \int_0^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}\big[f\big(s, \alpha(s)\big) + \lambda\alpha(s)\big]\,ds.$$

This and (19) give us

$$\alpha(t)e^{\lambda t} \leq \int_0^t \frac{e^{\lambda(T+s)}}{e^{\lambda T} - 1}\big[f\big(s, \alpha(s)\big) + \lambda\alpha(s)\big]\,ds + \int_t^T \frac{e^{\lambda s}}{e^{\lambda T} - 1}\big[f\big(s, \alpha(s)\big) + \lambda\alpha(s)\big]\,ds$$

and, consequently,

$$\begin{aligned}
\alpha(t) &\leq \int_0^t \frac{e^{\lambda(T+s-t)}}{e^{\lambda T} - 1}\,ds + \int_0^t \frac{e^{\lambda(s-t)}}{e^{\lambda T} - 1}\big[f\big(s, \alpha(s)\big) + \lambda\alpha(s)\big]\,ds \\
&= \int_0^T G(t, s)\big[f\big(s, \alpha(s)\big) + \lambda\alpha(s)\big]\,ds \\
&= (F\alpha)(t), \quad \text{for } t \in I.
\end{aligned}$$

Finally, Theorems 2.3 and 2.4 show that F has an unique fixed point. □

Example 3.3 In Theorem 3.2, we can choose the function $g(t)$ as follows:

(1) $g_1(t) = \ln(t^2 + 1)$;

(2)

$$g_2(t) = \begin{cases} t^3, & 0 \leq t < 1, \\ \frac{1}{2}, & t = 1, \\ t, & 1 < t < +\infty. \end{cases}$$

(3)

$$g_3(t) = \begin{cases} t^3, & 0 \le t \le \frac{1}{2}, \\ t - \frac{3}{8}, & \frac{1}{2} < t < +\infty. \end{cases}$$

The functions $g_1(t)$, $g_2(t)$ are continuous and non-decreasing. The function $g_3(t)$ is right upper semi-continuous. If we choose $g(t) = g_1(t)$ in Theorem 3.2, we obtain the result of [5].

Example 3.4 Consider the following first-order periodic problem:

$$\begin{cases} u'(t) = \frac{\sin t}{e^t} - \beta x, & t \in [0,T], \\ u(0) = u(T). \end{cases} \tag{20}$$

Let

$$f(t,x) = \frac{\sin t}{e^t} - \beta x, \quad x \in [0,\infty), t \in [0,1],$$

then $f(t,x)$ is continuous. Further, for $x \ge y$, we have

$$f(t,x) + \lambda x - \bigl[f(t,y) + \lambda y\bigr] = 2(\lambda - \beta)\frac{x-y}{2}.$$

We chose $\beta \in [0,\lambda]$ such that

$$2(\lambda - \beta) \le \left(\frac{2\lambda(e^{\lambda T} - 1)}{T(e^{\lambda T} + 1)}\right)^{\frac{1}{2}}.$$

Taking $g(t) = (\frac{t}{2})^2$ for all $t \in [0,+\infty)$, we have

$$f(t,x) + \lambda x - \bigl[f(t,y) + \lambda y\bigr] = 2(\lambda - \beta)\sqrt{g(x-y)}.$$

By using Theorem 3.2, we know that the first-order periodic problem (20) has a unique solution.

A second example where our results can be applied is the following two-point boundary value problem of the second order differential equation:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t,x), & x \in [0,\infty), t \in [0,1], \\ x(0) = x(1) = 0. \end{cases} \tag{21}$$

It is well known that $x \in C^2[0,1]$ is a solution of (20) that is equivalent to $x \in C[0,1]$ being a solution of the integral equation

$$x(t) = \int_0^1 G(t,s)f\bigl(s,x(s)\bigr)\,ds, \quad \text{for } t \in [0,1],$$

where $G(t,s)$ is the Green function given by

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{22}$$

Theorem 3.5 *Consider problem* (21) *with* $f : I \times R \to [0,\infty)$ *continuous and non-decreasing with respect to the second variable and suppose that there exists* $0 \leq \alpha \leq 8$ *such that for* $x, y \in R$ *with* $x \geq y$

$$f(t,x) - f(t,y) \leq \alpha \sqrt{g(x-y)}, \tag{23}$$

where $g(t) : [0,+\infty) \to [0,+\infty)$ *is a light upper semi-continuous function with* $g(0) = 0$, $g(t) < t^2$, $\forall t > 0$. *Then our problem* (21) *has a unique non-negative solution.*

Proof Consider the cone

$$P = \{x \in C[0,1] : x(t) \geq 0\}.$$

Obviously, (P,d) with $d(x,y) = \sup\{|x(t) - y(t)| : t \in [0,1]\}$ is a complete metric space. Consider the operator given by

$$(Tx)(t) = \int_0^1 G(t,s) f\bigl(s, x(s)\bigr)\,ds, \quad \text{for } x \in P,$$

where $G(t,s)$ is the Green function appearing in (22).

As f is non-decreasing with respect to the second variable, for $x, y \in P$ with $y \geq x$ and $t \in [0,1]$, we have

$$(Ty)(t) = \int_0^1 G(t,s) f\bigl(s, y(s)\bigr)\,ds \geq \int_0^1 G(t,s) f\bigl(s, x(s)\bigr)\,ds \geq (Tx)(t),$$

and this proves that T is a non-decreasing operator.

Besides, for $y \geq x$ and taking into account (23), we obtain

$$\begin{aligned} d(Ty, Tx) &= \sup_{t\in[0,1]} \bigl|(Tx)(t) - (Ty)(t)\bigr| \\ &= \sup_{t\in[0,1]} \bigl((Tx)(t) - (Ty)(t)\bigr) \\ &= \sup_{t\in[0,1]} \int_0^1 G(t,s)\bigl(f\bigl(s,x(s)\bigr) - f\bigl(s,y(s)\bigr)\bigr)\,ds \\ &\leq \sup_{t\in[0,1]} \int_0^1 G(t,s)\alpha\sqrt{g\bigl(x(s) - y(s)\bigr)} \\ &\leq \sup_{t\in[0,1]} \int_0^1 G(t,s)\alpha\sqrt{g\bigl(x(s) - y(s)\bigr)}\,ds \\ &= \alpha\sqrt{\ln\bigl[\|y-x\|^2 + 1\bigr]} \sup_{t\in[0,1]} \int_0^1 G(t,s)\,ds. \end{aligned} \tag{24}$$

It is easy to verify that

$$\int_0^1 G(t,s)\,ds = \frac{-t^2}{2} + \frac{t}{2}$$

and that

$$\sup_{t\in[0,1]} \int_0^1 G(t,s)\,ds = \frac{1}{8}.$$

These facts, the inequality (24), and the hypothesis $0 < \alpha \leq 8$ give us

$$\begin{aligned} d(Tx,Ty) &\leq \frac{\alpha}{8}\sqrt{g(x-y)} \\ &\leq \sqrt{g\bigl(\|x-y\|\bigr)} = \sqrt{g\bigl(d(x,y)\bigr)}. \end{aligned}$$

Hence

$$d(Ty,Tx)^2 \leq g\bigl(d(x,y)\bigr).$$

Put $\psi(t) = t^2$, $\phi(t) = g(t)$, obviously ψ is an altering distance function, ψ and ϕ satisfy the condition of $\psi(t) > \phi(t)$, for $t > 0$. From the last inequality, we have

$$\psi\bigl(d(Tx,Ty)\bigr) \leq \phi\bigl(d(x,y)\bigr).$$

Finally, as f and G are non-negative functions

$$T0 = \int_0^1 G(t,s)f(s,0)\,ds \geq 0$$

and Theorems 2.3 and 2.4 tell us that F has a unique non-negative solution. □

Remark 3.6 In Theorem 3.5, we can choose $g(t)$ as $g_1(1)$, $g_2(t)$, and $g_3(t)$ as well as in Theorem 3.2.

Example 3.7 Consider the following two-point boundary value problem of the second order differential equation:

$$\begin{cases} -\frac{d^2x}{dt^2} = \frac{\sin t}{e^t} + \frac{x}{1+\cos t\pi}, & x \in [0,\infty), t \in [0,1], \\ x(0) = x(1) = 0. \end{cases} \tag{25}$$

Let

$$f(t,x) = \frac{\sin t}{e^t} + \frac{x}{1+\cos t\pi}, \quad x \in [0,\infty), t \in [0,1],$$

then $f(t,x)$ is continuous and non-decreasing with respect to the second variable. Further, for $x \geq y$, we have

$$f(t,x) - f(t,y) = \frac{x}{1+\cos t\pi} - \frac{y}{1+\cos t\pi} \leq \sqrt{\left(\frac{x-y}{2}\right)^2}.$$

Taking $g(t) = \frac{t}{2}$ for all $t \in [0, +\infty)$. By using Theorem 3.2, we know that the two-point boundary value problem (25) has a unique non-negative solution.

Competing interests
The author declares that they have no competing interests.

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