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Coincidence point theorems for generalized cyclic $(\kappa h, \varphi L)_S$-weak contractions in partially ordered Menger PM-spaces

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Abstract

In this paper, we introduce a new concept of generalized cyclic $(\kappa h, \varphi L)_S$-weak contraction mappings and establish some coincidence point results for such mappings in complete partially ordered Menger PM-spaces. Our results generalize the main results of Nashine (Nonlinear Anal. 75:6160-6169, 2012) and Gopal *et al.* (Appl. Math. Comput. 233:955-967, 2014). We also obtain the corresponding coincidence point theorems for generalized cyclic $(\kappa h, \varphi L)_S$-weak contractions in partially ordered metric spaces.

MSC: 47H10; 54H25

Keywords: Menger PM-space; partially ordered; cyclic weak contraction; coincidence point

1 Introduction and preliminaries

Fixed point theory is a very useful tool in many fields such as nonlinear operator theory, control theory, game theory, dynamics and economic theory. One of the most fundamental fixed point theorems is the Banach contraction mapping principle. Due to its simplicity and importance, this classical result has been generalized by many authors in different directions (see [1–5]).

In 2004, Ran and Reurings [6] established a fixed point theorem for Banach's contraction mappings in partially ordered metric spaces. In 2009, Harjani and Sadarangani [7] proved some fixed point theorems for weakly contractive mappings in complete metric spaces endowed with a partial order. In 2012, Nashine [8] presented some fixed point results for cyclic generalized ψ-weakly contractive mappings in complete metric spaces. Other authors also obtained some important results in this area (see [9–14]). On the other hand, in 2012, Samet *et al.* [15] introduced the concept of α-ψ-contractive and α-admissible mappings in metric spaces. In 2013, Berzig and Karapinar [16] proved some fixed point results for $(\alpha\psi, \beta\varphi)$-contractive mappings for a generalized altering distance in complete metric spaces.

In 1942, Menger [17] introduced the concept of a probabilistic metric space, and a large number of authors have done considerable work in such field (see, *e.g.* [18–22]). Recently, the extension of fixed point theory to generalized structures as partially ordered probabilistic metric spaces has attracted much attention (see, *e.g.* [23–25]). Gopal *et al.* [26]

established some fixed point results for α-ψ-type contractive mappings and generalized β-type contractive mappings in Menger PM-spaces.

In this paper, we introduce the notion of generalized cyclic $(\kappa h, \varphi L)_S$-weak contraction mappings to establish some corresponding coincidence point theorems in complete partially ordered Menger PM-spaces. Also, an application is given to show the validity of our results. It is worth pointing out that our results extend and generalize the main results of [8] and [26].

First, we recall some notions, lemmas and examples which will be used in the sequel.

Let R denote the set of reals and R^+ the nonnegative reals. A mapping $F: R \to R^+$ is called a distribution function if it is nondecreasing and left continuous with $\inf_{t\in R} F(t) = 0$ and $\sup_{t\in R} F(t) = 1$. We will denote by $\mathcal{D}$ the set of all distribution functions and $\mathcal{D}^+ = \{F \in \mathcal{D} : F(t) = 0, t \leq 0\}$.

Let H denote the specific distribution function defined by

$$H(x) = \begin{cases} 0, & x \leq 0, \\ 1, & x > 0. \end{cases}$$

Definition 1.1 ([18]) The mapping $\Delta : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (for short, a t-norm) if the following conditions are satisfied:

(Δ-1) $\Delta(a,1) = a$, for all $a \in [0,1]$;
(Δ-2) $\Delta(a,b) = \Delta(b,a)$;
(Δ-3) $\Delta(a,b) \leq \Delta(c,d)$, for $c \geq a$, $d \geq b$;
(Δ-4) $\Delta(a,\Delta(b,c)) = \Delta(\Delta(a,b),c)$.

Three typical examples of continuous t-norms are $\Delta_1(a,b) = \max\{a+b-1,0\}$, $\Delta_2(a,b) = ab$ and $\Delta_M(a,b) = \min\{a,b\}$, for all $a,b \in [0,1]$.

Definition 1.2 ([18]) A triplet $(X,\mathcal{F},\Delta)$ is called a Menger probabilistic metric space (shortly, a Menger PM-space), if X is a nonempty set, Δ is a t-norm and $\mathcal{F}$ is a mapping from $X \times X \to \mathcal{D}^+$ satisfying the following conditions (for $x,y \in X$, we denote $\mathcal{F}(x,y)$ by $F_{x,y}$):

(MS-1) $F_{x,y}(t) = H(t)$, for all $t \in R$, if and only if $x = y$;
(MS-2) $F_{x,y}(t) = F_{y,x}(t)$, for all $x,y \in X$ and $t \in R$;
(MS-3) $F_{x,z}(s+t) \geq \Delta(F_{x,y}(s), F_{y,z}(t))$, for all $x,y,z \in X$ and $s,t \geq 0$.

Definition 1.3 ([19]) $(X,\mathcal{F},\Delta)$ is called a non-Archimedean Menger PM-space (shortly, a N.A Menger PM-space), if $(X,\mathcal{F},\Delta)$ is a Menger PM-space and Δ satisfies the following condition: for all $x,y,z \in X$ and $t_1,t_2 \geq 0$,

$$F_{x,z}\big(\max\{t_1,t_2\}\big) \geq \Delta\big(F_{x,y}(t_1), F_{y,z}(t_2)\big). \tag{1.1}$$

Definition 1.4 ([19]) A non-Archimedean Menger PM-space $(X,\mathcal{F},\Delta)$ is said to be type of $(D)_g$, if there exists a $g \in \Omega$, such that

$$g\big(\Delta(s,t)\big) \leq g(s) + g(t),$$

for all $s,t \in [0,1]$, where $\Omega = \{g : g : [0,1] \to [0,+\infty)$ is continuous, strictly decreasing, $g(1) = 0\}$. In fact, we obtain $g(F_{x,z}(t)) \leq g(F_{x,y}(t)) + g(F_{y,z}(t))$, for all $x,y,z \in X$ and $t \in R^+$.

Example 1.1 Let $(X,\mathcal{F},\Delta)$ be a N.A Menger PM-space, and $\Delta \geq \Delta_1$ (or $\Delta(s,t) \geq \frac{ts}{t+s-ts}$ for all $s,t \in [0,1]$). Then $(X,\mathcal{F},\Delta)$ is of $(D)_g$-type for $g \in \Omega$ defined by $g(t) = 1-t$ (or $g(t) = \frac{1}{t} - 1$ for $0 < t \leq 1$ and $g(0) = +\infty$).

Remark 1.1 Schweizer and Sklar [18] point out that if $(X,\mathcal{F},\Delta)$ is a Menger probabilistic metric space and Δ is continuous, then $(X,\mathcal{F},\Delta)$ is a Hausdorff topological space in the (ε,λ)-topology T, *i.e.*, the family of sets $\{U_x(\varepsilon,\lambda) : \varepsilon > 0, \lambda \in (0,1]\}$ $(x \in X)$ is a basis of neighborhoods of a point x for T, where $U_x(\varepsilon,\lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1-\lambda\}$.

Definition 1.5 ([1]) The function $h : [0,\infty) \to [0,\infty)$ is called an altering distance function, if the following properties are satisfied: (a) h is continuous and nondecreasing; (b) $h(t) = 0$ if and only if $t = 0$.

Definition 1.6 ([21]) A function $\psi : R^+ \to R^+$ is said to be a Ψ-function, if it is nondecreasing and continuous in R^+, $\psi(t) \to \infty$ as $t \to \infty$, $\psi(t) = 0$ if and only if $t = 0$.

In the sequel, the class of all Ψ functions will be denoted by Ψ. Any altering distance function h with the additional property $\lim_{t\to+\infty} h(t) = +\infty$ generalizes a Ψ-function ψ through $\psi(0) = 0$ and $\psi(t) = \sup\{s : h(s) < t\}$ whenever $t > 0$.

Lemma 1.1 ([19]) *Let $\{x_n\}$ be a sequence in $(X,\mathcal{F},\Delta)$ such that $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1$ for all $t > 0$. If the sequence $\{x_n\}$ is not a Cauchy sequence in X, then there exist $\varepsilon_0 > 0$, $t_0 > 0$ and two sequences $\{k(i)\}$, $\{m(i)\}$ of positive integers such that*

(1) *$m(i) > k(i)$, and $m(i) \to \infty$ as $i \to \infty$;*
(2) *$F_{x_{m(i)},x_{k(i)}}(t_0) < 1-\varepsilon_0$ and $F_{x_{m(i)-1},x_{k(i)}}(t_0) \geq 1-\varepsilon_0$, for $i = 1,2,\ldots$.*

2 Main results

In this section, we first introduce the new notions of generalized κ-admissible mappings, weakly comparable mappings and generalized cyclic $(\kappa h,\varphi L)_S$-weak contraction mappings in Menger PM-spaces.

Definition 2.1 Let X be a nonempty set, $S,T : X \to X$ be two self-maps and $\kappa : X \times X \times (0,\infty) \to R^+$ be a function. S and T are called generalized κ-admissible, if for all $x,y \in X$, $t > 0$, and $\kappa(Sx,Sy,t) \leq 1$, we have $\kappa(Tx,Ty,t) \leq 1$. κ is called m-transitive on X, if for all $t > 0$, $x_0,x_1,\ldots,x_m,x_{m+1} \in X$, $\kappa(x_0,x_1,t) \leq 1$, $\kappa(x_1,x_2,t) \leq 1,\ldots,\kappa(x_m,x_{m+1},t) \leq 1$, we have $\kappa(x_0,x_{m+1},t) \leq 1$.

Example 2.1 Let $X = [0,\infty)$, $Sx = \ln(1+x^2)$ for all $x \in X$,

$$Tx = \begin{cases} \frac{x}{2} + \frac{1}{2}, & x \in [0,3], \\ x^2, & x \in (3,\infty) \end{cases} \quad \text{and} \quad \kappa(x,y,t) = \begin{cases} 1, & x,y \in [0,2], \\ 2, & \text{otherwise.} \end{cases}$$

In fact, if $x,y \in X$, for all $t > 0$, $\kappa(Sx,Sy,t) = \kappa(\ln(1+x^2),\ln(1+y^2),t) \leq 1$, then $x,y \in [0,\sqrt{e^2-1}] \subset [0,3]$. Hence, $Tx,Ty \in [0,2]$, and so $\kappa(Tx,Ty,t) \leq 1$. Thus, S and T are generalized κ-admissible. Also, we can verify that κ is m-transitive.

Let $(X,\leq)$ be a partially ordered set, we will write $x \asymp y$ whenever x and y are comparable (that is, $x \leq y$ or $y \leq x$ holds).

Definition 2.2 Let $(X,\leq)$ be a partially ordered set, $S,T : X \to X$ be two self-maps and $T(X) \subset S(X)$. T is called weakly comparable with respect to S, if $x,y \in X$ such that $Sx \asymp Tx = Sy$ implies Tx and Ty are comparable (that is, $Tx \asymp Ty$). $\asymp$ is called m-transitive on X, if $x_0, x_1, \dots, x_{m+1} \in X$ such that $x_i \asymp x_{i+1}$ for all $i \in \{0,1,2,\dots,m\}$ implies $x_0 \asymp x_{m+1}$.

Example 2.2 Let $X = \{2,4,6,8,10\}$, $\preceq = \{\langle 2,2\rangle, \langle 4,4\rangle, \langle 6,6\rangle, \langle 8,8\rangle, \langle 10,10\rangle, \langle 2,4\rangle, \langle 2,6\rangle, \langle 8,4\rangle, \langle 10,6\rangle\}$, and

$$S : \begin{pmatrix} 2 & 4 & 6 & 8 & 10 \\ 6 & 10 & 2 & 8 & 4 \end{pmatrix}, \qquad T : \begin{pmatrix} 2 & 4 & 6 & 8 & 10 \\ 2 & 6 & 4 & 10 & 4 \end{pmatrix}.$$

Since $S(2) = 6 \asymp T(2) = 2 = S(6)$, we have $T(2) = 2 \asymp T(6) = 4$. Since $S(4) = 10 \asymp T(4) = 6 = S(2)$, we have $T(4) \asymp T(2) = 2$. Since $S(6) \asymp T(6) = 4 = S(10)$, we have $T(6) \asymp T(10)$. Since $S(10) \asymp T(10) = S(10)$, we have $T(10) \asymp T(10)$. Note that $S(8)$ and $T(8)$ are not comparable. Hence, T is weakly comparable with respect to S.

Definition 2.3 Let X be a nonempty set, m be a positive integer, $A_1, A_2, \dots, A_m$ be subsets of X, $Y = \bigcup_{i=1}^m A_i$ and $S,T : Y \to Y$ be two self-maps. Then Y is said to be a cyclic representation of Y with respect to S and T, if the following two conditions are satisfied:

(i) $S(A_i)$, $i = 1,2,\dots,m$, are nonempty closed sets;

(ii) $T(A_1) \subseteq S(A_2), T(A_2) \subseteq S(A_3), \dots, T(A_m) \subseteq S(A_1)$.

Example 2.3 Let $X = R^+$, $A_1 = [0,4]$, $A_2 = [0,\frac{7}{2}]$, $A_3 = [0,3]$, and $Y = \bigcup_{i=1}^3 A_i$. Define $S,T : Y \to Y$ by $Sx = \frac{1}{3} + \frac{11}{12}x$ and $Tx = 1 + \frac{1}{2}x$, for all $x \in Y$. Then it is easy to verify that $Y = \bigcup_{i=1}^3 A_i$ is a cyclic representation of Y with respect to S and T.

Definition 2.4 Let $(X,\leq)$ be a partially ordered set and $(X,\mathcal{F},\Delta)$ be a N.A Menger PM-space of type $(D)_g$. Let $\kappa : X \times X \times (0,\infty) \to [0,\infty)$ be a function and $\phi : X \to [0,\infty)$ be a lower semi-continuous function. Let m be a positive integer, $A_1, A_2, \dots, A_m$ be subsets of X, $Y = \bigcup_{i=1}^m A_i$, and $S,T : Y \to Y$ be two self-maps. T is said to be a generalized cyclic $(\kappa h, \varphi L)_S$-weak contraction, if Y is a cyclic representation of Y with respect to S and T, $A_{m+1} = A_1$, and for $k \in \{1,2,\dots,m\}$ and for all $x,y \in X$, $Sx \in S(A_k)$ and $Sy \in S(A_{k+1})$ are comparable such that

$$\begin{aligned} &h\bigl[g\bigl(F_{Tx,Ty}(t)\bigr) + \phi(Tx) + \phi(Ty)\bigr] \\ &\quad \leq \kappa(Sx,Sy,t)\bigl[h\bigl(M_t(Sx,Sy)\bigr) - \varphi\bigl(M_t(Sx,Sy)\bigr)\bigr] + L\bigl(N_t(Sx,Sy)\bigr), \end{aligned} \tag{2.1}$$

for all $t > 0$ and $\beta \in (0,1]$, where h is an altering distance function, $\varphi, L : [0,\infty) \to [0,\infty)$ are two continuous functions such that $L(0) = 0$, $\varphi(s) = 0$ if and only if $s = 0$, $\varphi(s) \leq h(s)$ for all $s \in R^+$, $N_t(Sx,Sy) = \min\{g(F_{Sx,Tx}(t)), g(F_{Sx,Ty}((2-\beta)t)), g(F_{Sy,Tx}(\beta t))\}$, and

$$\begin{aligned} M_t(Sx,Sy) = \max\Bigl\{ & g\bigl(F_{Sx,Sy}(t)\bigr) + \phi(Sx) + \phi(Sy), g\bigl(F_{Sx,Tx}(t)\bigr) \\ & + \phi(Sx) + \phi(Tx), g\bigl(F_{Sy,Ty}(t)\bigr) + \phi(Sy) + \phi(Ty), \end{aligned}$$

$$\frac{1}{2}\big[g\big(F_{Sx,Ty}\big((2-\beta)t\big)\big)+g\big(F_{Sy,Tx}(\beta t)\big)$$
$$+\phi(Sx)+\phi(Sy)+\phi(Tx)+\phi(Ty)\big]\Big\}.$$

Now we are ready to state our main results.

Theorem 2.1 *Let* $(X,\leq)$ *be a partially ordered set and* $(X,\mathcal{F},\Delta)$ *be a complete N.A Menger PM-space of type* $(D)_g$. *Let* m *be a positive integer,* $A_1,A_2,\ldots,A_m$ *be subsets of* X, $Y=\bigcup_{i=1}^m A_i$, $T:Y\to Y$ *be a generalized cyclic* $(\kappa h,\varphi L)_S$*-weak contraction satisfying* (2.1). *Suppose that the following conditions hold*:

(i) S *and* T *are generalized* κ*-admissible*;
(ii) κ *and* $\asymp$ *are* m*-transitive*;
(iii) T *is weakly comparable with respect to* S;
(iv) *there exists* $x_0\in A_1$ *such that* $Sx_0\asymp Tx_0$ *and* $\kappa(Sx_0,Tx_0,t)\leq 1$ *for all* $t>0$;
(v) *if a sequence* $\{y_n\}\subset Y$ *satisfies* $y_n\asymp y_{n+1}$, $\kappa(y_n,y_{n+1},t)\leq 1$ *for all* $n\in N$, *and* $t>0$ *and* $y_n\to y$ *as* $n\to\infty$, *then* $y_n\asymp y$ *and* $\kappa(y_n,y,t)\leq 1$ *for* n *sufficiently large and for all* $t>0$.

Then S *and* T *have a coincidence point in* X, *that is, there exists* $x\in X$ *such that* $Sx=Tx$.

Proof Since $T(A_1)\subset S(A_2)$ and $x_0\in A_1$, there exists an $x_1\in A_2$, such that $Sx_1=Tx_0$. Since $T(A_2)\subset S(A_3)$ and $x_1\in A_2$, there exists an $x_2\in A_3$, such that $Sx_2=Tx_1$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ defined by $y_{n+1}=Sx_{n+1}=Tx_n$, for all $n\in N$, and there exists $i_n\in\{1,2,\ldots,m\}$ such that $x_n\in A_{i_n}$ and $x_{n+1}\in A_{i_n+1}$.

By condition (iv), we get $Sx_0\asymp Tx_0=Sx_1$ and $\kappa(Sx_0,Sx_1,t)\leq 1$ for all $t>0$. It follows from (i) and (iii) that $Sx_1=Tx_0\asymp Tx_1=Sx_2$ and $\kappa(Sx_1,Sx_2,t)\leq 1$ for all $t>0$. By induction, we obtain

$$y_n=Sx_n\asymp Sx_{n+1}=y_{n+1}\quad\text{and}\quad\kappa(y_n,y_{n+1},t)\leq 1,\quad\text{for all }t>0.\tag{2.2}$$

We will complete the proof by the following three steps.

Step 1. We prove that

$$\lim_{n\to\infty}g\big(F_{y_n,y_{n+1}}(t)\big)=0\quad\text{and}\quad\lim_{n\to\infty}\phi(y_n)=0,\quad\text{for all }t>0.\tag{2.3}$$

Without loss of generality, assume that $y_{n+1}\neq y_n$, for all $n\in N$ (otherwise, $Tx_{n_0}=Sx_{n_0+1}=y_{n_0+1}=y_{n_0}=Sx_{n_0}$ for some $n_0\in N$, then x_{n_0} is the coincidence point of S and T. Hence, the conclusion holds).

Since $x_n\in A_{i_n}$, $x_{n+1}\in A_{i_n+1}$, $y_n\in S(A_{i_n})$, and $y_{n+1}\in S(A_{i_n+1})$ are comparable, for $i_n\in\{1,2,\ldots,m\}$, by (2.1) and (2.2), we get

$$\begin{aligned}&h\big[g\big(F_{y_n,y_{n+1}}(t)\big)+\phi(y_n)+\phi(y_{n+1})\big]\\&\quad\leq\kappa(y_{n-1},y_n,t)\big[h\big(M_t(y_{n-1},y_n)\big)-\varphi\big(M_t(y_{n-1},y_n)\big)\big]+L\big(N_t(y_{n-1},y_n)\big)\\&\quad\leq h\big(M_t(y_{n-1},y_n)\big)-\varphi\big(M_t(y_{n-1},y_n)\big)+L\big(N_t(y_{n-1},y_n)\big),\end{aligned}\tag{2.4}$$

for all $t > 0$, where $N_t(y_{n-1}, y_n) = \min\{g(F_{y_{n-1},y_n}(t)), g(F_{y_{n-1},y_{n+1}}((2-\beta)t)), 0\} = 0$ and

$$\begin{aligned} M_t(y_{n-1}, y_n) &= \max\Big\{g\big(F_{y_{n-1},y_n}(t)\big) + \phi(y_{n-1}) + \phi(y_n), g\big(F_{y_{n-1},y_n}(t)\big) \\ &\quad + \phi(y_{n-1}) + \phi(y_n), g\big(F_{y_n,y_{n+1}}(t)\big) + \phi(y_n) + \phi(y_{n+1}), \\ &\quad \frac{1}{2}\big[g\big(F_{y_{n-1},y_{n+1}}\big((2-\beta)t\big)\big) + \phi(y_{n-1}) + 2\phi(y_n) + \phi(y_{n+1})\big]\Big\} \\ &\leq \max\Big\{g\big(F_{y_{n-1},y_n}(t)\big) + \phi(y_{n-1}) + \phi(y_n), g\big(F_{y_n,y_{n+1}}(t)\big) + \phi(y_n) + \phi(y_{n+1}), \\ &\quad \frac{1}{2}\big[g\big(F_{y_{n-1},y_n}\big((2-\beta)t\big)\big) + g\big(F_{y_n,y_{n+1}}\big((2-\beta)t\big)\big) \\ &\quad + \phi(y_{n-1}) + 2\phi(y_n) + \phi(y_{n+1})\big]\Big\}. \end{aligned}$$

For all $\beta \in (0,1]$, we have $2 - \beta \geq 1$. Since $F_{x,y}$ is nondecreasing and g is strictly decreasing, we have $g(F_{y_{n-1},y_n}((2-\beta)t)) \leq g(F_{y_{n-1},y_n}(t))$ for all $n \in N$. Hence, $M_t(y_{n-1}, y_n) \leq \max\{g(F_{y_{n-1},y_n}(t)) + \phi(y_{n-1}) + \phi(y_n), g(F_{y_n,y_{n+1}}(t)) + \phi(y_n) + \phi(y_{n+1})\}$. On the other hand, it is obvious that $M_t(y_{n-1}, y_n) \geq \max\{g(F_{y_{n-1},y_n}(t)) + \phi(y_{n-1}) + \phi(y_n), g(F_{y_n,y_{n+1}}(t)) + \phi(y_n) + \phi(y_{n+1})\}$. Thus, $M_t(y_{n-1}, y_n) = \max\{g(F_{y_{n-1},y_n}(t)) + \phi(y_{n-1}) + \phi(y_n), g(F_{y_n,y_{n+1}}(t)) + \phi(y_n) + \phi(y_{n+1})\}$.

Suppose that $M_t(y_{n-1}, y_n) = g(F_{y_n,y_{n+1}}(t)) + \phi(y_n) + \phi(y_{n+1})$, by (2.4), we have

$$h\big[M_t(y_{n-1}, y_n)\big] \leq h\big[M_t(y_{n-1}, y_n)\big] - \varphi\big(M_t(y_{n-1}, y_n)\big), \quad \text{for all } t > 0,$$

which implies that $\varphi(g(F_{y_n,y_{n+1}}(t))) + \phi(y_n) + \phi(y_{n+1}) = 0$. Thus, $g(F_{y_n,y_{n+1}}(t)) = 0$, that is, $F_{y_n,y_{n+1}}(t) = 1$ for all $t > 0$. Then $y_n = y_{n+1}$, which is in contradiction to $y_n \neq y_{n+1}$ for all $n \in N$.

Hence, $M_t(y_{n-1}, y_n) = g(F_{y_{n-1},y_n}(t)) + \phi(y_{n-1}) + \phi(y_n)$. Let $Q_n(t) = g(F_{y_{n-1},y_n}(t)) + \phi(y_{n-1}) + \phi(y_n)$. By (2.4), we get

$$h\big[Q_{n+1}(t)\big] \leq h\big[Q_n(t)\big] - \varphi\big(Q_n(t)\big) \leq h\big[Q_n(t)\big], \quad \text{for all } t > 0. \tag{2.5}$$

Since h is nondecreasing, it follows from (2.5) that $\{Q_n(t)\}$ is a decreasing sequence and bounded from below, for every $t > 0$. Hence, there exists $r_t \geq 0$, such that $\lim_{n\to\infty} Q_n(t) = r_t$.

By using the continuities of h and φ, letting $n \to \infty$ in (2.5), we get $h(r_t) \leq h(r_t) - \varphi(r_t)$, which implies that $\varphi(r_t) = 0$. Thus $r_t = 0$, that is, $\lim_{n\to\infty} g(F_{y_{n-1},y_n}(t)) + \phi(y_{n-1}) + \phi(y_n) = 0$ for all $t > 0$. Hence, (2.3) holds.

Step 2. We prove that $\{y_n\}$ is a Cauchy sequence. To prove this fact, we first prove the following claim.

Claim: for every $t > 0$ and $\varepsilon > 0$, there exists $n_0 \in N$, such that $p, q \geq n_0$ with $p - q \equiv 1 \mod m$ then $F_{y_p,y_q}(t) > 1 - \varepsilon$, that is, $g(F_{y_p,y_q}(t)) < g(1-\varepsilon)$.

In fact, suppose this is not true, then there exist $t_0 > 0$ and $\varepsilon_0 > 0$, such that for any $n \in N$, we can find $p(n) > q(n) \geq n$ with $p(n) - q(n) \equiv 1 \mod m$ satisfying $F_{y_{p(n)},y_{q(n)}}(t_0) \leq 1 - \varepsilon_0$, that is, $g(F_{y_{p(n)},y_{q(n)}}(t_0)) \geq g(1-\varepsilon_0)$.

Now, take $n > 2m$. Then corresponding to $q(n) \geq n$, we can choose $p(n)$ in such a way that it is the smallest integer with $p(n) > q(n)$ satisfying $p(n) - q(n) \equiv 1 \mod m$

and $g(F_{y_{p(n)},y_{q(n)}}(t_0)) \geq g(1-\varepsilon_0)$. Therefore, $g(F_{y_{p(n)-m},y_{q(n)}}(t_0)) < g(1-\varepsilon_0)$. Using the non-Archimedean Menger triangular inequality and Definition 1.4, we have

$$\begin{aligned} g(1-\varepsilon_0) &\leq g\big(F_{y_{q(n)},y_{p(n)}}(t_0)\big) \leq g\big(\Delta\big(F_{y_{q(n)},y_{q(n)+1}}(t_0), F_{y_{q(n)+1},y_{p(n)}}(t_0)\big)\big) \\ &\leq g\big(F_{y_{q(n)},y_{q(n)+1}}(t_0)\big) + g\big(F_{y_{q(n)+1},y_{p(n)}}(t_0)\big) \\ &\leq g\big(F_{y_{q(n)},y_{q(n)+1}}(t_0)\big) + g\big(F_{y_{q(n)+1},y_{p(n)+1}}(t_0)\big) + g\big(F_{y_{p(n)+1},y_{p(n)}}(t_0)\big) \\ &\leq 2g\big(F_{y_{q(n)},y_{q(n)+1}}(t_0)\big) + g\big(F_{y_{q(n)},y_{p(n)+1}}(t_0)\big) + g\big(F_{y_{p(n)+1},y_{p(n)}}(t_0)\big) \\ &\leq 2g\big(F_{y_{q(n)},y_{q(n)+1}}(t_0)\big) + g\big(F_{y_{q(n)},y_{p(n)}}(t_0)\big) + 2g\big(F_{y_{p(n)+1},y_{p(n)}}(t_0)\big) \\ &\leq 2g\big(F_{y_{q(n)},y_{q(n)+1}}(t_0)\big) + g\big(F_{y_{q(n)},y_{p(n)-m}}(t_0)\big) \\ &\quad + g\big(F_{y_{p(n)-m},y_{p(n)}}(t_0)\big) + 2g\big(F_{x_{p(n)+1},x_{p(n)}}(t_0)\big) \\ &\leq 2g\big(F_{y_{q(n)},y_{q(n)+1}}(t_0)\big) + g(1-\varepsilon_0) \\ &\quad + \sum_{i=1}^{m} g\big(F_{y_{p(n)-i},y_{p(n)-i+1}}(t_0)\big) + 2g\big(F_{y_{p(n)+1}y_{p(n)}}(t_0)\big). \end{aligned} \tag{2.6}$$

Since $\lim_{n\to\infty} g(F_{y_n,y_{n+1}}(t)) = 0$ for all $t > 0$, letting $n \to \infty$ in (2.6), we obtain

$$\begin{aligned} g(1-\varepsilon_0) &= \lim_{n\to\infty} g\big(F_{y_{q(n)},y_{p(n)}}(t_0)\big) = \lim_{n\to\infty} g\big(F_{y_{q(n)},y_{p(n)+1}}(t_0)\big) \\ &= \lim_{n\to\infty} g\big(F_{y_{q(n)+1},x_{p(n)}}(t_0)\big) = \lim_{n\to\infty} g\big(F_{y_{q(n)+1},y_{p(n)+1}}(t_0)\big). \end{aligned} \tag{2.7}$$

By $p(n) - q(n) \equiv 1 \mod m$, we know that $y_{q(n)}$ and $y_{p(n)}$ lie in different adjacently labeled sets $S(A_i)$ and $S(A_{i+1})$, for $1 \leq i \leq m$. Since $\asymp$ and κ are m-transitive, we obtain $y_{p(n)} \asymp y_{q(n)}$ and $\kappa(y_{p(n)}, y_{q(n)}, t) \leq 1$ for all $t > 0$. Using the fact that T is a generalized cyclic $(\kappa h, \varphi L)_S$-weak contraction and letting $\beta = 1$, we have

$$\begin{aligned} &h\big[g\big(F_{y_{q(n)+1},y_{p(n)+1}}(t_0)\big) + \phi(y_{q(n)+1}) + \phi(y_{p(n)+1})\big] \\ &\quad\leq \kappa(y_{q(n)}, y_{p(n)}, t)\big[h\big(M_{t_0}(y_{q(n)}, y_{p(n)})\big) - \varphi\big(M_{t_0}(y_{q(n)}, y_{p(n)})\big)\big] + L\big(N_{t_0}(y_{q(n)}, y_{p(n)})\big) \\ &\quad\leq h\big(M_{t_0}(y_{q(n)}, y_{p(n)})\big) - \varphi\big(M_{t_0}(y_{q(n)}, y_{p(n)})\big) + L\big(N_{t_0}(y_{q(n)}, y_{p(n)})\big), \end{aligned} \tag{2.8}$$

where $N_{t_0}(y_{q(n)}, y_{p(n)}) = \min\{g(F_{y_{q(n)},y_{q(n)+1}}(t_0)), g(F_{y_{q(n)},y_{p(n)+1}}(t_0)), g(F_{y_{p(n)},y_{q(n)+1}}(t_0))\}$ and

$$\begin{aligned} M_{t_0}(y_{q(n)}, y_{p(n)}) = \max\bigg\{ &g\big(F_{y_{q(n)},y_{p(n)}}(t_0)\big) + \phi(y_{q(n)}) + \phi(y_{p(n)}), g\big(F_{y_{q(n)},y_{q(n)+1}}(t_0)\big) \\ &+ \phi(y_{q(n)}) + \phi(y_{q(n)+1}), g\big(F_{y_{p(n)},y_{p(n)+1}}(t_0)\big) + \phi(y_{p(n)}) + \phi(y_{p(n)+1}), \\ &\frac{1}{2}\big[g\big(F_{y_{q(n)},y_{p(n)+1}}(t_0)\big) + g\big(F_{y_{p(n)},y_{q(n)+1}}(t_0)\big) \\ &+ \phi(y_{q(n)}) + \phi(y_{p(n)}) + \phi(y_{q(n)+1}) + \phi(y_{p(n)+1})\big]\bigg\}. \end{aligned}$$

By (2.3) and (2.7), we have $\lim_{n\to\infty} M_{t_0}(x_{q(n)}, x_{p(n)}) = \max\{g(1-\varepsilon_0), 0, 0, \frac{1}{2}[g(1-\varepsilon_0) + g(1-\varepsilon_0)]\} = g(1-\varepsilon_0)$ and $\lim_{n\to\infty} N_{t_0}(y_{q(n)}, y_{p(n)}) = \min\{0, g(1-\varepsilon_0), g(1-\varepsilon_0)\} = 0$. According to the continuities of h and φ, letting $n \to \infty$ in (2.8), we get

$$h\big[g(1-\varepsilon_0)\big] \le h\big[g(1-\varepsilon_0)\big] - \varphi\big(g(1-\varepsilon_0)\big).$$

Thus, $\varphi(g(1-\varepsilon_0)) = 0$, that is, $g(1-\varepsilon_0) = 0$. Then $\varepsilon_0 = 0$, which is in contradiction to $\varepsilon_0 > 0$.

Therefore, our claim is proved. Next, we will prove that $\{y_n\}$ is a Cauchy sequence.

By the continuity of g and $g(1) = 0$, we have $\lim_{a\to 0^+} g(1-a\varepsilon) = 0$, for any given $\varepsilon > 0$. Since g is strictly decreasing, there exists $a > 0$, such that $g(1-a\varepsilon) \le \frac{g(1-\varepsilon)}{2}$.

For any given $t > 0$ and $\varepsilon > 0$, there exists $a > 0$, such that $g(1-a\varepsilon) \le \frac{g(1-\varepsilon)}{2}$. By the claim, we can find $n_0 \in N$, such that if $p, q \ge n_0$ with $p - q \equiv 1 \mod m$, then

$$F_{y_p,y_q}(t) > 1 - a\varepsilon \quad \text{and} \quad g\big(F_{y_p,y_q}(t)\big) < g(1-a\varepsilon) \le \frac{g(1-\varepsilon)}{2}. \tag{2.9}$$

Since $\lim_{n\to\infty} g(F_{y_n,y_{n+1}}(t)) = 0$, we can also find $n_1 \in N$, such that

$$g\big(F_{y_n,y_{n+1}}(t)\big) \le \frac{g(1-\varepsilon)}{2m}, \tag{2.10}$$

for any $n > n_1$.

Suppose that $r, s \ge \max\{n_0, n_1\}$ and $s > r$. Then there exists $k \in \{1, 2, \ldots, m\}$ such that $s - r \equiv k \mod m$. Therefore, $s - r + j \equiv 1 \mod m$, for $j = m - k + 1$, $j \in \{0, 1, \ldots, m-1\}$. Thus we have

$$g\big(F_{y_r,y_s}(t)\big) \le g\big(F_{y_r,y_{s+j}}(t)\big) + g\big(F_{y_{s+j},y_{s+j-1}}(t)\big) + \cdots + g\big(F_{y_{s+1},y_s}(t)\big). \tag{2.11}$$

By (2.9), (2.10), and (2.11), we get

$$g\big(F_{y_r,y_s}(t)\big) < \frac{g(1-\varepsilon)}{2} + j \cdot \frac{g(1-\varepsilon)}{2m} \le \frac{g(1-\varepsilon)}{2} + \frac{g(1-\varepsilon)}{2} = g(1-\varepsilon). \tag{2.12}$$

Since g is strictly decreasing, by (2.12), we have $F_{y_r,y_s}(t) > 1 - \varepsilon$. This proves that $\{y_n\}$ is a Cauchy sequence.

Step 3. We show that S and T have a coincidence point in X.

Since $\{y_n\} \subset X$ is a Cauchy sequence and $(X, \mathcal{F}, \Delta)$ is a complete Menger PM-space, there exists $y^* \in X$, such that $y_n \to y^*$. Since $S(Y) = S(\bigcup_{i=0}^m A_i) = \bigcup_{i=0}^m S(A_i)$ is closed and $\{y_n\} \subset S(Y)$, we know that $y^* \in S(Y)$. Hence, there exists $z \in Y$, such that $y^* = Sz$. As $Y = \bigcup_{i=1}^m A_i$ is a cyclic representation of Y with respect to S and T, the sequence $\{y_n\}$ has infinite terms in each $S(A_i)$ for $i \in \{1, 2, \ldots, m\}$.

First, suppose that $y^* \in S(A_i)$, then $Tz \in S(A_{i+1})$, and we can choose a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ with $y_{n_k} \in S(A_{i-1})$ (the existence of this subsequence is guaranteed by the above discussion). Since $y_n \asymp y_{n+1}$ and $y_n \to y^* = Sz$, by (v), we obtain the result that $y_{n_k} \in S(A_{i-1})$ and $y^* \in S(A_i)$ are comparable, $\kappa(y_{n_k}, y^*, t) \le 1$ for all $t > 0$ and k sufficiently large. Letting $\beta = 1$, by (2.1), we have

$$\begin{aligned} &h\big[g\big(F_{y_{n_k+1},Tz}(t)\big) + \phi(y_{n_k+1}) + \phi(Tz)\big] \\ &\quad \le \kappa\big(y_{n_k}, y^*, t\big)\big[h\big(M_t\big(y_{n_k}, y^*\big)\big) - \varphi\big(M_t\big(y_{n_k}, y^*\big)\big)\big] + L\big(N_t\big(y_{n_k}, y^*\big)\big) \\ &\quad \le h\big[M_t\big(y_{n_k}, y^*\big)\big] - \varphi\big(M_t\big(y_{n_k}, y^*\big)\big) + L\big(N_t\big(y_{n_k}, y^*\big)\big), \end{aligned} \tag{2.13}$$

where $N_t(y_{n_k}, y^*) = \min\{g(F_{y_{n_k}, y_{n_k+1}}(t)), g(F_{y_{n_k}, Tz}(t)), g(F_{y_{n_k+1}, y^*}(t))\}$ and

$$M_t(y_{n_k}, y^*) = \max\Big\{g(F_{y_{n_k}, y^*}(t)) + \phi(y_{n_k}) + \phi(y^*), g(F_{y_{n_k}, y_{n_k+1}}(t)) + \phi(y_{n_k}) + \phi(y_{n_k+1}),$$

$$g(F_{y^*, Tz}(t)) + \phi(y^*) + \phi(Tz), \frac{1}{2}\big[g(F_{y_{n_k}, Tz}(t)) + g(F_{y_{n_k+1}, y^*}(t)) + \phi(y_{n_k})$$

$$+ \phi(y_{n_k+1}) + \phi(y^*) + \phi(Tz)\big]\Big\}.$$

Since ϕ is lower semi-continuous and $y_n \to y^*$, by (2.3), we have

$$\phi(y^*) \le \liminf_{n\to\infty} \phi(y_n) = 0.$$

Let G_0 be the set of all discontinuous points of $F_{x^*, Tx^*}(\cdot)$. Since g, h, φ, and L are continuous, we find that G_0 is also the set of all discontinuous points of $g(F_{x^*, Tx^*}(\cdot))$, $h[g(F_{x^*, Tx^*}(\cdot)) + \phi(Tz)]$, $\varphi(g(F_{x^*, Tx^*}(\cdot)) + \phi(Tz))$, and $L(g(F_{x^*, Tx^*}(\cdot)))$. Moreover, we know that G_0 is a countable set. Let $G = R^+ \backslash G_0$. If $t \in G \backslash \{0\}$ (t is a continuous point of $F_{x^*, Tx^*}(\cdot)$), by (2.3), we have $\lim_{k\to\infty} N_t(y_{n_k}, y^*) = \min\{0, g(F_{y^*, Tz}(t)), 0\} = 0$ and

$$\lim_{k\to\infty} M_t(y_{n_k}, y^*) = \max\Big\{0, 0, g(F_{y^*, Tz}(t)) + \phi(Tz), \frac{1}{2}\big[g(F_{y^*, Tz}(t)) + \phi(Tz)\big]\Big\}$$

$$= g(F_{y^*, Tz}(t)) + \phi(Tz).$$

Letting $n \to \infty$ in (2.13), we get

$$h\big[g(F_{y^*, Tz}(t)) + \phi(Tz)\big] \le h\big[g(F_{y^*, Tz}(t)) + \phi(Tz)\big] - \varphi\big(g(F_{y^*, Tz}(t)) + \phi(Tz)\big),$$

which implies that $\varphi(g(F_{y^*, Tz}(t)) + \phi(Tz)) = 0$. Hence, $g(F_{y^*, Tz}(t)) = 0$ and $\phi(Tz) = 0$. Then

$$F_{y^*, Tz}(t) = 1, \quad \text{for all } t \in G \backslash \{0\}. \tag{2.14}$$

If $t \in G_0$ with $t > 0$, by the density of real numbers, there exist $t_1, t_2 \in G$, such that $0 < t_1 < t < t_2$. Since the distribution is nondecreasing, we have

$$1 = H(t_1) = F_{y^*, Tz}(t_1) \le F_{y^*, Tz}(t) \le F_{y^*, Tz}(t_2) = 1. \tag{2.15}$$

Hence, from (2.14) and (2.15), we have $F_{y^*, Tz}(t) = 1$ for any $t > 0$. Thus, $Sz = y^* = Tz$, that is, z is the coincidence point of S and T. □

Corollary 2.1 *Let $(X, \le)$ be a partially ordered set and $(X, \mathcal{F}, \Delta)$ be a complete N.A Menger PM-space of type $(D)_g$, $\kappa : X \times X \times (0, \infty) \to [0, \infty)$ be a function, $S, T : X \to X$ be two self-maps and $T(X) \subset S(X)$. Suppose that for $x, y \in X$, Sx and Sy are comparable, we have*

$$g(F_{Tx, Ty}(t)) \le \kappa(Sx, Sy, t)\Phi\big(M_t(Sx, Sy)\big) + L\big(N_t(Sx, Sy)\big), \tag{2.16}$$

for all $t > 0$, where $\Phi : [0, \infty) \to [0, \infty)$ is a continuous function, $\Phi(t) < t$ for $t > 0$ and $\Phi(0) = 0$, L is the same as the one in Theorem 2.1,

$$M_t(Sx, Sy) = \max\Big\{g(F_{Sx, Sy}(t)), g(F_{Sx, Tx}(t)), g(F_{Sy, Ty}(t)), \frac{1}{2}\big[g(F_{Sx, Ty}(t)) + g(F_{Sy, Tx}(t))\big]\Big\}$$

and

$$N_t(Sx,Sy) = \min\{g(F_{Sx,Tx}(t)), g(F_{Sx,Ty}(t)), g(F_{Sy,Tx}(t))\}.$$

Also, assume that the conditions (i)-(v) *of Theorem* 2.1 *are satisfied, where* $m = 1$.

Then S and T have a coincidence point in X, that is, there exists $x \in X$, such that $Sx = Tx$.

Proof Letting $h(x) = x$, $\phi(x) \equiv 0$, $\beta \equiv 1$, and $\varphi(t) = t - \Phi(t)$ in Theorem 2.1, the conclusion follows immediately. □

Definition 2.5 Let X be a nonempty set, $S, T : X \to X$ be two self-maps and $\alpha : X \times X \times (0,\infty) \to (0,\infty)$ be a function. S and T are called generalized α-admissible, if for all $x, y \in X$, $t > 0$, $\alpha(Sx, Sy, t) \geq 1$ implies $\alpha(Tx, Ty, t) \geq 1$. α is called 1-transitive on X, if for all $t > 0$, $x_0, x_1, x_2 \in X$, $\alpha(x_0, x_1, t) \geq 1$, $\alpha(x_1, x_2, t) \geq 1$ implies $\alpha(x_0, x_2, t) \geq 1$.

Theorem 2.2 *Let $(X, \leq)$ be a partially ordered set and $(X, \mathcal{F}, \Delta)$ be a complete N.A Menger PM-space, Δ be a continuous t-norm and $\Delta = \Delta_M$, $\alpha : X \times X \times (0,\infty) \to (0,\infty)$ be a function, $S, T : X \to X$ be two self-maps and $T(X) \subset S(X)$. Suppose that for $x, y \in X$, Sx and Sy are comparable, we have*

$$\alpha(Sx,Sy,t)\left[\left(\frac{1}{F_{Tx,Ty}(\psi(ct))} - 1\right) - L\big(N_{\psi(t)}(Sx,Sy)\big)\right] \leq \Phi\big(M_{\psi(t)}(Sx,Sy)\big), \tag{2.17}$$

for all $t > 0$, such that $\min\{F_{Sx,Sy}(\psi(t)), F_{Sx,Tx}(\psi(t)), F_{Sy,Ty}(\psi(t)), F_{Sx,Ty}(\psi(t)), F_{Sy,Tx}(\psi(t))\} > 0$, *where $c \in (0,1)$, $\psi \in \Psi$, L and Φ are the same as the ones in Corollary* 2.1,

$$\begin{aligned} M_{\psi(t)}(Sx,Sy) = \max\biggl\{ &\frac{1}{F_{Sx,Sy}(\psi(t))} - 1, \frac{1}{F_{Sx,Tx}(\psi(t))} - 1, \frac{1}{F_{Sy,Ty}(\psi(t))} - 1, \\ &\frac{1}{2}\left[\frac{1}{F_{Sx,Ty}(\psi(t))} + \frac{1}{F_{Sy,Tx}(\psi(t))} - 2\right]\biggr\} \end{aligned}$$

and

$$N_{\psi(t)}(Sx,Sy) = \min\left\{\frac{1}{F_{Sx,Tx}(\psi(t))} - 1, \frac{1}{F_{Sx,Ty}(\psi(t))} - 1, \frac{1}{F_{Sy,Tx}(\psi(t))} - 1\right\}.$$

Also assume that the following conditions hold:

(i) *S and T are generalized α-admissible*;
(ii) *α and $\asymp$ are* 1*-transitive*;
(iii) *T is weakly comparable with respect to S*;
(iv) *there exists $x_0 \in A_1$ such that $Sx_0 \asymp Tx_0$ and $\alpha(Sx_0, Tx_0, t) \geq 1$ for all $t > 0$*;
(v) *if a sequence $\{y_n\} \subset Y$ satisfies $y_n \asymp y_{n+1}$, $\alpha(y_n, y_{n+1}, t) \geq 1$ for all $n \in N$ and $t > 0$, and $y_n \to y$ as $n \to \infty$, then $y_n \asymp y$ and $\alpha(y_n, y, t) \geq 1$ for n sufficiently large and for all $t > 0$.*

Then S and T have a coincidence point in X.

Proof Let $g : [0,1] \to [0,+\infty)$ be a function defined by $g(t) = \frac{1}{t} - 1$ for $t > 0$ and $g(0) = +\infty$. Then $g \in \Omega$. Since $\Delta = \Delta_M$, we have $g(\Delta(s,t)) \leq g(s) + g(t)$ for all $s, t \in [0,1]$. Hence,

$(X, \mathcal{F}, \Delta)$ is a N.A Menger PM-space of $(D)_g$. Let $\kappa : X \times X \times (0,\infty) \to [0,\infty)$ be a function and $\kappa(x,y,t) = \frac{1}{\alpha(x,y,t)}$ for all $x, y \in X$ and $t > 0$.

Since ψ and $F_{x,y}$ are both nondecreasing, we have $F_{Tx,Ty}(\psi(ct)) \le F_{Tx,Ty}(\psi(t))$. Hence, $\frac{1}{F_{Tx,Ty}(\psi(ct))} - 1 \ge \frac{1}{F_{Tx,Ty}(\psi(t))} - 1$. By the definition of g, we have

$$g\big(F_{Tx,Ty}\big(\psi(t)\big)\big) \le g\big(F_{Tx,Ty}\big(\psi(ct)\big)\big). \tag{2.18}$$

For $x, y \in X$, Sx and Sy are comparable, by (2.17) and (2.18), we get

$$g\big(F_{Tx,Ty}\big(\psi(t)\big)\big) \le \kappa\big(Sx, Sy, \psi(t)\big)\Phi\big(M_{\psi(t)}(Sx,Sy)\big) + L\big(N_{\psi(t)}(Sx,Sy)\big). \tag{2.19}$$

Since ψ is continuous and $\psi(t) \to +\infty$ as $t \to +\infty$, it follows from (2.19) that (2.16) holds. Also, S and T also satisfy (i)-(v) of Theorem 2.1.

Thus, all the conditions of Corollary 2.1 are satisfied. Therefore, S and T have a coincidence point in X. □

3 Coincidence point results in partially ordered metric spaces

In this section, we will apply the results obtained in Section 2 to establish the corresponding coincidence point theorems for generalized cyclic $(\kappa h, \varphi L)_S$-weak contractions in partially ordered metric spaces. We first introduce a new notion in metric spaces that we will use in Theorem 3.1.

Definition 3.1 Let S and T be two self-maps of a metric space (X,d), $\kappa : X \times X \to R^+$ be a function. S and T are called generalized κ-admissible, if for all $x, y \in X$, $\kappa(Sx, Sy) \le 1$ implies $\kappa(Tx, Ty) \le 1$. κ is called m-transitive on X, if $x_0, x_1, \dots, x_m, x_{m+1} \in X$, $\kappa(x_0, x_1) \le 1, \kappa(x_1, x_2) \le 1, \dots, \kappa(x_m, x_{m+1}) \le 1$ implies $\kappa(x_0, x_{m+1}) \le 1$.

Theorem 3.1 *Let $(X, d, \le)$ be an ordered complete metric space and $\kappa : X \times X \times (0,\infty) \to [0,\infty)$ be a function. Let m be a positive integer, $A_1, A_2, \dots, A_m$ be subsets of X, $Y = \bigcup_{i=1}^m A_i$, S and $T : Y \to Y$ be two self-maps, Y be a cyclic representation of Y with respect to S and T. Suppose that $A_{m+1} = A_1$, and for $k \in \{1, 2, \dots, m\}$ and for all $x, y \in X$, $Sx \in S(A_k)$ and $Sy \in S(A_{k+1})$ are comparable, we have*

$$h\big(d(Tx,Ty)\big) \le \kappa(Sx,Sy)\big[h\big(M(Sx,Sy)\big) - \varphi\big(M(Sx,Sy)\big)\big] + L\big(N(Sx,Sy)\big), \tag{3.1}$$

for all $t > 0$, where h is a continuous and nondecreasing linear function, $h(s) = 0$ if and only if $s = 0$, $\varphi, L : [0,\infty) \to [0,\infty)$ are two continuous functions such that $L(0) = 0$, $\varphi(t) = 0$ if and only if $t = 0$, $\frac{\varphi(s)}{t} \ge \varphi(\frac{s}{t})$ and $\frac{L(s)}{t} \le L(\frac{s}{t})$ for all $t > 0$, $\varphi(t) \le h(t)$ for all $t \in R^+$,

$$N(Sx,Sy) = \min\big\{d(Sx,Tx), d(Sx,Ty), d(Sy,Tx)\big\}$$

and

$$M(Sx,Sy) = \max\left\{d(Sx,Sy), d(Sx,Tx), d(Sy,Ty), \frac{1}{2}\big[d(Sx,Ty) + d(Sy,Tx)\big]\right\}.$$

Also, assume that the following conditions hold:

(i) *S and T are generalized κ-admissible*;
(ii) *κ and $\asymp$ are m-transitive*;
(iii) *T is weakly comparable with respect to S*;
(iv) *there exists $x_0 \in A_1$ such that $Sx_0 \asymp Tx_0$ and $\kappa(Sx_0, Tx_0) \leq 1$*;
(v) *if a sequence $\{y_n\} \subset Y$ satisfies $y_n \asymp y_{n+1}$, $\kappa(y_n, y_{n+1}) \leq 1$ for all $n \in N$, and $y_n \to y$ as $n \to \infty$, then $y_n \asymp y$ and $\kappa(y_n, y) \leq 1$ for n sufficiently large.*

Then S and T have a coincidence point in X, that is, there exists $x \in X$ such that $Sx = Tx$.

Proof Let $(X, \mathcal{F}, \Delta)$ be the induced PM-space, where $\mathcal{F}$ is defined by $\mathcal{F}(x,y)(t) = F_{x,y}(t) = \frac{t}{t+d(x,y)}$, for all $t > 0$ and $x, y \in X$.

In fact, for $0 < t_1 \leq t_2$, $F_{x,z}(\max\{t_1, t_2\}) = F_{x,z}(t_2) = \frac{t_2}{t_2+d(x,z)}$ and $\frac{F_{x,y}(t_1)F_{y,z}(t_2)}{F_{x,y}(t_1)+F_{y,z}(t_2)-F_{x,y}(t_1)F_{y,z}(t_2)} = \frac{t_1t_2}{t_1t_2+t_1d(y,z)+t_2d(x,y)}$, by $d(x,z) \leq d(x,y) + d(y,z)$, we have

$$\frac{t_1t_2 + t_1d(y,z) + t_2d(x,y)}{t_1t_2} = 1 + \frac{d(x,y)}{t_1} + \frac{d(y,z)}{t_2} \geq 1 + \frac{d(x,z)}{t_2} = \frac{t_2 + d(x,z)}{t_2},$$

which implies that (1.1) holds and $\Delta(s,t) \geq \frac{ts}{t+s-ts}$. Hence, by Example 1.1, we know that $(X, \mathcal{F}, \Delta)$ is a N.A Menger PM-space of $(D)_g$-type for $g \in \Omega$ defined by $g(t) = \frac{1}{t} - 1$ for $0 < t \leq 1$ and $g(0) = +\infty$. It is not difficult to prove that a sequence $\{x_n\}$ in (X, d) converges to a point $x^* \in X$ if and only $\{x_n\}$ in $(X, \mathcal{F}, \Delta)$ τ-converges to x^*. Since (X, d) is a complete metric space, $(X, \mathcal{F}, \Delta)$ is a τ-complete N.A Menger PM-space of type $(D)_g$.

For $x, y \in X$, $Sx \in S(A_i)$ and $Sy \in S(A_{i+1})$ are comparable, by (3.1) and the properties of h, φ, L, for $t > 0$, we have

$$\begin{aligned} h\left(\frac{d(Tx,Ty)}{t}\right) &= \frac{h(d(Tx,Ty))}{t} \\ &\leq \kappa(Sx,Sy)\left[\frac{h(M(Sx,Sy))}{t} - \frac{\varphi(M(Sx,Sy))}{t}\right] + \frac{L(N(Sx,Sy))}{t} \\ &\leq \kappa(Sx,Sy)\left[h\left(\frac{M(Sx,Sy)}{t}\right) - \varphi\left(\frac{M(Sx,Sy)}{t}\right)\right] + L\left(\frac{N(Sx,Sy)}{t}\right). \end{aligned} \tag{3.2}$$

Since $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for $t > 0$, we have $g(F_{x,y}(t)) = \frac{d(x,y)}{t}$ for $t > 0$. It follows from (3.2) that (2.1) holds. In fact, S and T also satisfy (i)-(v) of Theorem 2.1.

Thus, all the conditions of Theorem 2.1 are satisfied when $\phi(x) \equiv 0$. Therefore, the conclusion holds. □

4 An illustration

In this section, we give an example to demonstrate Theorem 2.1.

Example 4.1 Let $X = R^+$, $\Delta(s,t) = \frac{ts}{t+s-ts}$ for all $s, t \in [0,1]$, $g(t) = \frac{1}{t} - 1$ for all $0 < t \leq 1$ and $g(0) = +\infty$. Define $\mathcal{F} : X \times X \to \mathcal{D}^+$ by

$$\mathcal{F}(x,y)(t) = F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

for all $x, y \in X$. Then $(X, \mathcal{F}, \Delta)$ is a complete N.A Menger PM-space of $(D)_g$-type. Suppose that $A_1 = [0,4]$, $A_2 = [0,2]$, $A_3 = [0, \sqrt{2}]$, and $Y = \bigcup_{i=1}^{3} A_i = [0,4]$. Define $S, T : Y \to Y$ and

$\kappa : X \times X \times (0,\infty) \to (0,\infty)$ by

$$Sx = x, \qquad Tx = \begin{cases} \frac{x}{2}, & x \in [0,1), \\ \sqrt{x}, & x \in [1,4], \end{cases} \qquad \kappa(x,y,t) = \begin{cases} 1, & x,y \in [0,1) \text{ or } x,y \in [1,4], \\ 4, & \text{otherwise.} \end{cases}$$

Then S and T are generalized κ-admissible and Y is a cyclic representation of Y with respect to S and T, and T is weakly comparable with respect to S, and κ and $\asymp$ are 3-transitive.

Let $\phi : X \to [0,\infty)$, $\phi(x) \equiv 0$ for all $x \in X$, $\beta \equiv 1$, $L(t) = 0$, $\varphi(t) = \frac{t}{2}$, $h(t) = t$, for all $t \in [0,\infty)$. Now, we verify inequality (2.1) in Theorem 2.1. By the definitions of F, g, ϕ, h, φ, and L, we only need to prove that

$$\frac{|Tx - Ty|}{t} \le \frac{1}{2} \cdot \kappa(x,y,t) \cdot \max\left\{\frac{|x-y|}{t}, \frac{|Tx-x|}{t}, \frac{|Ty-y|}{t}, \frac{1}{2}\left[\frac{|Tx-y|}{t} + \frac{|Ty-x|}{t}\right]\right\},$$

for all $t > 0$, that is,

$$|Tx - Ty| \le \frac{1}{2} \cdot \kappa(x,y,t) \cdot \max\left\{|x-y|, |Tx-x|, |Ty-y|, \frac{1}{2}\big[|Tx-y| + |Ty-x|\big]\right\}, \qquad (4.1)$$

where $\kappa(x,y,t) = 1$ if $x,y \in [0,1)$ or $x,y \in [1,4]$, and $\kappa(x,y,t) = 4$ if otherwise.

We consider the following cases:

Case 1. If $x,y \in [0,1)$, then $\frac{1}{2} \cdot \kappa(x,y,t) = \frac{1}{2}$. By the definition of T, we have

$$|Tx - Ty| = \frac{1}{2}|x-y| \le \frac{1}{2} \cdot \kappa(x,y,t) \cdot |x-y|,$$

which implies that (4.1) holds.

Case 2. If $x \in [0,1)$ and $y \in [1,4]$, then $\frac{1}{2} \cdot \kappa(x,y,t) = 2$ and $x < \sqrt{y} \le y$. By the definition of T, we have

$$\begin{aligned} |Tx - Ty| &= \sqrt{y} - \frac{1}{2}x \le y - x + \sqrt{y} - \frac{1}{2}x = \left|y - \frac{1}{2}x\right| + |\sqrt{y} - x| \\ &= \frac{1}{2} \cdot \kappa(x,y,t) \cdot \left(\frac{1}{2}\big[|Tx-y| + |Ty-x|\big]\right), \end{aligned}$$

which implies that (4.1) holds. Similarly, if $x \in [1,4]$ and $y \in [0,1)$, we also have (4.1) holds.

Case 3. If $x,y \in [1,4]$, then $\frac{1}{2} \cdot \kappa(x,y,t) = \frac{1}{2}$ and $\sqrt{x} + \sqrt{y} \ge 2$. By the definition of T, we have

$$|Tx - Ty| = |\sqrt{x} - \sqrt{y}| \le \frac{\sqrt{x} + \sqrt{y}}{2}|\sqrt{x} - \sqrt{y}| = \frac{1}{2}|x-y|,$$

which implies that (4.1) holds.

Thus, all the conditions of Theorem 2.1 are satisfied. Therefore, S and T have a coincidence point in X, indeed, $x = 0$ and $x = 1$ are coincidence points of S and T.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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