Bnouhachem *Fixed Point Theory and Applications* 2014, 2014:22
http://www.fixedpointtheoryandapplications.com/content/2014/1/22

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

A modified projection method for a common solution of a system of variational inequalities, a split equilibrium problem and a hierarchical fixed-point problem

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Abstract

In this paper, we suggest and analyze a modified projection method for finding a common solution of a system of variational inequalities, a split equilibrium problem, and a hierarchical fixed-point problem in the setting of real Hilbert spaces. We prove the strong convergence of the sequence generated by the proposed method to a common solution of a system of variational inequalities, a split equilibrium problem, and a hierarchical fixed-point problem. Several special cases are also discussed. The results presented in this paper extend and improve some well-known results in the literature.

MSC: 49J30; 47H09; 47J20

Keywords: system of variational inequalities; equilibrium problem; hierarchical fixed-point problem; fixed-point problem; projection method

1 Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Recently, Ceng *et al.* [1] considered the general system of variational inequalities, which involves finding $(x^*,y^*)\in C\times C$ such that

$$\begin{cases} \langle \mu_1 B_1 y^* + x^* - y^*, x - x^*\rangle \geq 0; & \forall x\in C \text{ and } \mu_1 > 0,\\ \langle \mu_2 B_2 x^* + y^* - x^*, x - y^*\rangle \geq 0; & \forall x\in C \text{ and } \mu_2 > 0,\end{cases} \tag{1.1}$$

where $B_i : C \to C$ is a nonlinear mapping for each $i = 1,2$. The solution set of (1.1) is denoted by S^*. As pointed out in [2] the system of variational inequalities is used as a tool to study the Nash equilibrium problem, see, for example, [3–6] and the references therein. We believe that the problem (1.1) could be used to study the Nash equilibrium problem for a two players game. The theory of variational inequalities is well established and it has a wide range of applications in science, engineering, management, and social sciences, see, for example, [4–7] and the references therein.

Ceng *et al.* [1] transformed problem (1.1) into a fixed-point problem (see Lemma 2.2) and introduced an iterative method for finding the common element of the set $\mathrm{Fix}(T)\cap S^*$. Based on the one-step iterative method [8] and the multi-step iterative method [9], Latif *et al.* [10] proposed a multi-step hybrid viscosity method that generates a sequence via an explicit iterative algorithm to compute the approximate solutions of a system of variational inequalities defined over the intersection of the set of solutions of an equilibrium problem, the set of common fixed points of a finite family of nonexpansive mappings, and the solution set of a nonexpansive mapping. Under very mild conditions, they proved that the sequence converges strongly to a unique solution of system of variational inequalities defined over the set consisting of the set of solutions of an equilibrium problem, the set of common fixed points of nonexpansive mappings, and the set of fixed points of a mapping, and to a unique solution of the triple hierarchical variational inequality problem.

On the other hand, by combining the regularization method, Korpelevich's extragradient method, the hybrid steepest-descent method, and the viscosity approximation method, Ceng *et al.* [2] introduced and analyzed implicit and explicit iterative schemes for computing a common element of the solution set of system of variational inequalities and a set of zeros of an accretive operator in Banach space. Under suitable assumptions, they proved the strong convergence of the sequences generated by the proposed schemes.

If $B_1 = B_2 = B$, then the problem (1.1) reduces to finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \mu_1 By^* + x^* - y^*, x - x^* \rangle \geq 0; \quad \forall x \in C \text{ and } \mu_1 > 0, \\ \langle \mu_2 Bx^* + y^* - x^*, x - y^* \rangle \geq 0; \quad \forall x \in C \text{ and } \mu_2 > 0, \end{cases} \tag{1.2}$$

which has been introduced and studied by Verma [11, 12].

If $x^* = y^*$ and $\mu_1 = \mu_2$, then the problem (1.2) collapses to the classical variational inequality: finding $x^* \in C$, such that

$$\langle Bx^*, x - x^* \rangle \geq 0, \quad \forall x \in C.$$

For the recent applications, numerical techniques, and physical formulation, see [1–45].

We introduce the following definitions, which are useful in the following analysis.

Definition 1.1 The mapping $T : C \to H$ is said to be

(a) monotone, if

$$\langle Tx - Ty, x - y \rangle \geq 0, \quad \forall x, y \in C;$$

(b) strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in C;$$

(c) α-inverse strongly monotone, if there exists an $\alpha > 0$ such that

$$\langle Tx - Ty, x - y \rangle \geq \alpha \|Tx - Ty\|^2, \quad \forall x, y \in C;$$

(d) nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(e) k-Lipschitz continuous, if there exists a constant $k > 0$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C;$$

(f) a contraction on C, if there exists a constant $0 \leq k < 1$ such that

$$\|Tx - Ty\| \leq k\|x - y\|, \quad \forall x, y \in C;$$

It is easy to observe that every α-inverse strongly monotone T is monotone and Lipschitz continuous. It is well known that every nonexpansive operator $T : H \to H$ satisfies, for all $(x, y) \in H \times H$, the inequality

$$\big\langle \big(x - T(x)\big) - \big(y - T(y)\big), T(y) - T(x) \big\rangle \leq \frac{1}{2} \big\| \big(T(x) - x\big) - \big(T(y) - y\big) \big\|^2, \tag{1.3}$$

and therefore, we get, for all $(x, y) \in H \times \mathrm{Fix}(T)$,

$$\big\langle x - T(x), y - T(x) \big\rangle \leq \frac{1}{2} \big\| T(x) - x \big\|^2. \tag{1.4}$$

The fixed-point problem for the mapping T is to find $x \in C$ such that

$$Tx = x. \tag{1.5}$$

We denote by $F(T)$ the set of solutions of (1.5). It is well known that $F(T)$ is closed and convex, and $P_F(T)$ is well defined.

The equilibrium problem, denoted by *EP*, is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.6}$$

The solution set of (1.6) is denoted by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to finding a solution of (1.6), see [25, 38]. In 1994, Censor and Elfving [19] introduced and studied the following split feasibility problem.

Let C and Q be nonempty closed convex subsets of the infinite-dimensional real Hilbert spaces H_1 and H_2, respectively, and let $A \in B(H_1, H_2)$, where $B(H_1, H_2)$ denotes the collection of all bounded linear operators from H_1 to H_2. The problem is to find x^* such that

$$x^* \in C \quad \text{such that} \quad Ax^* \in Q.$$

Very recently, Ceng *et al.* [22] introduced and analyzed an extragradient method with regularization for finding a common element of the solution set of the split feasibility problem and the set of fixed points of a nonexpansive mapping S in the setting of infinite-dimensional Hilbert spaces. By combining Mann's iterative method and the extragradient method, Ceng *et al.* [21] proposed three different kinds of Mann type iterative methods for finding a common element of the solution set of the split feasibility problem and the set of fixed points of a nonexpansive mapping S in the setting of infinite-dimensional Hilbert spaces.

Recently, Censor *et al.* [23] introduced a new variational inequality problem which we call the split variational inequality problem (SVIP). Let H_1 and H_2 be two real Hilbert spaces. Given the operators $f : H_1 \to H_1$ and $g : H_2 \to H_2$, a bounded linear operator $A : H_1 \to H_2$, and the nonempty, closed, and convex subsets $C \subseteq H_1$ and $Q \subseteq H_2$, the SVIP is formulated as follows: find a point $x^* \in C$ such that

$$\big\langle f\big(x^*\big), x - x^* \big\rangle \geq 0 \quad \text{for all } x \in C \tag{1.7}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad \langle g(y^*), y - y^* \rangle \geq 0 \quad \text{for all } y \in Q. \tag{1.8}$$

In [36], Moudafi introduced an iterative method which can be regarded as an extension of the method given by Censor *et al.* [23] for the following split monotone variational inclusions:

$$\text{Find} \quad x^* \in H_1 \quad \text{such that} \quad 0 \in f(x^*) + B_1(x^*)$$

and such that

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in g(y^*) + B_2(y^*),$$

where $B_i : H_i \to 2^{H_i}$ is a set-valued mapping for $i = 1, 2$. Later Byrne *et al.* [18] generalized and extended the work of Censor *et al.* [23] and Moudafi [36].

In this paper, we consider the following pair of equilibrium problems, called split equilibrium problems: Let $F_1 : C \times C \to R$ and $F_2 : Q \times Q \to R$ be nonlinear bifunctions and $A : H_1 \to H_2$ be a bounded linear operator, then the split equilibrium problem (SEP) is to find $x^* \in C$ such that

$$F_1(x^*, x) \geq 0, \quad \forall x \in C, \tag{1.9}$$

and such that

$$y^* = Ax^* \in Q \quad \text{solves} \quad F_2(y^*, y) \geq 0, \quad \forall y \in Q. \tag{1.10}$$

The solution set of SEP (1.9)-(1.10) is denoted by $\Lambda = \{p \in EP(F_1) : Ap \in EP(F_2)\}$.

Let $S : C \to H$ be a nonexpansive mapping. The following problem is called a hierarchical fixed-point problem: find $x \in F(T)$ such that

$$\langle x - Sx, y - x \rangle \geq 0, \quad \forall y \in F(T). \tag{1.11}$$

It is known that the hierarchical fixed-point problem (1.11) links with some monotone variational inequalities and convex programming problems; see [45]. Various methods have been proposed to solve the hierarchical fixed-point problem; see Mainge and Moudafi [34] and Cianciaruso *et al.* [26]. In 2010, Yao *et al.* [45] introduced the following strong convergence iterative algorithm to solve the problem (1.11):

$$\begin{aligned} y_n &= \beta_n S x_n + (1 - \beta_n) x_n, \\ x_{n+1} &= P_C\left[\alpha_n f(x_n) + (1 - \alpha_n) T y_n\right], \quad \forall n \geq 0, \end{aligned} \tag{1.12}$$

where $f : C \to H$ is a contraction mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $(0, 1)$. Under some certain restrictions on the parameters, Yao *et al.* proved that the sequence $\{x_n\}$ generated by (1.12) converges strongly to $z \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (I - f) z, y - z \rangle \geq 0, \quad \forall y \in F(T). \tag{1.13}$$

In 2011, Ceng *et al.* [20] investigated the following iterative method:

$$x_{n+1} = P_C\big[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)\big(T(y_n)\big)\big], \quad \forall n \geq 0, \tag{1.14}$$

where U is a Lipschitzian mapping, and F is a Lipschitzian and strongly monotone mapping. They proved that under some approximate assumptions on the operators and parameters, the sequence $\{x_n\}$ generated by (1.14) converges strongly to the unique solution of the variational inequality

$$\big\langle \rho U(z) - \mu F(z), x - z\big\rangle \geq 0, \quad \forall x \in \operatorname{Fix}(T).$$

In this paper, motivated by the work of Censor *et al.* [23], Moudafi [36], Byrne *et al.* [18], Yao *et al.* [45], Ceng *et al.* [20], Bnouhachem [15–17] and by the recent work going in this direction, we give an iterative method for finding the approximate element of the common set of solutions of (1.1), (1.9)-(1.10), and (1.11) in real Hilbert space. We establish a strong convergence theorem based on this method. We would like to mention that our proposed method is quite general and flexible and includes many known results for solving a system of variational inequality problems, split equilibrium problems, and hierarchical fixed-point problems, see, *e.g.* [20, 26, 34, 40, 45] and relevant references cited therein.

2 Preliminaries

In this section, we list some fundamental lemmas that are useful in the consequent analysis. The first lemma provides some basic properties of projection onto C.

Lemma 2.1 *Let P_C denote the projection of H onto C. Then we have the following inequalities*:

$$\big\langle z - P_C[z], P_C[z] - v\big\rangle \geq 0, \quad \forall z \in H, v \in C; \tag{2.1}$$

$$\big\langle u - v, P_C[u] - P_C[v]\big\rangle \geq \big\|P_C[u] - P_C[v]\big\|^2, \quad \forall u, v \in H; \tag{2.2}$$

$$\big\|P_C[u] - P_C[v]\big\| \leq \|u - v\|, \quad \forall u, v \in H; \tag{2.3}$$

$$\big\|u - P_C[z]\big\|^2 \leq \|z - u\|^2 - \big\|z - P_C[z]\big\|^2, \quad \forall z \in H, u \in C. \tag{2.4}$$

Lemma 2.2 [1] *For any $(x^*, y^*) \in C \times C$, (x^*, y^*) is a solution of* (1.1) *if and only if x^* is a fixed point of the mapping $Q : C \to C$ defined by*

$$Q(x) = P_C\big[P_C[x - \mu_2 B_2 x] - \mu_1 B_1 P_C[x - \mu_2 B_2 x]\big], \quad \forall x \in C, \tag{2.5}$$

where $y^* = P_C[x^* - \mu_2 B_2 x^*]$, $\mu_i \in (0, 2\theta_i)$ and $B_i : C \to C$ is for the θ_i-inverse strongly monotone mappings for each $i = 1, 2$.

Assumption 2.1 [14] Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying the following assumptions:

(i) $F(x, x) = 0$, $\forall x \in C$;
(ii) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$, $\forall x, y \in C$;
(iii) for each $x, y, z \in C$, $\lim_{t \to 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(iv) for each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous;
(v) for fixed $r > 0$ and $z \in C$, there exists a bounded subset K of H_1 and $x \in C \cap K$ such that

$$F(y,x) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \quad \forall y \in C \backslash K.$$

Lemma 2.3 [28] *Assume that* $F_1 : C \times C \to \mathbb{R}$ *satisfies Assumption* 2.1. *For* $r > 0$ *and* $\forall x \in H_1$, *define a mapping* $T_r^{F_1} : H_1 \to C$ *as follows*:

$$T_r^{F_1}(x) = \left\{ z \in C : F_1(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}.$$

Then the following hold:
(i) $T_r^{F_1}$ *is nonempty and single-valued*;
(ii) $T_r^{F_1}$ *is firmly nonexpansive, i.e.*,

$$\left\| T_r^{F_1}(x) - T_r^{F_1}(y) \right\|^2 \leq \left\langle T_r^{F_1}(x) - T_r^{F_1}(y), x - y \right\rangle, \quad \forall x, y \in H_1;$$

(iii) $F(T_r^{F_1}) = EP(F_1)$;
(iv) $EP(F_1)$ *is closed and convex.*

Assume that $F_2 : Q \times Q \to \mathbb{R}$ satisfies Assumption 2.1, and for $s > 0$ and $\forall u \in H_2$, define a mapping $T_s^{F_2} : H_2 \to Q$ as follows:

$$T_s^{F_2}(u) = \left\{ v \in Q : F_2(v,w) + \frac{1}{s}\langle w - v, v - u\rangle \geq 0, \forall w \in Q \right\}.$$

Then $T_s^{F_2}$ satisfies conditions (i)-(iv) of Lemma 2.3. $F(T_s^{F_2}) = EP(F_2, Q)$, where $EP(F_2, Q)$ is the solution set of the following equilibrium problem:

$$\text{Find} \quad y^* \in Q \quad \text{such that} \quad F_2\left(y^*, y\right) \geq 0, \quad \forall y \in Q.$$

Lemma 2.4 [27] *Assume that* $F_1 : C \times C \to \mathbb{R}$ *satisfies Assumption* 2.1, *and let* $T_r^{F_1}$ *be defined as in Lemma* 2.3. *Let* $x, y \in H_1$ *and* $r_1, r_2 > 0$. *Then*

$$\left\| T_{r_2}^{F_1}(y) - T_{r_1}^{F_1}(x) \right\| \leq \|y - x\| + \left| \frac{r_2 - r_1}{r_2} \right| \left\| T_{r_2}^{F_1}(y) - y \right\|.$$

Lemma 2.5 [31] *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *If* $T : C \to C$ *is a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$, *then the mapping* $I - T$ *is demiclosed at* 0, *i.e., if* $\{x_n\}$ *is a sequence in* C *weakly converges to* x *and if* $\{(I - T)x_n\}$ *converges strongly to* 0, *then* $(I - T)x = 0$.

Lemma 2.6 [20] *Let* $U : C \to H$ *be a* τ*-Lipschitzian mapping and let* $F : C \to H$ *be a* k*-Lipschitzian and* η*-strongly monotone mapping, then for* $0 \leq \rho\tau < \mu\eta$, $\mu F - \rho U$ *is* $\mu\eta - \rho\tau$*-strongly monotone i.e.*,

$$\left\langle (\mu F - \rho U)x - (\mu F - \rho U)y, x - y \right\rangle \geq (\mu\eta - \rho\tau)\|x - y\|^2, \quad \forall x, y \in C.$$

Lemma 2.7 [39] *Suppose that $\lambda \in (0,1)$ and $\mu > 0$. Let $F : C \to H$ be a k-Lipschitzian and η-strongly monotone operator. In association with a nonexpansive mapping $T : C \to C$, define the mapping $T^\lambda : C \to H$ by*

$$T^\lambda x = Tx - \lambda\mu FT(x), \quad \forall x \in C.$$

Then T^λ is a contraction provided $\mu < \frac{2\eta}{k^2}$, that is,

$$\left\| T^\lambda x - T^\lambda y \right\| \le (1 - \lambda\nu)\|x - y\|, \quad \forall x, y \in C,$$

where $\nu = 1 - \sqrt{1 - \mu(2\eta - \mu k^2)}$.

Lemma 2.8 [43] *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and δ_n is a sequence such that
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9 [13] *Let C be a closed convex subset of H. Let $\{x_n\}$ be a bounded sequence in H. Assume that*
(i) *the weak w-limit set $w_w(x_n) \subset C$ where $w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\}$;*
(ii) *for each $z \in C$, $\lim_{n\to\infty} \|x_n - z\|$ exists.*
Then $\{x_n\}$ is weakly convergent to a point in C.

3 The proposed method and some properties

In this section, we suggest and analyze our method for finding the common solutions of the system of the variational inequality problem (1.1), the split equilibrium problem (1.9)-(1.10), and the hierarchical fixed-point problem (1.11).

Let H_1 and H_2 be two real Hilbert spaces and $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets of Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Assume that $F_1 : C \times C \to \mathbb{R}$ and $F_2 : Q \times Q \to \mathbb{R}$ are the bifunctions satisfying Assumption 2.1 and F_2 is upper semicontinuous in the first argument. Let $B_i : C \to H$ be a θ_i-inverse strongly monotone mapping for each $i = 1, 2$ and $S, T : C \to C$ a nonexpansive mappings such that $S^* \cap \Lambda \cap F(T) \neq \emptyset$. Let $F : C \to C$ be a k-Lipschitzian mapping and be η-strongly monotone, and let $U : C \to C$ be a τ-Lipschitzian mapping.

Algorithm 3.1 For an arbitrary given $x_0 \in C$, let the iterative sequences $\{u_n\}$, $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be generated by

$$\begin{cases} u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n); \\ z_n = P_C[P_C[u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C[u_n - \mu_2 B_2 u_n]]; \\ y_n = \beta_n S x_n + (1 - \beta_n) z_n; \\ x_{n+1} = P_C[\alpha_n \rho U(x_n) + (I - \alpha_n \mu F)(T(y_n))], \quad \forall n \ge 0, \end{cases} \tag{3.1}$$

where $\mu_i \in (0, 2\theta_i)$ for each $i = 1, 2$, $\{r_n\} \subset (0, 2\varsigma)$ and $\gamma \in (0, 1/L)$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A. Suppose the parameters satisfy $0 < \mu < \frac{2\eta}{k^2}$,

$0 \le \rho\tau < \nu$, where $\nu = 1 - \sqrt{1-\mu(2\eta - \mu k^2)}$. Also, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying the following conditions:

(a) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(b) $\lim_{n\to\infty}(\beta_n/\alpha_n) = 0$,
(c) $\sum_{n=1}^{\infty} |\alpha_{n-1} - \alpha_n| < \infty$ and $\sum_{n=1}^{\infty} |\beta_{n-1} - \beta_n| < \infty$,
(d) $\liminf_{n\to\infty} r_n < \limsup_{n\to\infty} r_n < 2\varsigma$ and $\sum_{n=1}^{\infty} |r_{n-1} - r_n| < \infty$.

Remark 3.1 Our method can be viewed as an extension and improvement for some well-known results, for example the following.

- The proposed method is an extension and improvement of the method of Wang and Xu [42] for finding the approximate element of the common set of solutions of a split equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.
- If we have the Lipschitzian mapping $U = f$, $F = I$, $\rho = \mu = 1$, and $B_1 = B_2 = 0$, we obtain an extension and improvement of the method of Yao *et al.* [45] for finding the approximate element of the common set of solutions of a split equilibrium problem and a hierarchical fixed-point problem in a real Hilbert space.
- The contractive mapping f with a coefficient $\alpha \in [0,1)$ in other papers [39, 40, 45] is extended to the cases of the Lipschitzian mapping U with a coefficient constant $\gamma \in [0,\infty)$.

This shows that Algorithm 3.1 is quite general and unifying.

Lemma 3.1 *Let $x^* \in S^* \cap \Lambda \cap F(T)$. Then $\{x_n\}$, $\{u_n\}$, $\{z_n\}$, and $\{y_n\}$ are bounded.*

Proof Let $x^* \in S^* \cap \Lambda \cap F(T)$; we have $x^* = T_{r_n}^{F_1}(x^*)$ and $Ax^* = T_{r_n}^{F_2}(Ax^*)$. Then

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - x^*\|^2 \\
&= \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - T_{r_n}^{F_1}(x^*)\|^2 \\
&\le \|x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^*\|^2 \\
&= \|x_n - x^*\|^2 + \gamma^2\|A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 + 2\gamma\langle x_n - x^*, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&= \|x_n - x^*\|^2 + \gamma^2\langle (T_{r_n}^{F_2} - I)Ax_n, AA^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\quad + 2\gamma\langle x_n - x^*, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle.
\end{aligned} \tag{3.2}$$

From the definition of L it follows that

$$\begin{aligned}
\gamma^2\langle (T_{r_n}^{F_2} - I)Ax_n, AA^*(T_{r_n}^{F_2} - I)Ax_n\rangle &\le L\gamma^2\langle (T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle \\
&= L\gamma^2\|(T_{r_n}^{F_2} - I)Ax_n\|^2.
\end{aligned} \tag{3.3}$$

It follows from (1.4) that

$$\begin{aligned}
&2\gamma\langle x_n - x^*, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\quad = 2\gamma\langle A(x_n - x^*), (T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\quad = 2\gamma\langle A(x_n - x^*) + (T_{r_n}^{F_2} - I)Ax_n - (T_{r_n}^{F_2} - I)Ax_n, (T_{r_n}^{F_2} - I)Ax_n\rangle \\
&\quad = 2\gamma(\langle T_{r_n}^{F_2}Ax_n - Ax^*, (T_{r_n}^{F_2} - I)Ax_n\rangle - \|(T_{r_n}^{F_2} - I)Ax_n\|^2)
\end{aligned}$$

$$\begin{aligned}&\le 2\gamma\left(\frac{1}{2}\left\|\left(T_{r_n}^{F_2}-I\right)Ax_n\right\|^2-\left\|\left(T_{r_n}^{F_2}-I\right)Ax_n\right\|^2\right)\\&=-\gamma\left\|\left(T_{r_n}^{F_2}-I\right)Ax_n\right\|^2.\end{aligned}\tag{3.4}$$

Applying (3.4) and (3.3) to (3.2) and from the definition of γ, we get

$$\begin{aligned}\left\|u_n-x^*\right\|^2&\le\left\|x_n-x^*\right\|^2+\gamma(L\gamma-1)\left\|\left(T_{r_n}^{F_2}-I\right)Ax_n\right\|^2\\&\le\left\|x_n-x^*\right\|^2.\end{aligned}\tag{3.5}$$

Let $x^*\in S^*\cap\Lambda\cap F(T)$; we have

$$x^*=P_C\left[y^*-\mu_1B_1y^*\right],$$

where

$$y^*=P_C\left[x^*-\mu_2B_2x^*\right].$$

We set $v_n=P_C[u_n-\mu_2B_2u_n]$. Since B_2 is a θ_2-inverse strongly monotone mapping, it follows that

$$\begin{aligned}\left\|v_n-y^*\right\|^2&=\left\|P_C[u_n-\mu_2B_2u_n]-P_C\left[x^*-\mu_2B_2x^*\right]\right\|^2\\&\le\left\|u_n-x^*-\mu_2\left(B_2u_n-B_2x^*\right)\right\|^2\\&\le\left\|x_n-x^*\right\|^2-\mu_2(2\theta_2-\mu_2)\left\|B_2u_n-B_2x^*\right\|^2\\&\le\left\|x_n-x^*\right\|^2.\end{aligned}\tag{3.6}$$

Since B_i is θ_i-inverse strongly monotone mappings, for each $i=1,2$, we get

$$\begin{aligned}\left\|z_n-x^*\right\|^2&=\big\|P_C\left[P_C[u_n-\mu_2B_2u_n]-\mu_1B_1P_C[u_n-\mu_2B_2u_n]\right]\\&\quad-P_C\left[P_C\left[x^*-\mu_2B_2x^*\right]-\mu_1B_1P_C\left[x^*-\mu_2B_2x^*\right]\right]\big\|^2\\&\le\big\|P_C[u_n-\mu_2B_2u_n]-\mu_1B_1P_C[u_n-\mu_2B_2u_n]\\&\quad-\left(P_C\left[x^*-\mu_2B_2x^*\right]-\mu_1B_1P_C\left[x^*-\mu_2B_2x^*\right]\right)\big\|^2\\&=\big\|P_C[u_n-\mu_2B_2u_n]-P_C\left[x^*-\mu_2B_2x^*\right]\\&\quad-\mu_1\left(B_1P_C[u_n-\mu_2B_2u_n]-B_1P_C\left[x^*-\mu_2B_2x^*\right]\right)\big\|^2\\&\le\left\|P_C[u_n-\mu_2B_2u_n]-P_C\left[x^*-\mu_2B_2x^*\right]\right\|^2\\&\quad-\mu_1(2\theta_1-\mu_1)\left\|B_1P_C[u_n-\mu_2B_2u_n]-B_1P_C\left[x^*-\mu_2B_2x^*\right]\right\|^2\\&\le\left\|(u_n-\mu_2B_2u_n)-\left(x^*-\mu_2B_2x^*\right)\right\|^2\\&\quad-\mu_1(2\theta_1-\mu_1)\left\|B_1P_C[u_n-\mu_2B_2u_n]-B_1P_C\left[x^*-\mu_2B_2x^*\right]\right\|^2\\&\le\left\|u_n-x^*\right\|^2-\mu_2(2\theta_2-\mu_2)\left\|B_2u_n-B_2x^*\right\|^2-\mu_1(2\theta_1-\mu_1)\left\|B_1v_n-B_1y^*\right\|^2\\&\le\left\|u_n-x^*\right\|^2\\&\le\left\|x_n-x^*\right\|^2.\end{aligned}\tag{3.7}$$

We denote $V_n = \alpha_n\rho U(x_n) + (I - \alpha_n\mu F)(T(y_n))$. Next, we prove that the sequence $\{x_n\}$ is bounded, and without loss of generality we can assume that $\beta_n \le \alpha_n$ for all $n \ge 1$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|P_C[V_n] - P_C[x^*]\| \\
&\le \|\alpha_n\rho U(x_n) + (I - \alpha_n\mu F)(T(y_n)) - x^*\| \\
&\le \alpha_n\|\rho U(x_n) - \mu F(x^*)\| + \|(I - \alpha_n\mu F)(T(y_n)) - (I - \alpha_n\mu F)T(x^*)\| \\
&= \alpha_n\|\rho U(x_n) - \rho U(x^*) + (\rho U - \mu F)x^*\| \\
&\quad + \|(I - \alpha_n\mu F)(T(y_n)) - (I - \alpha_n\mu F)T(x^*)\| \\
&\le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + (1 - \alpha_n\nu)\|y_n - x^*\| \\
&\le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| \\
&\quad + (1 - \alpha_n\nu)\|\beta_n Sx_n + (1 - \beta_n)z_n - x^*\| \\
&\le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + (1 - \alpha_n\nu)(\beta_n\|Sx_n - Sx^*\| \\
&\quad + \beta_n\|Sx^* - x^*\| + (1 - \beta_n)\|z_n - x^*\|) \\
&\le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + (1 - \alpha_n\nu)(\beta_n\|Sx_n - Sx^*\| \\
&\quad + \beta_n\|Sx^* - x^*\| + (1 - \beta_n)\|x_n - x^*\|) \\
&\le \alpha_n\rho\tau\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + (1 - \alpha_n\nu)(\beta_n\|x_n - x^*\| \\
&\quad + \beta_n\|Sx^* - x^*\| + (1 - \beta_n)\|x_n - x^*\|) \\
&= (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| \\
&\quad + (1 - \alpha_n\nu)\beta_n\|Sx^* - x^*\| \\
&\le (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \alpha_n\|(\rho U - \mu F)x^*\| + \beta_n\|Sx^* - x^*\| \\
&\le (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| + \alpha_n(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|) \\
&= (1 - \alpha_n(\nu - \rho\tau))\|x_n - x^*\| \\
&\quad + \frac{\alpha_n(\nu - \rho\tau)}{\nu - \rho\tau}(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|) \\
&\le \max\left\{\|x_n - x^*\|, \frac{1}{\nu - \rho\tau}(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|)\right\},
\end{aligned}$$

where the third inequality follows from Lemma 2.7.

By induction on n, we obtain $\|x_n - x^*\| \le \max\{\|x_0 - x^*\|, \frac{1}{\nu-\rho\tau}(\|(\rho U - \mu F)x^*\| + \|Sx^* - x^*\|)\}$, for $n \ge 0$ and $x_0 \in C$. Hence $\{x_n\}$ is bounded and, consequently, we deduce that $\{u_n\}$, $\{z_n\}$, $\{v_n\}$, $\{y_n\}$, $\{S(x_n)\}$, $\{T(x_n)\}$, $\{F(T(y_n))\}$, and $\{U(x_n)\}$ are bounded. □

Lemma 3.2 *Let $x^* \in S^* \cap \Lambda \cap F(T)$ and $\{x_n\}$ be the sequence generated by Algorithm* 3.1. *Then we have*:

(a) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.
(b) *The weak w-limit set $w_w(x_n) \subset F(T)$ $(w_w(x_n) = \{x : x_{n_i} \rightharpoonup x\})$.*

Proof Since $u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)$ and $u_{n-1} = T_{r_{n-1}}^{F_1}(x_{n-1} + \gamma A^*(T_{r_{n-1}}^{F_2} - I)Ax_{n-1})$. It follows from Lemma 2.4 that

$$\begin{aligned}
\|u_n - u_{n-1}\| &\le \big\|x_n - x_{n-1} + \gamma\big(A^*\big(T_{r_n}^{F_2} - I\big)Ax_n - A^*\big(T_{r_{n-1}}^{F_2} - I\big)Ax_{n-1}\big)\big\| \\
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&\le \big\|x_n - x_{n-1} - \gamma A^*A(x_n - x_{n-1})\big\| + \gamma\|A\|\big\|T_{r_n}^{F_2}Ax_n - T_{r_{n-1}}^{F_2}Ax_{n-1}\big\| \\
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&\le \big(\|x_n - x_{n-1}\|^2 - 2\gamma\big\|A(x_n - x_{n-1})\big\|^2 + \gamma^2\|A\|^4\|x_n - x_{n-1}\|^2\big)^{\frac{1}{2}} \\
&\quad + \gamma\|A\|\left(\big\|A(x_n - x_{n-1})\big\| + \left|1 - \frac{r_{n-1}}{r_n}\right|\big\|T_{r_n}^{F_2}Ax_n - Ax_n\big\|\right) \\
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&\le \big(1 - 2\gamma\|A\|^2 + \gamma^2\|A\|^4\big)^{\frac{1}{2}}\|x_n - x_{n-1}\| + \gamma\|A\|^2\|x_n - x_{n-1}\| \\
&\quad + \gamma\|A\|\left|1 - \frac{r_{n-1}}{r_n}\right|\big\|T_{r_n}^{F_2}Ax_n - Ax_n\big\| \\
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&= \big(1 - \gamma\|A\|^2\big)\|x_n - x_{n-1}\| + \gamma\|A\|^2\|x_n - x_{n-1}\| \\
&\quad + \gamma\|A\|\left|1 - \frac{r_{n-1}}{r_n}\right|\big\|T_{r_n}^{F_2}Ax_n - Ax_n\big\| \\
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&= \|x_n - x_{n-1}\| + \gamma\|A\|\left|1 - \frac{r_{n-1}}{r_n}\right|\big\|T_{r_n}^{F_2}Ax_n - Ax_n\big\| \\
&\quad + \left|1 - \frac{r_{n-1}}{r_n}\right| \big\|T_{r_n}^{F_1}\big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big) - \big(x_n + \gamma A^*\big(T_{r_n}^{F_2} - I\big)Ax_n\big)\big\| \\
&= \|x_n - x_{n-1}\| + \left|\frac{r_n - r_{n-1}}{r_n}\right|\big(\gamma\|A\|\sigma_n + \chi_n\big),
\end{aligned}$$

where $\sigma_n := \|T_{r_n}^{F_2}Ax_n - Ax_n\|$ and $\chi_n := \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - (x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)\|$. Without loss of generality, let us assume that there exists a real number μ such that $r_n > \mu > 0$, for all positive integers n. Then we get

$$\|u_{n-1} - u_n\| \le \|x_{n-1} - x_n\| + \frac{1}{\mu}|r_{n-1} - r_n|\big(\gamma\|A\|\sigma_n + \chi_n\big). \tag{3.8}$$

Next, we estimate

$$\begin{aligned}
\|z_n - z_{n-1}\|^2 &= \big\|P_C\big[P_C[u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C[u_n - \mu_2 B_2 u_n]\big] \\
&\quad - P_C\big[P_C[u_{n-1} - \mu_2 B_2 u_{n-1}] - \mu_1 B_1 P_C[u_{n-1} - \mu_2 B_2 u_{n-1}]\big]\big\|^2 \\
&\le \big\|P_C[u_n - \mu_2 B_2 u_n] - \mu_1 B_1 P_C[u_n - \mu_2 B_2 u_n]
\end{aligned}$$

$$
\begin{aligned}
&\quad - \big(P_C[u_{n-1} - \mu_2 B_2 u_{n-1}] - \mu_1 B_1 P_C[u_{n-1} - \mu_2 B_2 u_{n-1}]\big)\big\|^2 \\
&= \big\|P_C[u_n - \mu_2 B_2 u_n] - P_C[u_{n-1} - \mu_2 B_2 u_{n-1}] \\
&\quad - \mu_1\big(B_1 P_C[u_n - \mu_2 B_2 u_n] - B_1 P_C[u_{n-1} - \mu_2 B_2 u_{n-1}]\big)\big\|^2 \\
&\le \big\|P_C[u_n - \mu_2 B_2 u_n] - P_C[u_{n-1} - \mu_2 B_2 u_{n-1}]\big\|^2 \\
&\quad - \mu_1(2\theta_1 - \mu_1)\big\|B_1 P_C[u_n - \mu_2 B_2 u_n] - B_1 P_C[u_{n-1} - \mu_2 B_2 u_{n-1}]\big\|^2 \\
&\le \big\|P_C[u_n - \mu_2 B_2 u_n] - P_C[u_{n-1} - \mu_2 B_2 u_{n-1}]\big\|^2 \\
&\le \big\|(u_n - u_{n-1}) - \mu_2(B_2 u_n - B_2 u_{n-1})\big\|^2 \\
&\le \|u_n - u_{n-1}\|^2 - \mu_2(2\theta_2 - \mu_2)\|B_2 u_n - B_2 u_{n-1}\|^2 \\
&\le \|u_n - u_{n-1}\|^2.
\end{aligned} \tag{3.9}
$$

It follows from (3.8) and (3.9) that

$$
\|z_n - z_{n-1}\| \le \|x_{n-1} - x_n\| + \frac{1}{\mu}|r_{n-1} - r_n|\big(\gamma\|A\|\sigma_n + \chi_n\big).
$$

From (3.1) and the above inequality, we get

$$
\begin{aligned}
\|y_n - y_{n-1}\| &= \big\|\beta_n S x_n + (1-\beta_n) z_n - \big(\beta_{n-1} S x_{n-1} + (1-\beta_{n-1}) z_{n-1}\big)\big\| \\
&= \big\|\beta_n(S x_n - S x_{n-1}) + (\beta_n - \beta_{n-1}) S x_{n-1} \\
&\quad + (1-\beta_n)(z_n - z_{n-1}) + (\beta_{n-1} - \beta_n) z_{n-1}\big\| \\
&\le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|z_n - z_{n-1}\| + |\beta_n - \beta_{n-1}|\big(\|S x_{n-1}\| + \|z_{n-1}\|\big) \\
&\le \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\left\{\|x_{n-1} - x_n\| + \frac{1}{\mu}|r_{n-1} - r_n|\big(\gamma\|A\|\sigma_n + \chi_n\big)\right\} \\
&\quad + |\beta_n - \beta_{n-1}|\big(\|S x_{n-1}\| + \|z_{n-1}\|\big) \\
&\le \|x_n - x_{n-1}\| + \frac{1}{\mu}|r_{n-1} - r_n|\big(\gamma\|A\|\sigma_n + \chi_n\big) \\
&\quad + |\beta_n - \beta_{n-1}|\big(\|S x_{n-1}\| + \|z_{n-1}\|\big).
\end{aligned} \tag{3.10}
$$

Next, we estimate

$$
\begin{aligned}
\|x_{n+1} - x_n\| &= \big\|P_C[V_n] - P_C[V_{n-1}]\big\| \\
&\le \big\|\alpha_n\rho\big(U(x_n) - U(x_{n-1})\big) + (\alpha_n - \alpha_{n-1})\rho U(x_{n-1}) + (I - \alpha_n\mu F)\big(T(y_n)\big) \\
&\quad - (I - \alpha_n\mu F)T(y_{n-1}) + (I - \alpha_n\mu F)\big(T(y_{n-1})\big) - (I - \alpha_{n-1}\mu F)\big(T(y_{n-1})\big)\big\| \\
&\le \alpha_n\rho\tau\|x_n - x_{n-1}\| + (1 - \alpha_n\nu)\|y_n - y_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\big(\big\|\rho U(x_{n-1})\big\| + \big\|\mu F\big(T(y_{n-1})\big)\big\|\big),
\end{aligned} \tag{3.11}
$$

where the second inequality follows from Lemma 2.7. From (3.10) and (3.11), we have

$$
\begin{aligned}
\|x_{n+1} - x_n\| \le \alpha_n\rho\tau\|x_n - x_{n-1}\| + (1 - \alpha_n\nu)\bigg\{&\|x_n - x_{n-1}\| + \frac{1}{\mu}|r_{n-1} - r_n|\big(\gamma\|A\|\sigma_n + \chi_n\big) \\
&+ |\beta_n - \beta_{n-1}|\big(\|S x_{n-1}\| + \|z_{n-1}\|\big)\bigg\}
\end{aligned}
$$

$$\begin{aligned}
&+ |\alpha_n - \alpha_{n-1}|\big(\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|\big) \\
&\le \big(1 - (\nu - \rho\tau)\alpha_n\big)\|x_n - x_{n-1}\| \\
&+ \frac{1}{\mu}|r_{n-1} - r_n|\big(\gamma\|A\|\sigma_n + \chi_n\big) + |\beta_n - \beta_{n-1}|\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big) \\
&+ |\alpha_n - \alpha_{n-1}|\big(\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|\big) \\
&\le \big(1 - (\nu - \rho\tau)\alpha_n\big)\|x_n - x_{n-1}\| \\
&+ M\left(\frac{1}{\mu}|r_{n-1} - r_n| + |\beta_n - \beta_{n-1}| + |\alpha_n - \alpha_{n-1}|\right).
\end{aligned} \tag{3.12}$$

Here

$$\begin{aligned}
M = \max\Big\{&\sup_{n\ge 1}\big(\gamma\|A\|\sigma_n + \chi_n\big), \sup_{n\ge 1}\big(\|Sx_{n-1}\| + \|z_{n-1}\|\big), \\
&\sup_{n\ge 1}\big(\|\rho U(x_{n-1})\| + \|\mu F(T(y_{n-1}))\|\big)\Big\}.
\end{aligned}$$

It follows by conditions (a)-(d) of Algorithm 3.1 and Lemma 2.8 that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Next, we show that $\lim_{n\to\infty}\|u_n - x_n\| = 0$. Since $x^* \in S^* \cap \Lambda \cap F(T)$ by using (3.2), (3.5), and (3.7), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle P_C[V_n] - x^*, x_{n+1} - x^*\rangle \\
&= \langle P_C[V_n] - V_n, P_C[V_n] - x^*\rangle + \langle V_n - x^*, x_{n+1} - x^*\rangle \\
&\le \big\langle \alpha_n\big(\rho U(x_n) - \mu F(x^*)\big) + (I - \alpha_n\mu F)\big(T(y_n)\big) \\
&\quad - (I - \alpha_n\mu F)\big(T(x^*)\big), x_{n+1} - x^*\big\rangle \\
&= \big\langle \alpha_n\rho\big(U(x_n) - U(x^*)\big), x_{n+1} - x^*\big\rangle + \alpha_n\big\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\big\rangle \\
&\quad + \big\langle (I - \alpha_n\mu F)\big(T(y_n)\big) - (I - \alpha_n\mu F)\big(T(x^*)\big), x_{n+1} - x^*\big\rangle \\
&\le \alpha_n\rho\tau\|x_n - x^*\|\|x_{n+1} - x^*\| + \alpha_n\big\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\big\rangle \\
&\quad + (1 - \alpha_n\nu)\|y_n - x^*\|\|x_{n+1} - x^*\| \\
&\le \frac{\alpha_n\rho\tau}{2}\big(\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\big) + \alpha_n\big\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\big\rangle \\
&\quad + \frac{(1 - \alpha_n\nu)}{2}\big(\|y_n - x^*\|^2 + \|x_{n+1} - x^*\|^2\big) \\
&\le \frac{(1 - \alpha_n(\nu - \rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\big\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\big\rangle \\
&\quad + \frac{(1 - \alpha_n\nu)}{2}\big(\beta_n\|Sx_n - x^*\|^2 + (1 - \beta_n)\|z_n - x^*\|^2\big) \\
&\le \frac{(1 - \alpha_n(\nu - \rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\big\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\big\rangle + \frac{(1 - \alpha_n\nu)\beta_n}{2}\|Sx_n - x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1-\alpha_n\nu)(1-\beta_n)}{2}\bigl\{\|u_n-x^*\|^2-\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2\\
&-\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\bigr\}\\
\le{}& \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1}-x^*\|^2+\frac{\alpha_n\rho\tau}{2}\|x_n-x^*\|^2\\
&+\alpha_n\bigl\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\bigr\rangle+\frac{(1-\alpha_n\nu)\beta_n}{2}\|Sx_n-x^*\|^2\\
&+\frac{(1-\alpha_n\nu)(1-\beta_n)}{2}\bigl\{\|x_n-x^*\|^2+\gamma(L\gamma-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2\\
&-\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2-\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\bigr\}, \quad (3.13)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2\le{}& \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2\\
&+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\bigl\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\bigr\rangle\\
&+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2\\
&+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\bigl\{\|x_n-x^*\|^2+\gamma(L\gamma-1)\|(T_{r_n}^{F_2}-I)Ax_n\|^2\\
&-\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2-\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\bigr\}\\
\le{}& \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2\\
&+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\bigl\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\bigr\rangle\\
&+\|x_n-x^*\|^2+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2\\
&-\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\bigl\{\gamma(1-L\gamma)\|(T_{r_n}^{F_2}-I)Ax_n\|^2\\
&+\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2+\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\bigr\}.
\end{aligned}
$$

Then from the above inequality, we get

$$
\begin{aligned}
&\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\bigl\{\gamma(1-L\gamma)\|(T_{r_n}^{F_2}-I)Ax_n\|^2+\mu_2(2\theta_2-\mu_2)\|B_2u_n-B_2x^*\|^2\\
&\quad+\mu_1(2\theta_1-\mu_1)\|B_1v_n-B_1y^*\|^2\bigr\}\\
&\le\frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\bigl\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\bigr\rangle\\
&\quad+\beta_n\|Sx_n-x^*\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\le\frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\bigl\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\bigr\rangle\\
&\quad+\beta_n\|Sx_n-x^*\|^2+\bigl(\|x_n-x^*\|+\|x_{n+1}-x^*\|\bigr)\|x_{n+1}-x_n\|.
\end{aligned}
$$

Since $\gamma(1-L\gamma) > 0$, $2\theta_1 - \mu_1 > 0$, $2\theta_2 - \mu_2 > 0$, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, $\alpha_n \to 0$, and $\beta_n \to 0$, we obtain

$$\lim_{n\to\infty} \|B_2 u_n - B_2 x^*\| = 0,$$
$$\lim_{n\to\infty} \|B_1 v_n - B_1 y^*\| = 0$$

and

$$\lim_{n\to\infty} \|(T_{r_n}^{F_2} - I)Ax_n\| = 0. \tag{3.14}$$

Since $T_{r_n}^{F_1}$ is firmly nonexpansive, we have

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n) - T_{r_n}^{F_1}(x^*)\|^2 \\
&\le \langle u_n - x^*, x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^*\rangle \\
&= \frac{1}{2}\{\|u_n - x^*\|^2 + \|x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^*\|^2 \\
&\quad - \|u_n - x^* - [x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^*]\|^2\} \\
&= \frac{1}{2}\{\|u_n - x^*\|^2 + \|x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n - x^*\|^2 \\
&\quad - \|u_n - x_n - \gamma A^*(T_{r_n}^{F_2} - I)Ax_n\|^2\} \\
&\le \frac{1}{2}\{\|u_n - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad - \|u_n - x_n - \gamma A^*(T_{r_n}^{F_2} - I)Ax_n\|^2\} \\
&= \frac{1}{2}\{\|u_n - x^*\|^2 + \|x_n - x^*\|^2 \\
&\quad - [\|u_n - x_n\|^2 + \gamma^2\|A^*(T_{r_n}^{F_2} - I)Ax_n\|^2 - 2\gamma\langle u_n - x_n, A^*(T_{r_n}^{F_2} - I)Ax_n\rangle]\}.
\end{aligned}$$

Hence, we get

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\gamma\|Au_n - Ax_n\|\|(T_{r_n}^{F_2} - I)Ax_n\|. \tag{3.15}$$

From (3.13), (3.7), and the above inequality, we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)}{2}(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|z_n - x^*\|^2) \\
&\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\langle\rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle \\
&\quad + \frac{(1-\alpha_n\nu)}{2}(\beta_n\|Sx_n - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2)
\end{aligned}$$

$$\begin{aligned}
&\le \frac{(1-\alpha_n(\nu-\rho\tau))}{2}\|x_{n+1}-x^*\|^2+\frac{\alpha_n\rho\tau}{2}\|x_n-x^*\|^2 \\
&\quad+\alpha_n\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&\quad+\frac{(1-\alpha_n\nu)}{2}\bigl\{\beta_n\|Sx_n-x^*\|^2+(1-\beta_n)\bigl(\|x_n-x^*\|^2-\|u_n-x_n\|^2 \\
&\quad+2\gamma\|Au_n-Ax_n\|\bigl\|\bigl(T_{r_n}^{F_2}-I\bigr)Ax_n\bigr\|\bigr)\bigr\},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&\quad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&\quad+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\bigl\{\|x_n-x^*\|^2-\|u_n-x_n\|^2 \\
&\quad+2\gamma\|Au_n-Ax_n\|\bigl\|\bigl(T_{r_n}^{F_2}-I\bigr)Ax_n\bigr\|\bigr\} \\
&\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&\quad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&\quad+\|x_n-x^*\|^2+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\bigl\{-\|u_n-x_n\|^2 \\
&\quad+2\gamma\|Au_n-Ax_n\|\bigl\|\bigl(T_{r_n}^{F_2}-I\bigr)Ax_n\bigr\|\bigr\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\|u_n-x_n\|^2 &\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&\quad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&\quad+\frac{2(1-\alpha_n\nu)(1-\beta_n)\gamma}{1+\alpha_n(\nu-\rho\tau)}\|Au_n-Ax_n\|\bigl\|\bigl(T_{r_n}^{F_2}-I\bigr)Ax_n\bigr\| \\
&\quad+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2 \\
&\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\quad+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&\quad+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2
\end{aligned}$$

$$+\frac{2(1-\alpha_n\nu)(1-\beta_n)\gamma}{1+\alpha_n(\nu-\rho\tau)}\|Au_n-Ax_n\|\big\|\big(T_{r_n}^{F_2}-I\big)Ax_n\big\| + \big(\big\|x_n-x^*\big\|+\big\|x_{n+1}-x^*\big\|\big)\|x_{n+1}-x_n\|.$$

Since $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\alpha_n\to 0$, $\beta_n\to 0$, and $\lim_{n\to\infty}\|(T_{r_n}^{F_2}-I)Ax_n\|=0$, we obtain

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.16}$$

From (2.2), we get

$$\begin{aligned}
\big\|v_n-y^*\big\|^2 &= \big\|P_C[u_n-\mu_2B_2u_n]-P_C\big[x^*-\mu_2B_2x^*\big]\big\|^2\\
&\le \big\langle v_n-y^*,(u_n-\mu_2B_2u_n)-\big(x^*-\mu_2B_2x^*\big)\big\rangle\\
&= \frac{1}{2}\big\{\big\|v_n-y^*\big\|^2+\big\|u_n-x^*-\mu_2\big(B_2u_n-B_2x^*\big)\big\|^2\\
&\quad -\big\|u_n-x^*-\mu_2\big(B_2u_n-B_2x^*\big)-\big(v_n-y^*\big)\big\|^2\big\}\\
&\le \frac{1}{2}\big\{\big\|v_n-y^*\big\|^2+\big\|u_n-x^*\big\|^2-\mu_2(2\theta_2-\mu_2)\big\|B_2u_n-B_2x^*\big\|^2\\
&\quad -\big\|u_n-x^*-\mu_2\big(B_2u_n-B_2x^*\big)-\big(v_n-y^*\big)\big\|^2\big\}\\
&\le \frac{1}{2}\big\{\big\|v_n-y^*\big\|^2+\big\|u_n-x^*\big\|^2-\big\|u_n-v_n-\mu_2\big(B_2u_n-B_2x^*\big)-\big(x^*-y^*\big)\big\|^2\big\}\\
&= \frac{1}{2}\big\{\big\|v_n-y^*\big\|^2+\big\|u_n-x^*\big\|^2-\big\|u_n-v_n-\big(x^*-y^*\big)\big\|^2\\
&\quad +2\mu_2\big\langle u_n-v_n-\big(x^*-y^*\big),B_2u_n-B_2x^*\big\rangle-\mu_2^2\big\|B_2u_n-B_2x^*\big\|^2\big\}\\
&\le \frac{1}{2}\big\{\big\|v_n-y^*\big\|^2+\big\|u_n-x^*\big\|^2-\big\|u_n-v_n-\big(x^*-y^*\big)\big\|^2\\
&\quad +2\mu_2\big\|u_n-v_n-\big(x^*-y^*\big)\big\|\big\|B_2u_n-B_2x^*\big\|\big\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\big\|v_n-y^*\big\|^2 &\le \big\|u_n-x^*\big\|^2-\big\|u_n-v_n-\big(x^*-y^*\big)\big\|^2\\
&\quad +2\mu_2\big\|u_n-v_n-\big(x^*-y^*\big)\big\|\big\|B_2u_n-B_2x^*\big\|\\
&\le \big\|x_n-x^*\big\|^2-\|u_n-x_n\|^2+2\gamma\|Au_n-Ax_n\|\big\|\big(T_{r_n}^{F_2}-I\big)Ax_n\big\|\\
&\quad -\big\|u_n-v_n-\big(x^*-y^*\big)\big\|^2\\
&\quad +2\mu_2\big\|u_n-v_n-\big(x^*-y^*\big)\big\|\big\|B_2u_n-B_2x^*\big\|,
\end{aligned} \tag{3.17}$$

where the last inequality follows from (3.15). On the other hand, from (3.1) and (2.2), we obtain

$$\begin{aligned}
\big\|z_n-x^*\big\|^2 &= \big\|P_C[v_n-\mu_1B_1v_n]-P_C\big[y^*-\mu_1B_1y^*\big]\big\|^2\\
&\le \big\langle z_n-x^*,(v_n-\mu_1B_1v_n)-\big(y^*-\mu_1B_1y^*\big)\big\rangle\\
&= \frac{1}{2}\big\{\big\|z_n-x^*\big\|^2+\big\|v_n-y^*-\mu_1\big(B_1v_n-B_1y^*\big)\big\|^2\\
&\quad -\big\|v_n-y^*-\mu_1\big(B_1v_n-B_1y^*\big)-\big(z_n-x^*\big)\big\|^2\big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\bigl\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - 2\mu_1\langle v_n - y^*, B_1 v_n - B_1 y^*\rangle \\
&\quad + \mu_1^2\|B_1 v_n - B_1 y^*\|^2 - \bigl\|v_n - y^* - \mu_1(B_1 v_n - B_1 y^*) - (z_n - x^*)\bigr\|^2\bigr\} \\
&\le \frac{1}{2}\bigl\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \mu_1(2\theta_1 - \mu_1)\|B_1 v_n - B_1 y^*\|^2 \\
&\quad - \bigl\|v_n - y^* - \mu_1(B_1 v_n - B_1 y^*) - (z_n - x^*)\bigr\|^2\bigr\} \\
&\le \frac{1}{2}\bigl\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 \\
&\quad - \bigl\|v_n - z_n - \mu_1(B_1 v_n - B_1 y^*) + (x^* - y^*)\bigr\|^2\bigr\} \\
&\le \frac{1}{2}\bigl\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \bigl\|v_n - z_n + (x^* - y^*)\bigr\|^2 \\
&\quad + 2\mu_1\bigl\langle v_n - z_n + (x^* - y^*), B_1 v_n - B_1 y^*\bigr\rangle\bigr\} \\
&\le \frac{1}{2}\bigl\{\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \bigl\|v_n - z_n + (x^* - y^*)\bigr\|^2 \\
&\quad + 2\mu_1\bigl\|v_n - z_n + (x^* - y^*)\bigr\|\|B_1 v_n - B_1 y^*\|\bigr\}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|z_n - x^*\|^2 &\le \|v_n - y^*\|^2 - \bigl\|v_n - z_n + (x^* - y^*)\bigr\|^2 \\
&\quad + 2\mu_1\bigl\|v_n - z_n + (x^* - y^*)\bigr\|\|B_1 v_n - B_1 y^*\| \\
&\le \|x_n - x^*\|^2 - \|u_n - x_n\|^2 + 2\gamma\|Au_n - Ax_n\|\bigl\|(T_{r_n}^{F_2} - I)Ax_n\bigr\| \\
&\quad - \bigl\|u_n - v_n - (x^* - y^*)\bigr\|^2 + 2\mu_2\bigl\|u_n - v_n - (x^* - y^*)\bigr\|\|B_2 u_n - B_2 x^*\| \\
&\quad - \bigl\|v_n - z_n + (x^* - y^*)\bigr\|^2 + 2\mu_1\bigl\|v_n - z_n + (x^* - y^*)\bigr\|\|B_1 v_n - B_1 y^*\|,
\end{aligned}
$$

where the last inequality follows from (3.17). From (3.13) and the above inequality, we have

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \frac{(1 - \alpha_n(\nu - \rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\bigl\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\bigr\rangle \\
&\quad + \frac{(1 - \alpha_n\nu)}{2}\bigl(\beta_n\|Sx_n - x^*\|^2 + (1 - \beta_n)\|z_n - x^*\|^2\bigr) \\
&\le \frac{(1 - \alpha_n(\nu - \rho\tau))}{2}\|x_{n+1} - x^*\|^2 + \frac{\alpha_n\rho\tau}{2}\|x_n - x^*\|^2 \\
&\quad + \alpha_n\bigl\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\bigr\rangle \\
&\quad + \frac{(1 - \alpha_n\nu)}{2}\bigl\{\beta_n\|Sx_n - x^*\|^2 + (1 - \beta_n)\bigl(\|x_n - x^*\|^2 - \|u_n - x_n\|^2 \\
&\quad + 2\gamma\|Au_n - Ax_n\|\bigl\|(T_{r_n}^{F_2} - I)Ax_n\bigr\|\bigr) \\
&\quad + (1 - \beta_n)\bigl(-\bigl\|u_n - v_n - (x^* - y^*)\bigr\|^2 \\
&\quad + 2\mu_2\bigl\|u_n - v_n - (x^* - y^*)\bigr\|\|B_2 u_n - B_2 x^*\|\bigr) \\
&\quad + (1 - \beta_n)\bigl(-\bigl\|v_n - z_n + (x^* - y^*)\bigr\|^2 \\
&\quad + 2\mu_1\bigl\|v_n - z_n + (x^* - y^*)\bigr\|\|B_1 v_n - B_1 y^*\|\bigr)\bigr\},
\end{aligned}
$$

which implies that

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 \le{}& \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}\{\|x_n-x^*\|^2-\|u_n-x_n\|^2 \\
&+2\gamma\|Au_n-Ax_n\|\|(T_{r_n}^{F_2}-I)Ax_n\|\} \\
&+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}(-\|u_n-v_n-(x^*-y^*)\|^2 \\
&+2\mu_2\|u_n-v_n-(x^*-y^*)\|\|B_2u_n-B_2x^*\|) \\
&+\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}(-\|v_n-z_n+(x^*-y^*)\|^2 \\
&+2\mu_1\|v_n-z_n+(x^*-y^*)\|\|B_1v_n-B_1y^*\|) \\
\le{}& \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle \\
&+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&+\|x_n-x^*\|^2+2\gamma\|Au_n-Ax_n\|\|(T_{r_n}^{F_2}-I)Ax_n\| \\
&+2\mu_2\|u_n-v_n-(x^*-y^*)\|\|B_2u_n-B_2x^*\| \\
&+2\mu_1\|v_n-z_n+(x^*-y^*)\|\|B_1v_n-B_1y^*\| \\
&-\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}(\|u_n-x_n\|^2+\|u_n-v_n-(x^*-y^*)\|^2 \\
&+\|v_n-z_n+(x^*-y^*)\|^2).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{(1-\alpha_n\nu)(1-\beta_n)}{1+\alpha_n(\nu-\rho\tau)}(\|u_n-x_n\|^2+\|u_n-v_n-(x^*-y^*)\|^2+\|v_n-z_n+(x^*-y^*)\|^2) \\
&\quad\le \frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2 \\
&\qquad+\frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle\rho U(x^*)-\mu F(x^*),x_{n+1}-x^*\rangle+\frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n-x^*\|^2 \\
&\qquad+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\gamma\|Au_n-Ax_n\|\|(T_{r_n}^{F_2}-I)Ax_n\| \\
&\qquad+2\mu_2\|u_n-v_n-(x^*-y^*)\|\|B_2u_n-B_2x^*\| \\
&\qquad+2\mu_1\|v_n-z_n+(x^*-y^*)\|\|B_1v_n-B_1y^*\| \\
&\quad=\frac{\alpha_n\rho\tau}{1+\alpha_n(\nu-\rho\tau)}\|x_n-x^*\|^2
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{2\alpha_n}{1+\alpha_n(\nu-\rho\tau)}\langle \rho U(x^*) - \mu F(x^*), x_{n+1} - x^*\rangle + \frac{(1-\alpha_n\nu)\beta_n}{1+\alpha_n(\nu-\rho\tau)}\|Sx_n - x^*\|^2 \\
&+ (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\| + 2\gamma\|Au_n - Ax_n\|\|(T_{r_n}^{F_2} - I)Ax_n\| \\
&+ 2\mu_2\|u_n - v_n - (x^* - y^*)\|\|B_2u_n - B_2x^*\| \\
&+ 2\mu_1\|v_n - z_n + (x^* - y^*)\|\|B_1v_n - B_1y^*\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\alpha_n \to 0$, $\beta_n \to 0$, and $\lim_{n\to\infty}\|(T_{r_n}^{F_2} - I)Ax_n\| = 0$, $\lim_{n\to\infty}\|B_2u_n - B_2x^*\| = 0$, $\lim_{n\to\infty}\|B_1v_n - B_1y^*\| = 0$, we obtain

$$\lim_{n\to\infty}\|u_n - v_n - (x^* - y^*)\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|v_n - z_n + (x^* - y^*)\| = 0.$$

Since

$$\|u_n - z_n\| \le \|u_n - v_n - (x^* - y^*)\| + \|v_n - z_n + (x^* - y^*)\|$$

we get

$$\lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{3.18}$$

It follows from (3.16) and (3.18) that

$$\lim_{n\to\infty}\|x_n - z_n\| = 0. \tag{3.19}$$

Since $T(x_n) \in C$, we have

$$
\begin{aligned}
\|x_n - T(x_n)\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T(x_n)\| \\
&= \|x_n - x_{n+1}\| + \|P_C[V_n] - P_C[T(x_n)]\| \\
&\le \|x_n - x_{n+1}\| + \|\alpha_n(\rho U(x_n) - \mu F(T(y_n))) + T(y_n) - T(x_n)\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|\rho U(x_n) - \mu F(T(y_n))\| + \|y_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|\rho U(x_n) - \mu F(T(y_n))\| + \|\beta_nSx_n + (1-\beta_n)z_n - x_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|\rho U(x_n) - \mu F(T(y_n))\| \\
&\quad + \beta_n\|Sx_n - x_n\| + (1-\beta_n)\|z_n - x_n\|.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, $\alpha_n \to 0$, $\beta_n \to 0$, and $\|\rho U(x_n) - \mu F(T(y_n))\|$ and $\|Sx_n - x_n\|$ are bounded and $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we obtain

$$\lim_{n\to\infty}\|x_n - T(x_n)\| = 0.$$

Since $\{x_n\}$ is bounded, without loss of generality we can assume that $x_n \rightharpoonup x^* \in C$. It follows from Lemma 2.5 that $x^* \in F(T)$. Therefore $w_w(x_n) \subset F(T)$. □

Theorem 3.1 *The sequence $\{x_n\}$ generated by Algorithm* 3.1 *converges strongly to z, which is the unique solution of the variational inequality*

$$\langle \rho U(z) - \mu F(z), x - z\rangle \le 0, \quad \forall x \in S^* \cap \Lambda \cap F(T). \tag{3.20}$$

Proof Since $\{x_n\}$ is bounded $x_n \rightharpoonup w$ and from Lemma 3.2, we have $w \in F(T)$. Next, we show that $w \in EP(F_1)$. Since $u_n = T_{r_n}^{F_1}(x_n + \gamma A^*(T_{r_n}^{F_2} - I)Ax_n)$, we have

$$F_1(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle - \frac{1}{r_n}\langle y - u_n, \gamma A^*(T_{r_n}^{F_2} - I)Ax_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from monotonicity of F_1 that

$$-\frac{1}{r_n}\langle y - u_n, \gamma A^*(T_{r_n}^{F_2} - I)Ax_n\rangle + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F_1(y, u_n), \quad \forall y \in C$$

and

$$-\frac{1}{r_{n_k}}\langle y - u_{n_k}, \gamma A^*(T_{r_{n_k}}^{F_2} - I)Ax_{n_k}\rangle + \left\langle y - u_{n_k}, \frac{u_{n_k} - x_{n_k}}{r_{n_k}}\right\rangle \geq F_1(y, u_{n_k}), \quad \forall y \in C. \tag{3.21}$$

Since $\lim_{n\to\infty}\|u_n - x_n\| = 0$, $\lim_{n\to\infty}\|(T_{r_n}^{F_2} - I)Ax_n\| = 0$, and $x_n \rightharpoonup w$, it is easy to observe that $u_{n_k} \to w$. It follows by Assumption 2.1(iv) that $F_1(y, w) \leq 0$, $\forall y \in C$.

For any $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$, and we have $y_t \in C$. Then, from Assumption 2.1(i) and (iv), we have

$$\begin{aligned} 0 = F_1(y_t, y_t) &\leq tF_1(y_t, y) + (1-t)F_1(y_t, w) \\ &\leq tF_1(y_t, y). \end{aligned}$$

Therefore $F_1(y_t, y) \geq 0$. From Assumption 2.1(iii), we have $F_1(w, y) \geq 0$, which implies that $w \in EP(F_1)$.

Next, we show that $Aw \in EP(F_2)$. Since $\{x_n\}$ is bounded and $x_n \rightharpoonup w$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to w$, and since A is a bounded linear operator $Ax_{n_k} \to Aw$. Now we set $v_{n_k} = Ax_{n_k} - T_{r_{n_k}}^{F_2}Ax_{n_k}$. It follows from (3.14) that $\lim_{k\to\infty} v_{n_k} = 0$ and $Ax_{n_k} - v_{n_k} = T_{r_{n_k}}^{F_2}Ax_{n_k}$. Therefore from the definition of $T_{r_{n_k}}^{F_2}$, we have

$$F_2(Ax_{n_k} - v_{n_k}, y) + \frac{1}{r_{n_k}}\langle y - (Ax_{n_k} - v_{n_k}), (Ax_{n_k} - v_{n_k}) - Ax_{n_k}\rangle \geq 0, \quad \forall y \in C.$$

Since F_2 is upper semicontinuous in the first argument, taking lim sup in the above inequality as $k \to \infty$ and using Assumption 2.1(iv), we obtain

$$F_2(Aw, y) \geq 0, \quad \forall y \in C,$$

which implies that $Aw \in EP(F_2)$ and hence $w \in \Lambda$.

Next, we show that $w \in S^*$. Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$ and there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to w$, it is easy to observe that $z_{n_k} \to w$. For any $x, y \in C$, using (2.5), we have

$$\begin{aligned} \|Q(x) - Q(y)\|^2 &= \|P_C[P_C[x - \mu_2 B_2 x] - \mu_1 B_1 P_C[x - \mu_2 B_2 x]] \\ &\quad - P_C[P_C[y - \mu_2 B_2 y] - \mu_1 B_1 P_C[y - \mu_2 B_2 y]]\|^2 \\ &\leq \|(P_C[x - \mu_2 B_2 x] - P_C[y - \mu_2 B_2 y]) \end{aligned}$$

$$
\begin{aligned}
&\quad - \mu_1\big(B_1P_C[x-\mu_2B_2x] - B_1P_C[y-\mu_2B_2y]\big)\big\|^2 \\
&\le \big\|P_C[x-\mu_2B_2x] - P_C[y-\mu_2B_2y]\big\|^2 \\
&\quad - \mu_1(2\theta_1-\mu_1)\big\|B_1P_C[x-\mu_2B_2x] - B_1P_C[y-\mu_2B_2y]\big\|^2 \\
&\le \big\|P_C[x-\mu_2B_2x] - P_C[y-\mu_2B_2y]\big\|^2 \\
&\le \big\|(x-\mu_2B_2x) - (y-\mu_2B_2y)\big\|^2 \\
&\le \|x-y\|^2 - \mu_2(2\theta_2-\mu_2)\|B_2x-B_2y\|^2 \\
&\le \|x-y\|^2.
\end{aligned}
$$

This implies that $Q: C \to C$ is nonexpansive. On the other hand

$$
\begin{aligned}
\big\|z_n - Q(z_n)\big\|^2 &= \big\|P_C\big[P_C[u_n-\mu_2B_2u_n] - \mu_1B_1P_C[u_n-\mu_2B_2u_n]\big] - Q(z_n)\big\|^2 \\
&= \big\|Q(u_n) - Q(z_n)\big\|^2 \\
&\le \|u_n - z_n\|^2.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|u_n - z_n\| = 0$ (see (3.18)), we have $\lim_{n\to\infty}\|z_n - Q(z_n)\| = 0$. It follows from Lemma 2.5 that $w = Q(w)$, which implies from Lemma 2.2 that $w \in S^*$.

Thus we have

$$
w \in S^* \cap \Lambda \cap F(T).
$$

Observe that the constants satisfy $0 \le \rho\tau < \nu$ and

$$
\begin{aligned}
k \ge \eta \quad &\Longleftrightarrow \quad k^2 \ge \eta^2 \\
&\Longleftrightarrow \quad 1 - 2\mu\eta + \mu^2k^2 \ge 1 - 2\mu\eta + \mu^2\eta^2 \\
&\Longleftrightarrow \quad \sqrt{1-\mu\big(2\eta-\mu k^2\big)} \ge 1 - \mu\eta \\
&\Longleftrightarrow \quad \mu\eta \ge 1 - \sqrt{1-\mu\big(2\eta-\mu k^2\big)} \\
&\Longleftrightarrow \quad \mu\eta \ge \nu;
\end{aligned}
$$

therefore, from Lemma 2.6, the operator $\mu F - \rho U$ is $\mu\eta - \rho\tau$ strongly monotone, and we get the uniqueness of the solution of the variational inequality (3.20) and denote it by $z \in S^* \cap \Lambda \cap F(T)$.

Next, we claim that $\limsup_{n\to\infty}\langle \rho U(z) - \mu F(z), x_n - z\rangle \le 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$
\begin{aligned}
\limsup_{n\to\infty}\big\langle \rho U(z) - \mu F(z), x_n - z\big\rangle &= \limsup_{k\to\infty}\big\langle \rho U(z) - \mu F(z), x_{n_k} - z\big\rangle \\
&= \big\langle \rho U(z) - \mu F(z), w - z\big\rangle \le 0.
\end{aligned}
$$

Next, we show that $x_n \to z$. We have

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &= \big\langle P_C[V_n] - z, x_{n+1} - z\big\rangle \\
&= \big\langle P_C[V_n] - V_n, P_C[V_n] - z\big\rangle + \langle V_n - z, x_{n+1} - z\rangle
\end{aligned}
$$

$$\begin{aligned}
&\le \langle \alpha_n(\rho U(x_n) - \mu F(z)) + (I - \alpha_n\mu F)(T(y_n)) - (I - \alpha_n\mu F)(T(z)), x_{n+1} - z\rangle \\
&\le \langle \alpha_n\rho(U(x_n) - U(z)), x_{n+1} - z\rangle + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + \langle (I - \alpha_n\mu F)(T(y_n)) - (I - \alpha_n\mu F)(T(z)), x_{n+1} - z\rangle \\
&\le \alpha_n\rho\tau\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n\nu)\|y_n - z\|\|x_{n+1} - z\| \\
&\le \alpha_n\rho\tau\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n\nu)\{\beta_n\|Sx_n - Sz\| + \beta_n\|Sz - z\| + (1 - \beta_n)\|z_n - z\|\}\|x_{n+1} - z\| \\
&\le \alpha_n\rho\tau\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n\nu)\{\beta_n\|x_n - z\| + \beta_n\|Sz - z\| + (1 - \beta_n)\|x_n - z\|\}\|x_{n+1} - z\| \\
&= (1 - \alpha_n(\nu - \rho\tau))\|x_n - z\|\|x_{n+1} - z\| + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n\nu)\beta_n\|Sz - z\|\|x_{n+1} - z\| \\
&\le \frac{1 - \alpha_n(\nu - \rho\tau)}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n\nu)\beta_n\|Sz - z\|\|x_{n+1} - z\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &\le \frac{1 - \alpha_n(\nu - \rho\tau)}{1 + \alpha_n(\nu - \rho\tau)}\|x_n - z\|^2 + \frac{2\alpha_n}{1 + \alpha_n(\nu - \rho\tau)}\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle \\
&\quad + \frac{2(1 - \alpha_n\nu)\beta_n}{1 + \alpha_n(\nu - \rho\tau)}\|Sz - z\|\|x_{n+1} - z\| \\
&\le (1 - \alpha_n(\nu - \rho\tau))\|x_n - z\|^2 \\
&\quad + \frac{2\alpha_n(\nu - \rho\tau)}{1 + \alpha_n(\nu - \rho\tau)}\left\{\frac{1}{\nu - \rho\tau}\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle\right. \\
&\quad \left. + \frac{(1 - \alpha_n\nu)\beta_n}{\alpha_n(\nu - \rho\tau)}\|Sz - z\|\|x_{n+1} - z\|\right\}.
\end{aligned}$$

Let $\gamma_n = \alpha_n(\nu - \rho\tau)$ and $\delta_n = \frac{2\alpha_n(\nu-\rho\tau)}{1+\alpha_n(\nu-\rho\tau)}\{\frac{1}{\nu-\rho\tau}\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle + \frac{(1-\alpha_n\nu)\beta_n}{\alpha_n(\nu-\rho\tau)}\|Sz - z\|\|x_{n+1} - z\|\}$.

We have

$$\sum_{n=1}^{\infty}\alpha_n = \infty$$

and

$$\limsup_{n\to\infty}\left\{\frac{1}{\nu - \rho\tau}\langle \rho U(z) - \mu F(z), x_{n+1} - z\rangle + \frac{(1 - \alpha_n\nu)\beta_n}{\alpha_n(\nu - \rho\tau)}\|Sz - z\|\|x_{n+1} - z\|\right\} \le 0.$$

It follows that

$$\sum_{n=1}^{\infty}\gamma_n = \infty \quad \text{and} \quad \limsup_{n\to\infty}\frac{\delta_n}{\gamma_n} \le 0.$$

Thus all the conditions of Lemma 2.8 are satisfied. Hence we deduce that $x_n \to z$. This completes the proof. □

Remark 3.2 In hierarchical fixed-point problem (1.11), if $S = I - (\rho U - \mu F)$, then we can get the variational inequality (3.20). In (3.20), if $U = 0$ then we get the variational inequality $\langle F(z), x - z\rangle \geq 0$, $\forall x \in S^* \cap \Lambda \cap F(T)$, which just is the variational inequality studied by Suzuki [39] extending the common set of solutions of a system of variational inequalities, a split equilibrium problem, and a hierarchical fixed-point problem.

4 Conclusions

In this paper, we suggest and analyze an iterative method for finding the approximate element of the common set of solutions of (1.1), (1.9)-(1.10), and (1.11) in real Hilbert space, which can be viewed as a refinement and improvement of some existing methods for solving a system of variational inequality problem, a split equilibrium problem, and a hierarchical fixed-point problem. Some existing methods (*e.g.* [20, 26, 34, 40, 45]) can be viewed as special cases of Algorithm 3.1. Therefore, the new algorithm is expected to be widely applicable.

Competing interests
The author declares that he has no competing interests.

Received: 2 September 2013 Accepted: 30 December 2013 Published: 23 Jan 2014

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10.1186/1687-1812-2014-22
Cite this article as: Bnouhachem: **A modified projection method for a common solution of a system of variational inequalities, a split equilibrium problem and a hierarchical fixed-point problem.** *Fixed Point Theory and Applications* 2014, 2014:22