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A new iteration scheme for a hybrid pair of generalized nonexpansive mappings

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Abstract
In this paper, we construct an iteration scheme involving a hybrid pair of nonexpansive mappings and utilize the same to prove some convergence theorems. In the process, we remove a restricted condition (called end-point condition) in Sokhuma and Kaewkhao's results (Fixed Point Theory Appl. 2010:618767, 2010). Thus, our results generalized and improved several results contained in Sokhuma and Kaewkhao (Fixed Point Theory Appl. 2010:618767, 2010), Akkasriworn *et al.* (Int. J. Math. Anal. 6(19):923-932, 2012), Uddin *et al.* (Bull. Malays. Math. Soc., accepted) and Sokhuma (J. Math. Anal. 4(2):23-31, 2013).
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1 Introduction

Let X be a Banach space and K be a nonempty subset of X. Let $CB(K)$ be the family of nonempty closed bounded subsets of K, while $C(K)$ be the family of nonempty compact convex subsets of K. A subset K of X is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = d(x,K) = \inf\{\|x-y\| : y \in K\}.$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal. We shall denote by $PB(K)$, the family of nonempty bounded proximinal subsets of K.

The Hausdorff metric H on $CB(K)$ is defined as

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\} \quad \text{for } A,B \in CB(K).$$

A mapping $f : K \to K$ is said to be nonexpansive if

$$\|fx - fy\| \le \|x-y\|, \quad \text{for all } x,y \in K,$$

while a multivalued mapping $T : K \to CB(K)$ is said to be nonexpansive if

$$H(Tx,Ty) \le \|x-y\|, \quad \text{for all } x,y \in K.$$

We use the notation $F(T)$ for the set of fixed points of the mapping T, while $F(f,T)$ denotes the set of common fixed points of f and T, *i.e.*, a point x is said to be a common fixed point of f and T if $fx = x \in Tx$.

On the other hand, in 2008, Suzuki [1] introduced a new class of mappings which is larger than the class of nonexpansive mappings and named the defining condition condition (C) and utilized them to prove some existence and convergence fixed point theorems.

It is well known that the sequence of the Picard iteration (*cf.* [2]) defined as (for any $x_1 \in K$)

$$x_{n+1} = f^n x, \quad n \in \mathbb{N}, \tag{1.1}$$

does not need to be convergent with respect to a nonexpansive mapping, *e.g.*, the sequence of iterates $x_{n+1} = fx_n$ for the mapping $f : [-1,1] \to [-1,1]$ defined by $fx = -x$ does not converge to 0 for any choice of non-zero initial point which is indeed the fixed point of f. In an attempt to construct a convergent sequence of iterates with respect to a nonexpansive mapping, Mann [3] defined an iteration method by (for any $x_1 \in K$)

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n f x_n, \quad n \in \mathbb{N}, \tag{1.2}$$

where $\alpha_n \in (0,1)$.

In 1974, with a view to approximate fixed point of pseudo-contractive compact mappings in Hilbert spaces Ishikawa [4] introduced a new iteration procedure as follows: (for $x_1 \in K$)

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n f x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n f y_n, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\alpha_n, \beta_n \in (0,1)$.

The study of fixed points for multivalued contractions as well as multivalued nonexpansive mappings was initiated by Nadler [5] and Markin [6] and by now there exists an extensive literature on multivalued fixed point theory which has applications in diverse areas, such as control theory, convex optimization, differential inclusion, and economics (see [7] and references cited therein). Moreover, the existence of fixed points for multivalued nonexpansive mappings in uniformly convex Banach spaces was proved by Lim [8]. In recent years, different iterative processes have been used to approximate the fixed points of multivalued nonexpansive mappings. Among these iterative procedures, iteration schemes due to Sastry and Babu [9], Panyanak [10], Song and Wang [11], and Shahzad and Zegeye [12] are notable generalizations of Mann and Ishikawa iteration process especially in the case of multivalued mappings. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings. For an almost up to date account of the literature on iterative fixed points, we refer the readers to Berinde [13].

Recently, Sokhuma and Kaewkhao [14] introduced the following modified Ishikawa iteration scheme for a pair of single valued and multivalued mapping.

Let K be a nonempty closed and bounded convex subset of Banach space X and let $f : K \to K$ be a single valued nonexpansive mapping and let $T : K \to CB(K)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = \beta_n z_n + (1-\beta_n)x_n, \\ x_{n+1} = \alpha_n f y_n + (1-\alpha_n)x_n, \end{cases} \tag{1.4}$$

where $x_0 \in K$, $z_n \in Tx_n$ and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

This scheme has been studied by several authors [14–18] with respect to various classes of mappings in different classes of spaces. All the authors proved their results with the end-point condition $Tw = w$ for all $w \in F(T)$, where T is multivalued mapping. With a motivation to remove this strong condition, in this paper we introduce a new iteration scheme for a pair of hybrid mapping and prove some convergence theorems for generalized nonexpansive mappings. In this way, we are not only able to remove a restricted condition but also to generalize the class of functions. In the process several relevant results, especially those contained in Sokhuma and Kaewkhao [14], Akkasriworn *et al.* [15], Uddin *et al.* [17], Sokhuma [16], and Sokhuma [18] are generalized and improved.

2 Preliminaries

With a view to make our presentation self contained, we collect some relevant basic definitions, results, and iterative methods, which will be used frequently in the text later.

In 2005, Sastry and Babu [9] defined Ishikawa iteration scheme for multivalued mappings. Let $T : K \to PB(K)$ be a multivalued mapping and $p \in F(T)$. Then the sequence of Ishikawa iteration is defined as follows:

Choose $x_0 \in K$,

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \quad \beta_n \in [0,1], n \geq 0,$$

where $z_n \in Tx_n$ such that $\|z_n - p\| = d(p, Tx_n)$ and

$$x_{n+1} = \alpha_n \acute{z}_n + (1-\alpha_n)x_n, \quad \alpha_n \in [0,1], n \geq 0,$$

where $\acute{z}_n \in Ty_n$ such that $\|\acute{z}_n - p\| = d(p, Ty_n)$.

Sastry and Babu [9] proved that the Ishikawa iteration scheme for a multivalued nonexpansive mapping T with a fixed point p converges to a fixed point q of T under certain conditions. In 2007, Panyanak [10] extended the results of Sastry and Babu to uniformly convex Banach space for multivalued nonexpansive mappings. Panyanak also modified the iteration scheme of Sastry and Babu and posed the question of convergence of this scheme. He introduced the following modified Ishikawa iteration method:

For $x_0 \in K$, write

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \quad \beta_n \in [a,b], 0 < a < b < 1, n \geq 0,$$

where $z_n \in Tx_n$ is such that $\|z_n - u_n\| = \operatorname{dist}(u_n, Tx_n)$, and $u_n \in F(T)$ such that $\|x_n - u_n\| = \operatorname{dist}(x_n, F(T))$, and

$$x_{n+1} = \alpha_n \acute{z}_n + (1-\alpha_n)x_n, \quad \alpha_n \in [a,b],$$

where $\acute{z}_n \in Ty_n$ such that $\|\acute{z}_n - v_n\| = \operatorname{dist}(v_n, Ty_n)$, and $v_n \in F(T)$ such that $\|y_n - v_n\| = \operatorname{dist}(y_n, F(T))$.

In 2009, Song and Wang [11] pointed out a gap in the result of Panyanak [10]. In an attempt to remove this gap, they gave a partial answer to the question raised by Panyanak by using the following iteration scheme.

Let $\alpha_n, \beta_n \in [0,1]$ and $\gamma_n \in (0,\infty)$ such that $\lim_{n\to\infty} \gamma_n = 0$ with $x_0 \in K$, write

$$y_n = \beta_n z_n + (1-\beta_n)x_n,$$
$$x_{n+1} = \alpha_n \acute{z}_n + (1-\alpha_n)x_n,$$

where $\|z_n - \acute{z}_n\| \leq H(Tx_n, Ty_n) + \gamma_n$ and $\|z_{n+1} - \acute{z}_n\| \leq H(Tx_{n+1}, Ty_n) + \gamma_n$ for $z_n \in Tx_n$ and $\acute{z}_n \in Ty_n$.

Simultaneously, Shahzad and Zegeye [12] extended the corresponding results of Sastry and Babu [9], Panyanak [10], and Song and Wang [11] to quasi-nonexpansive multivalued mappings and also relaxed the end-point condition and compactness of the domain by using the following modified iteration scheme and gave an affirmative answer to the Panyanak question in a more general setting wherein

$$y_n = \beta_n z_n + (1-\beta_n)x_n, \quad \beta_n \in [0,1], n \geq 0,$$
$$x_{n+1} = \alpha_n \acute{z}_n + (1-\alpha_n)x_n, \quad \alpha_n \in [0,1], n \geq 0,$$

where $z_n \in Tx_n$ and $\acute{z}_n \in Ty_n$.

Now, we collect some relevant definitions and results.

Definition 2.1 ([1]) A mapping f defined on a subset K of a Banach space X is said to satisfy condition (C) if (for all $x, y \in K$)

$$\frac{1}{2}\|x - fx\| \leq \|x - y\| \quad \Rightarrow \quad \|fx - fy\| \leq \|x - y\|.$$

Every nonexpansive mapping satisfies condition (C). If f satisfies condition (C) and has a fixed point, then f is a quasi-nonexpansive mapping. But the converse of the above statements does not need to be true in general. The following examples justify the converse fact.

Example 2.2 ([1]) Define a mapping f on $[0,3]$ by

$$fx = \begin{cases} 0, & \text{when } x \neq 3, \\ 1, & \text{when } x = 3. \end{cases}$$

Then f satisfies condition (C) but f is not a nonexpansive mapping.

Example 2.3 ([1]) Define a mapping f on $[0,3]$ by

$$fx = \begin{cases} 0, & \text{when } x \neq 3, \\ 2, & \text{when } x = 3. \end{cases}$$

Then $F(f) \neq \emptyset$ and f is a quasi-nonexpansive mapping but does not satisfy condition (C).

The following is a multivalued version of condition (C).

Definition 2.4 ([19]) Let T be a mapping defined on a subset K of a Banach space X. Then T is said to satisfy condition (C) if

$$\frac{1}{2}d(x, Tx) \leq \|x - y\| \quad \Rightarrow \quad H(Tx, Ty) \leq \|x - y\|$$

for all $x, y \in X$.

Similar to the case of a single valued mapping, every multivalued nonexpansive mapping satisfies condition (C). If T satisfies condition (C) and has a fixed point, then T is a quasi-nonexpansive mapping. But the converse of the above statements does not need to be true in general. The following examples justify the converse fact.

Example 2.5 ([19]) Define a mapping $T : [0,3] \to CB([0,3])$ by

$$Tx = \begin{cases} \{0\}, & \text{when } x \neq 3, \\ [0.5, 1], & \text{when } x = 3. \end{cases}$$

Then T satisfies condition (C) but T is not a nonexpansive mapping.

Example 2.6 ([19]) Define a mapping $T : [0,3] \to CB([0,3])$ by

$$Tx = \begin{cases} \{0\}, & \text{when } x \neq 3, \\ [1.5, 2], & \text{when } x = 3. \end{cases}$$

Then $F(T) \neq \emptyset$ and T is a quasi-nonexpansive mapping but it does not satisfy condition (C).

The following result is very important and will be used repeatedly.

Lemma 2.7 ([19]) *Let K be a subset of a Banach space X and $f : K \to K$ be a mapping which satisfies condition* (*C*), *then for all $x, y \in K$ the following holds*:

$$\|x - fy\| \leq 3\|x - fx\| + \|x - y\|.$$

García-Falset *et al.* [20] used the concept of strongly demiclosedness to prove a weaker version of the famous principle of demiclosedness for generalized nonexpansive mappings in which weak convergence has been replaced by strong convergence. The definition runs as follows.

Definition 2.8 ([20]) If $f : K \to K$ is a mapping, then $(I - f)$ is said to be strongly demiclosed at 0 if for every sequence $\{x_n\}$ in K strongly convergent to $z \in K$ and such that $x_n - fx_n \to 0$ we have $z = fz$.

Proposition 2.9 *Let K be a nonempty subset of a Banach space X. If $f : K \to K$ satisfies condition* (*C*), *then $(I - f)$ is strongly demiclosed at* 0.

Proof Let $\{x_n\}$ be a sequence such that $x_n \to z$ and $\lim_{n\to\infty} \|x_n - fx_n\| = 0$. In view of Lemma 2.7,

$$\|x_n - fz\| \leq 3\|x_n - fx_n\| + \|x_n - z\|.$$

On letting $n \to \infty$ we get $x_n \to fz$ and hence $fz = z$. □

Now, we list the following important property of a uniformly convex Banach space essentially due to Schu [21] and an important lemma due to Khaewkhao and Sokhuma [14].

Lemma 2.10 ([21]) *Let X be a uniformly convex Banach space, let $\{u_n\}$ be a sequence of real numbers such that $0 < b \leq u_n \leq c < 1$ for all $n \geq 1$, and let $\{x_n\}$ and $\{y_n\}$ be sequences in X such that $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|u_n x_n + (1-u_n)y_n\| = a$ for some $a > 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.11 ([14]) *Let X be a Banach space, and let K be a nonempty, closed, convex subset of X. Then*

$$d(y, Ty) \leq \|y - x\| + d(x, Tx) + H(Tx, Ty),$$

where $x, y \in K$ and T is a multivalued nonexpansive mapping from K into $CB(K)$.

The following very useful theorem is due to Song and Cho [22].

Lemma 2.12 *Let $T : K \to P(K)$ be a multivalued mapping and $P_T(x) = \{y \in Tx : \|x - y\| = d(x, Tx)\}$. Then the following are equivalent.*
(i) $x \in F(T)$,
(ii) $P_T(x) = \{x\}$,
(iii) $x \in F(P_T)$.
Moreover, $F(T) = F(P_T)$.

3 Main results

In this paper, we introduce the following iteration scheme: Let K be a nonempty closed, bounded, and convex subset of Banach space X and let $f : K \to K$ be a single valued nonexpansive mapping and let $T : K \to CB(K)$ be a multivalued nonexpansive mapping. The sequence $\{x_n\}$ of the modified Ishikawa iteration is defined by

$$\begin{cases} y_n = \alpha_n z_n + (1 - \alpha_n)x_n, \\ x_{n+1} = \beta_n f y_n + (1 - \beta_n)x_n, \end{cases} \tag{3.1}$$

where $x_0 \in K$, $z_n \in P_T x_n$, and $0 < a \leq \alpha_n, \beta_n \leq b < 1$.

Now, we start with the following lemma.

Lemma 3.1 *Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X which satisfies condition (C), and let $T : K \to P(K)$ be a multivalued mapping with $F(f, T) \neq \emptyset$ such that P_T enjoys condition (C). If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by (3.1), then $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(f, T)$.*

Proof Let $w \in F(f, T)$, in view of Lemma 2.12 we have

$$w \in P_T w = \{w\}.$$

Also,

$$\frac{1}{2}\|w - fw\| = 0 \leq \|fy_n - w\|,$$

owing to condition (C), we get

$$\|fy_n - fw\| \leq \|y_n - w\|.$$

Similarly, in view of $\frac{1}{2}d(w, Tw) = 0 \leq \|x_n - w\|$, we have $H(P_T x_n, P_T w) \leq \|x_n - w\|$.

Now, consider

$$\begin{aligned}\|x_{n+1} - w\| &= \left\|(1-\beta_n)x_n + \beta_n f y_n - w\right\| \\ &\leq (1-\beta_n)\|x_n - w\| + \beta_n\|fy_n - fw\| \\ &\leq (1-\beta_n)\|x_n - w\| + \beta_n\|y_n - w\|. \end{aligned} \tag{3.2}$$

But

$$\begin{aligned}\|y_n - w\| &= \left\|(1-\alpha_n)x_n + \alpha_n z_n - w\right\| \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|z_n - w\| \\ &= (1-\alpha_n)\|x_n - w\| + \alpha_n d(z_n, P_T w) \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n H(P_T x_n, P_T w) \\ &\leq (1-\alpha_n)\|x_n - w\| + \alpha_n\|x_n - w\| \\ &= \|x_n - w\|. \end{aligned} \tag{3.3}$$

In view of (3.2) and (3.3), we have

$$\|x_{n+1} - w\| \leq \|x_n - w\|, \tag{3.4}$$

which shows that $\{\|x_n - p\|\}$ is a decreasing sequence of non-negative reals. Thus in all, the sequence $\{\|x_n - p\|\}$ is bounded below and decreasing, therefore it remains convergent. □

Lemma 3.2 *Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X which satisfies condition* (*C*), *and let $T : K \to P(K)$ be a multivalued mapping with $F(f, T) \neq \emptyset$ such that P_T enjoys condition* (*C*). *If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by* (3.1), *then* $\lim_{n\to\infty} \|fy_n - x_n\| = 0$.

Proof In view of Lemma 3.1, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(f, T)$.

Write $\lim_{n\to\infty} \|x_n - w\| = c$.

Now, consider

$$\begin{aligned}\|fy_n - w\| &\le \|y_n - w\| \\ &\le \bigl\|(1-\alpha_n)x_n + \alpha_n z_n - w\bigr\| \\ &\le (1-\alpha_n)\|x_n - w\| + \alpha_n\|z_n - w\| \\ &= (1-\alpha_n)\|x_n - w\| + \alpha_n d(z_n, P_T w) \\ &\le (1-\alpha_n)\|x_n - w\| + \alpha_n H(P_T x_n, P_T w) \\ &\le (1-\alpha_n)\|x_n - w\| + \alpha_n\|x_n - w\| \\ &= \|x_n - w\|. \end{aligned} \tag{3.5}$$

On taking lim sup of both sides, we obtain

$$\limsup_{n\to\infty}\|fy_n - w\| \le c. \tag{3.6}$$

Also,

$$\begin{aligned} c &= \lim_{n\to\infty}\|x_{n+1} - w\| \\ &= \lim_{n\to\infty}\bigl\|(1-\beta_n)x_n + \beta_n fy_n - w\bigr\| \\ &= \lim_{n\to\infty}\bigl\|(1-\beta_n)(x_n - w) + \beta_n(fy_n - w)\bigr\|. \end{aligned} \tag{3.7}$$

In view of (3.5), (3.6), (3.7), and Lemma 2.12, we get

$$\lim_{n\to\infty}\bigl\|(fy_n - w) - (x_n - w)\bigr\| = \lim_{n\to\infty}\|fy_n - x_n\| = 0. \qquad \square$$

Lemma 3.3 *Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X which satisfies condition* (C) *and let $T : K \to P(K)$ be a multivalued mapping with $F(f,T) \ne \emptyset$ such that P_T enjoys condition* (C). *If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by* (3.1), *then* $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

Proof Let $w \in F(f,T)$ and $\{x_n\}$ be the sequence described by (3.1). Then, in view of Lemma 2.12, we have

$$w \in P_T(w) = \{w\}.$$

Now, consider

$$\begin{aligned}\|x_{n+1} - w\| &= \bigl\|(1-\beta_n)x_n + \beta_n fy_n - w\bigr\| \\ &\le (1-\beta_n)\|x_n - w\| + \beta_n\|fy_n - fw\| \\ &\le (1-\beta_n)\|x_n - w\| + \beta_n\|y_n - w\|, \end{aligned} \tag{3.8}$$

so that

$$\begin{aligned}&\|x_{n+1}-w\|-\|x_n-w\|\le\beta_n\bigl(\|y_n-w\|-\|x_n-w\|\bigr),\quad\text{or}\\&\frac{\|x_{n+1}-w\|-\|x_n-w\|}{\beta_n}\le\|y_n-w\|-\|x_n-w\|.\end{aligned}$$

Since $0<a\le\beta_n\le b<1$, we have

$$\liminf_{n\to\infty}\left\{\left(\frac{\|x_{n+1}-w\|-\|x_n-w\|}{\beta_n}\right)+\|x_n-w\|\right\}\le\liminf_{n\to\infty}\|y_n-w\|.$$

It follows that

$$c\le\liminf_{n\to\infty}\|y_n-w\|.$$

Owing to (3.3) $\limsup_{n\to\infty}\|y_n-w\|\le c$ so that

$$\begin{aligned}c&=\lim_{n\to\infty}\|y_n-w\|\\&=\lim_{n\to\infty}\bigl\|(1-\alpha_n)x_n+\alpha_nz_n-w\bigr\|\\&=\lim_{n\to\infty}\bigl\|(1-\alpha_n)(x_n-w)+\alpha_n(z_n-w)\bigr\|.\end{aligned}\tag{3.9}$$

As $\|z_n-w\|=d(z_n,P_Tw)\le H(P_Tx_n,P_Tw)\le\|x_n-w\|$, we have

$$\limsup_{n\to\infty}\|z_n-w\|\le\limsup_{n\to\infty}\|x_n-w\|=c.\tag{3.10}$$

Owing to Lemma 2.12, (3.9), and (3.10) we obtain $\lim_{n\to\infty}\|x_n-z_n\|=0$. □

Lemma 3.4 *Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X which satisfies condition* (*C*) *and let $T:K\to P(K)$ be a multivalued mapping with $F(f,T)\neq\emptyset$ such that P_T enjoys condition* (*C*). *If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by* (3.1), *then* $\lim_{n\to\infty}\|fx_n-x_n\|=0$.

Proof Owing to Lemma 2.7, we get

$$\begin{aligned}\|fx_n-x_n\|&=\|fx_n-y_n+y_n-x_n\|\\&\le\|fx_n-y_n\|+\|y_n-x_n\|\\&\le3\|fy_n-y_n\|+\|y_n-x_n\|+\|y_n-x_n\|\\&\le3\|fy_n-x_n\|+5\|y_n-x_n\|\\&=3\|fy_n-x_n\|+5\bigl\|\alpha_nz_n+(1-\alpha_n)x_n-x_n\bigr\|\\&\le3\|fy_n-x_n\|+5\alpha_n\|z_n-x_n\|;\end{aligned}$$

therefore,

$$\lim_{n\to\infty}\|fx_n-x_n\|\le\lim_{n\to\infty}5\alpha_n\|x_n-z_n\|+\lim_{n\to\infty}3\|fy_n-x_n\|.$$

On using Lemma 3.2 and Lemma 3.3, we get

$$\lim_{n\to\infty} \|fx_n - x_n\| = 0. \qquad \square$$

Theorem 3.5 *Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X which satisfies condition* (C) *and let $T : K \to P(K)$ be a multivalued mapping with $F(f,T) \neq \emptyset$ such that P_T enjoys condition* (C). *If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by* (3.1), *then $\{x_{n_i}\} \to y$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ implies $y \in F(f,T)$.*

Proof Assume that $\lim_{i\to\infty} \|x_{n_i} - y\| = 0$. By Lemma 3.4, we obtain $0 = \lim_{i\to\infty} \|fx_{n_i} - x_{n_i}\| = \lim_{i\to\infty} \|(I-f)(x_{n_i})\|$. Since $(I-f)$ is strongly demiclosed at 0 so that we have $(I-f)(y) = 0$. Thus $y = fy$, *i.e.*, $y \in F(f)$. By Lemma 2.11, we have

$$\begin{aligned} d(y, P_T y) &\le \|y - x_{n_i}\| + d(x_{n_i}, P_T x_{n_i}) + H(P_T x_{n_i}, P_T y) \\ &\le \|y - x_{n_i}\| + \|x_{n_i} - z_{n_i}\| + \|x_{n_i} - y\| \to 0 \end{aligned}$$

as $i \to \infty$. It follows that $y \in F(P_T) = F(T)$. Thus $y \in F(f,T)$. $\square$

Theorem 3.6 *Let f be a self-mapping of a nonempty compact convex subset K of a uniformly convex Banach space X which satisfies condition* (C) *and let $T : K \to P(K)$ be a multivalued mapping with $F(f,T) \neq \emptyset$ such that P_T enjoys condition* (C). *If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by* (3.1), *then $\{x_n\}$ converges strongly to a common fixed point of f and T.*

Proof Since $\{x_n\}$ is contained in a compact subset K, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $y \in K$, that is, $\lim_{i\to\infty} \|x_{n_i} - y\| = 0$. Now, in view of Theorem 3.5, $y \in F(f,T)$, while owing to Lemma 3.1 $\lim_{n\to\infty} \|x_n - y\|$ exists. Thus, in all, $\lim_{n\to\infty} \|x_n - y\| = \lim_{i\to\infty} \|x_{n_i} - y\| = 0$, so that $\{x_n\}$ converges strongly to $y \in F(f,T)$. $\square$

Khan and Fukhar-ud-din [23] introduced the so-called condition (A') for two mappings and gave an improved version in [24] of condition (I) of Senter and Dotson [25]. A hybrid version of condition (A'), involving a pair of single valued and multivalued mappings, which is weaker than compactness of the domain, is given as follows:

A pair of a single valued mapping $f : K \to K$ and a multivalued mapping $T : K \to CB(K)$ is said to satisfy condition (A') if there exists a nondecreasing function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$, $g(r) > 0$ for all $r \in (0,\infty)$ such that either $d(x,fx) \ge g(d(x,F(f,T)))$ or $d(x,Tx) \ge g(d(x,F(f,T)))$ for all $x \in K$.

Theorem 3.7 *Let f be a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space X which satisfies condition* (C), *and let $T : K \to P(K)$ be a multivalued mapping with $F(f,T) \neq \emptyset$ such that P_T enjoys condition* (C). *If $\{x_n\}$ is the sequence of the modified Ishikawa iteration defined by* (3.1) *and the pair (f,T) satisfies condition* (A'), *then $\{x_n\}$ converges strongly to a common fixed point of f and T.*

Proof Firstly, we show that $F(f,T)$ is closed. Let $\{x_n\}$ be a sequence in $F(f,T)$ converging to some point $z \in K$. We have

$$\begin{aligned}\|x_n - fz\| &= \|fx_n - fz\| \\ &\leq \|x_n - z\|,\end{aligned}$$

so that

$$\limsup_n \|x_n - fz\| \leq \limsup \|x_n - z\| = 0.$$

Owing to the uniqueness of the limit, we have $fz = z$. Also,

$$d(x_n, P_T z) \leq H(P_T x_n, P_T z) \leq \|x_n - z\| \to 0 \quad \text{as } n \to \infty.$$

This implies that $\{x_n\}$ converges to some point of $P_T z$ and hence $z \in F(P_T) = F(T)$.

By Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(f,T)$ and let us take it to be c. If $c = 0$, then there is nothing to prove. If $c > 0$, then in view of (3.4) for all $p \in F(f,T)$, we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|,$$

so that

$$\inf_{p\in F(f,T)} \|x_{n+1} - p\| \leq \inf_{p\in F(f,T)} \|x_n - p\|,$$

which amounts to saying that

$$d\big(x_{n+1}, F(f,T)\big) \leq d\big(x_n, F(f,T)\big),$$

and hence $\lim_{n\to\infty} d(x_n, F(f,T))$ exists. Owing to condition (A') there exists a nondecreasing function g such that

$$\lim_{n\to\infty} g\big(d\big(x_n, F(f,T)\big)\big) \leq \lim_{n\to\infty} \|x_n - fx_n\| = 0$$

or

$$\lim_{n\to\infty} g\big(d\big(x_n, F(f,T)\big)\big) \leq \lim_{n\to\infty} d(x_n, P_T x_n) \leq \lim_{n\to\infty} \|x_n - z_n\| = 0,$$

so that in both cases $\lim_{n\to\infty} g(d(x_n, F(f,T))) = 0$. Since g is a nondecreasing function and $g(0) = 0$, we have $\lim_{n\to\infty} d(x_n, F(f,T)) = 0$.

This implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\|x_{n_k} - p_k\| \leq \frac{1}{2^k} \quad \text{for all } k \geq 1$$

wherein $\{p_k\}$ is in $F(f,T)$. By Lemma 3.1, we have

$$\|x_{n_{k+1}} - p_k\| \leq \|x_{n_k} - p_k\| \leq \frac{1}{2^k},$$

so that

$$\begin{aligned}\|p_{k+1} - p_k\| &\le \|p_{k+1} - x_{n_{k+1}}\| + \|x_{n_{k+1}} - p_k\| \\ &\le \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}},\end{aligned}$$

which implies that $\{p_k\}$ is a Cauchy sequence. Since $F(f, T)$ is closed, therefore $\{p_k\}$ is a convergent sequence. Write $\lim_{k\to\infty} p_k = p$. Now, in order to show that $\{x_n\}$ converges to p let us proceed as follows:

$$\|x_{n_k} - p\| \le \|x_{n_k} - p_k\| + \|p_k - p\| \to 0 \quad \text{as } k \to \infty,$$

so that $\lim_{k\to\infty} \|x_{n_k} - p\| = 0$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, we have $x_n \to p$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed in this work on an equal basis. All authors read and approved the final manuscript.

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