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Strong convergence theorems for equilibrium problems involving a family of nonexpansive mappings

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Abstract

We give new hybrid variants of extragradient methods for finding a common solution of an equilibrium problem and a family of nonexpansive mappings. We present a scheme that combines the idea of an extragradient method and a successive iteration method as a hybrid variant. Then, this scheme is modified by projecting on a suitable convex set to get a better convergence property under certain assumptions in a real Hilbert space.

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1 Introduction

In this paper, we always assume that $\mathcal{H}$ is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset of $\mathcal{H}$ and the bifunction $f : C \times C \to \mathcal{R}$. Then f is called *strongly monotone* on C with $\beta > 0$ iff

$$f(x,y) + f(y,x) \leq -\beta \|x-y\|^2 \quad \forall x,y \in C;$$

monotone on C iff

$$f(x,y) + f(y,x) \leq 0 \quad \forall x,y \in C;$$

pseudomonotone on C iff

$$f(x,y) \geq 0 \quad \text{implies} \quad f(y,x) \leq 0 \quad \forall x,y \in C;$$

Lipschitz-type continuous on C in the sense of Mastroeni [1] iff there exist positive constants $c_1 > 0$, $c_2 > 0$ such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1 \|x-y\|^2 - c_2 \|y-z\|^2 \quad \forall x,y,z \in C.$$

An equilibrium problem, shortly $EP(f,C)$, is to find a point in

$$\mathrm{Sol}(f,C) = \{x^* \in C : f(x^*,y) \geq 0 \ \forall y \in C\}.$$

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Let a mapping T of C into itself. Then T is called *contractive* with constant $\delta \in (0,1)$ iff

$$\|T(x) - T(y)\| \leq \delta \|x - y\| \quad \forall x, y \in C.$$

The mapping T is called *strictly pseudocontractive* iff there exists a constant $k \in [0,1)$ such that

$$\|T(x) - T(y)\|^2 \leq \|x - y\|^2 + k\|(I - T)(x) - (I - T)(y)\|^2.$$

In the case $k = 0$, the mapping T is called *nonexpansive* on C. We denote by $\mathrm{Fix}(T)$ the set of fixed points of T.

Let $T_i : C \to C$, $i \in \Gamma$, be a family of nonexpansive mappings where Γ stands for an index set. In this paper, we are interested in the problem of finding a common element of the solution set of problem $EP(f, C)$ and the set of fixed points $F = \bigcap_{i \in \Gamma} \mathrm{Fix}(T_i)$, namely:

$$\text{Find } x^* \in F \cap \mathrm{Sol}(f, C), \tag{1.1}$$

where the function f and the mappings T_i, $i \in \Gamma$, satisfy the following conditions:

(A_1) $f(x, x) = 0$ for all $x \in C$ and f is pseudomonotone on C,
(A_2) f is Lipschitz-type continuous on C with constants $c_1 > 0$ and $c_2 > 0$,
(A_3) f is upper semicontinuous on C,
(A_4) For each $x \in C$, $f(x, \cdot)$ is convex and subdifferentiable on C,
(A_5) $F \cap \mathrm{Sol}(f, C) \neq \emptyset$.

Under these assumptions, for each $r > 0$ and $x \in C$, there exists a unique element $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \quad \forall y \in C. \tag{1.2}$$

Problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, equilibrium equilibriums, fixed point problems (see, *e.g.*, [2–7]). Recently, it has become an attractive field for many researchers in both theory and its solution methods (see, *e.g.*, [3, 4, 8–12] and the references therein). Most of these algorithms are based on inequality (1.2) for solving the underlying equilibrium problem when $F \cap \mathrm{Sol}(f, C) \neq \emptyset$. Motivated by this idea for finding a common point of $\mathrm{Sol}(f, C)$ and the fixed point set $\mathrm{Fix}(T)$ of a nonexpansive mapping T, Takahashi and Takahashi [13] first introduced an iterative scheme by the viscosity approximation method. The sequence $\{x^n\}$ is defined by

$$\begin{cases} x^0 \in C, \\ f(u^n, y) + \frac{1}{r_n}\langle y - u^n, u^n - x^n\rangle \geq 0 \quad \forall y \in C, \\ x^{n+1} = \alpha_n g(x^n) + (1 - \alpha_n)T(u^n) \quad \forall n \geq 0, \end{cases}$$

where $g : C \to C$ is contractive. Under certain conditions over the parameters $\{\alpha_n\}$ and $\{r_n\}$, they showed that the sequences $\{x^n\}$ and $\{u^n\}$ strongly converge to $z = \mathrm{Pr}_{\mathrm{Fix}(T) \cap \mathrm{Sol}(f,C)}\, g(z)$, where Pr_C denotes the projection on C. At each iteration n in all of these algorithms, it requires to solve approximation auxiliary equilibrium problems for finding a common solution of an equilibrium problem and a fixed point problem. In order to avoid this requirement, Anh [14] recently proposed a hybrid extragradient algorithm for finding a common point of the set $\mathrm{Fix}(T) \cap \mathrm{Sol}(f,C)$. Starting with an arbitrary initial point $x^0 \in C$, iteration sequences are defined by

$$\begin{cases} y^k = \mathrm{argmin}\{\lambda_k f(x^k, y) + \frac{1}{2}\|y - x^k\|^2 : y \in C\}, \\ t^k = \mathrm{argmin}\{\lambda_k f(y^k, t) + \frac{1}{2}\|t - x^k\|^2 : t \in C\}, \\ x^{k+1} = \alpha_k x^0 + (1 - \alpha_k) T(x^k). \end{cases} \tag{1.3}$$

Under certain conditions onto parameters $\{\lambda_k\}$ and $\{\alpha_k\}$, he showed that the sequences $\{x^k\}$, $\{y^k\}$ and $\{t^k\}$ weakly converge to the point $x \in \mathrm{Fix}(T) \cap \mathrm{Sol}(f,C)$ in a real Hilbert space. At each main iteration n of the scheme, he only solved strongly convex problems on C, but the proof of convergence was still done under the assumptions that $x^{n+1} - x^n \to 0$.

For finding a common point of a family of nonexpansive mappings T_i ($i \in \Gamma$), as a corollary of Theorem 2.1 in [15], Zhou proposed the following iteration scheme:

$$\begin{cases} x^0 \in \mathcal{H} \text{ chosen arbitrarily}, \\ C_{1,i} = C, C_1 = \bigcap_{i\in\Gamma} C_{1,i}, \\ x^1 = \mathrm{Pr}_{C_1}(x^0), \\ y^{n,i} = (1 - \alpha_{n,i})x^n + \alpha_{n,i} T_i(x^n), \\ C_{n+1,i} = \{z \in C_{n,i} : \alpha_{n,i}(1 - 2\alpha_{n,i})\|x^n - T_i(x^n)\|^2 \le \langle x^n - z, y^{n,i} - T_i(y^{n,i})\rangle\}, \\ C_{n+1} = \bigcap_{i\in\Gamma} C_{n+1,i}, \\ x^{n+1} = \mathrm{Pr}_{C_{n+1}}(x^0). \end{cases} \tag{1.4}$$

Under the restrictions of the control sequences $0 < \liminf_{n\to\infty} \alpha_{n,i} \le \limsup_{n\to\infty} \alpha_{n,i} \le a_i < \frac{1}{2}$, he showed that the sequence $\{x^n\}$ defined by (1.4) strongly converges to $x^* = \mathrm{Pr}_F(x^0)$ in a real Hilbert space $\mathcal{H}$, where $F = \bigcap_{i\in\Gamma} \mathrm{Fix}(T_i)$.

In this paper, motivated by Ceng *et al.* [16, 17], Wang and Guo [18], Zhou [15], Nadezhkina and Takahashi [10], Cho *et al.* [19], Takahashi and Takahashi [13], Anh [6, 12] and Anh *et al.* [20, 21], we introduce several modified hybrid extragradient schemes to modify the iteration schemes (1.3) and (1.4) to obtain new strong convergence theorems for a family of nonexpansive mappings and the equilibrium problem $EP(f,C)$ in the framework of a real Hilbert space $\mathcal{H}$.

To investigate the convergence of this scheme, we recall the following technical lemmas which will be used in the sequel.

Lemma 1.1 ([14], Lemma 3.1) *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $f : C \times C \to \mathcal{R}$ be a pseudomonotone and Lipschitz-type continuous bifunction. For each $x \in C$, let $f(x,\cdot)$ be convex and subdifferentiable on C. Suppose that the sequences $\{x^n\}$, $\{y^n\}$, $\{t^n\}$ are generated by scheme* (1.3) *and $x^* \in \mathrm{Sol}(f,C)$. Then*

$$\|t^n - x^*\|^2 \le \|x^n - x^*\|^2 - (1 - 2\lambda_n c_1)\|x^n - y^n\|^2 - (1 - 2\lambda_n c_2)\|y^n - t^n\|^2 \quad \forall n \ge 0.$$

Lemma 1.2 *Let C be a closed convex subset of a real Hilbert space $\mathcal{H}$, and let Pr_C be the metric projection from $\mathcal{H}$ on to C (i.e., for $x \in \mathcal{H}$, Pr_C is the only point in C such that $\|x - \mathrm{Pr}_C x\| = \inf\{\|x - z\| : z \in C\}$). Given $x \in \mathcal{H}$ and $z \in C$. Then $z = \mathrm{Pr}_C x$ if only if there holds the relation $\langle x - z, y - z\rangle \leq 0$ for all $y \in C$.*

Lemma 1.3 *Let $\mathcal{H}$ be a real Hilbert space. Then the following equations hold:*

(i) $\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ *for all* $x, y \in \mathcal{H}$.

(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2$ *for all* $t \in [0, 1]$ *and* $x, y \in \mathcal{H}$.

2 Convergence theorems

Now, we prove the main convergence theorem.

Theorem 2.1 *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Suppose that assumptions (A_1)-(A_5) are satisfied and $\{T_i\}_{i\in\Gamma}$ is a family of nonexpansive mappings from C into itself and a nonempty common fixed points set F. Let $\{x^n\}$ be a sequence generated by the following scheme:*

$$\begin{cases} x^0 \in \mathcal{H} \text{ chosen arbitrarily}, \\ C_{1,i} = D_{1,i} = C, C_1 = \bigcap_{i\in\Gamma} C_{1,i}, D_1 = \bigcap_{i\in\Gamma} D_{1,i}, \\ x^1 = \mathrm{Pr}_{C_1\cap D_1} x^0, \\ y^n = \operatorname{argmin}\{\lambda_n f(x^n, y) + \frac{1}{2}\|y - x^n\|^2 : y \in C\}, \\ z^n = \operatorname{argmin}\{\lambda_n f(y^n, y) + \frac{1}{2}\|z - x^n\|^2 : z \in C\}, \\ y^{n,i} = (1 - \alpha_{n,i})z^n + \alpha_{n,i} T_i z^n, \\ C_{n+1,i} = \{z \in C_{n,i} : \alpha_{n,i}(1 - 2\alpha_{n,i})\|z^n - T_i z^n\|^2 \leq \langle z^n - z, y^{n,i} - T_i y^{n,i}\rangle\}, \\ C_{n+1} = \bigcap_{i\in\Gamma} C_{n+1,i}, \\ D_{n+1,i} = \{z \in D_{n,i} : \|y^{n,i} - z\| \leq \|x^n - z\|\}, \\ D_{n+1} = \bigcap_{i\in\Gamma} D_{n+1,i}, \\ x^{n+1} = \mathrm{Pr}_{C_{n+1}\cap D_{n+1}} x^0, \\ 0 < \liminf \alpha_{n,i} \leq \limsup \alpha_{n,i} < 1, \\ \{\lambda_n\} \subset [a, b] \text{ for some } a, b \in (0, \frac{1}{L}), \text{ where } L = \max\{2c_1, 2c_2\}. \end{cases}$$

Then the sequences $\{x^n\}$, $\{y^n\}$ and $\{z^n\}$ strongly converge to the same point $\mathrm{Pr}_{F\cap \mathrm{Sol}(f,C)} x^0$.

Proof The proof of this theorem is divided into several steps.

Step 1. Claim that C_n and D_n are closed and convex for all $n \geq 0$.

We have to show that for any fixed point but arbitrary $i \in \Gamma$, $C_{n,i}$ is closed and convex for every $n \geq 0$. This can be proved by induction on n. It is obvious that $C_{1,i} = C$ is closed and convex. Assume that $C_{n,i}$ is closed and convex for some $n \in \mathcal{N}^* = \{1, 2, \ldots\}$. We have that the set

$$A = \{z \in C : \alpha_{n,i}(1 - 2\alpha_{n,i})\|z^n - T_i z^n\|^2 \leq \langle z^n - z, y^{n,i} - T_i y^{n,i}\rangle\}$$

is closed and convex, and $C_{n+1,i} = C_{n,i} \cap A$, hence $C_{n+1,i}$ is closed and convex. Then C_n is closed and convex for all $n \geq 0$. We can write $D_{n+1,i}$ under the form

$$D_{n+1,i} = \{z \in D_{n,i} : \|y^{n,i} - x^n\|^2 + 2\langle y^{n,i} - x^n, x^n - z\rangle \leq 0\}.$$

Then $D_{n+1,i}$ is closed and convex. Thus, D_n is closed and convex.

Step 2. Claim that $F \cap \mathrm{Sol}(f, C) \subseteq C_n \cap D_n$ for all $n \in \mathcal{N}^*$.

First, we show that $F \subseteq C_n$ by induction on n. It suffices to show that $F \subseteq C_{n,i}$.

We have $F \subseteq C = C_{1,i}$ is obvious. Suppose $F \subseteq C_{n,i}$ for some $n \in \mathcal{N}$. We have to show that $F \subseteq C_{n+1,i}$. Indeed, let $w \in F$, by inductive hypothesis, we have $w \in C_{n,i}$ and

$$\begin{aligned}
\|z^n - T_iz^n\|^2 &= \langle z^n - T_iz^n, z^n - T_iz^n\rangle \\
&= \frac{1}{\alpha_{n,i}}\langle z^n - y^{n,i}, z^n - T_iz^n\rangle \\
&= \frac{1}{\alpha_{n,i}}\langle z^n - y^{n,i}, z^n - T_iz^n - (y^{n,i} - T_iy^{n,i})\rangle + \frac{1}{\alpha_{n,i}}\langle z^n - y^{n,i}, y^{n,i} - T_iy^{n,i}\rangle \\
&= \frac{1}{\alpha_{n,i}}\langle z^n - y^{n,i}, z^n - T_iz^n - (y^{n,i} - T_iy^{n,i})\rangle \\
&\quad + \frac{1}{\alpha_{n,i}}\langle z^n - w + w - y^{n,i}, y^{n,i} - T_iy^{n,i}\rangle \\
&= \frac{1}{\alpha_{n,i}}\langle z^n - y^{n,i}, z^n - y^{n,i}\rangle + \frac{1}{\alpha_{n,i}}\langle z^n - y^{n,i}, T_iy^{n,i} - T_iz^n\rangle \\
&\quad + \frac{1}{\alpha_{n,i}}\langle z^n - w, y^{n,i} - T_iy^{n,i}\rangle + \frac{1}{\alpha_{n,i}}\langle w - y^{n,i}, y^{n,i} - T_iy^{n,i}\rangle \\
&\leq \frac{2}{\alpha_{n,i}}\|z^n - y^{n,i}\|^2 + \frac{1}{\alpha_{n,i}}\langle z^n - w, y^{n,i} - T_iy^{n,i}\rangle \\
&\quad + \frac{1}{\alpha_{n,i}}\langle w - y^{n,i}, y^{n,i} - T_iy^{n,i}\rangle. \qquad (2.1)
\end{aligned}$$

On the other hand, for all $w \in F$ and $y^{n,i} \in C$, we have

$$\begin{aligned}
\|w - y^{n,i}\|^2 &\geq \langle T_iw - T_iy^{n,i}, w - y^{n,i}\rangle \\
&= \langle w - T_iy^{n,i}, w - y^{n,i}\rangle \\
&= \langle w - y^{n,i} + y^{n,i} - T_iy^{n,i}, w - y^{n,i}\rangle \\
&= \|w - y^{n,i}\|^2 + \langle y^{n,i} - T_iy^{n,i}, w - y^{n,i}\rangle,
\end{aligned}$$

and hence

$$\langle w - y^{n,i}, y^{n,i} - T_iy^{n,i}\rangle \leq 0.$$

Combining this with (2.1), we obtain

$$\begin{aligned}
\|z^n - T_iz^n\|^2 &\leq \frac{2}{\alpha_{n,i}}\|z^n - y^{n,i}\|^2 + \frac{1}{\alpha_{n,i}}\langle z^n - w, y^{n,i} - T_iy^{n,i}\rangle \\
&\leq 2\alpha_{n,i}\|z^n - T_iz^n\|^2 + \frac{1}{\alpha_{n,i}}\langle z^n - w, y^{n,i} - T_iy^{n,i}\rangle.
\end{aligned}$$

This follows that

$$\alpha_{n,i}(1-2\alpha_{n,i})\|z^n - T_i z^n\|^2 \le \langle z^n - w, y^{n,i} - T_i y^{n,i}\rangle.$$

By the definition of $C_{n+1,i}$, we have $w \in C_{n+1,i}$, and so $F \subseteq C_{n+1,i}$ for all $i \in \Gamma$, which deduces that $F \subseteq C_n$. This shows that $F \cap \mathrm{Sol}(f,C) \subseteq C_n$ for all $n \in \mathcal{N}^*$.

Next, we will prove $F \cap \mathrm{Sol}(f,C) \subseteq D_n$ by induction on $n \in \mathcal{N}^*$. It suffices to show that $F \cap \mathrm{Sol}(f,C) \subseteq D_{n,i}$. Indeed, $F \subseteq C = D_{1,i}$ so $F \cap \mathrm{Sol}(f,C) \subseteq D_{1,i}$. Suppose that $F \cap \mathrm{Sol}(f,C) \subseteq D_{n,i}$. Let $x^* \in F \cap \mathrm{Sol}(f,C)$, then $x^* \in D_{n,i}$. Using Lemma 1.1, we get

$$\begin{aligned}\|y^{n,i} - x^*\|^2 &= \|(1-\alpha_{n,i})z^n + \alpha_{n,i}T_i z^n - x^*\|^2 \\ &\le (1-\alpha_{n,i})\|z^n - x^*\|^2 + \alpha_{n,i}\|T_i z^n - T_i x^*\|^2 \\ &\le \|z^n - x^*\|^2 \\ &\le \|x^n - x^*\|^2 - (1-2\lambda_n c_1)\|x^n - y^n\|^2 - (1-2\lambda_n c_2)\|y^n - z^n\|^2 \\ &\le \|x^n - x^*\|^2. \end{aligned} \tag{2.2}$$

Then we have $x^* \in D_{n+1,i}$ and hence $F \cap \mathrm{Sol}(f,C) \subseteq D_{n+1,i}$. This shows that $F \cap \mathrm{Sol}(f,C) \subseteq D_n$, which yields that $F \cap \mathrm{Sol}(f,C) \subseteq C_n \cap D_n$ for all $n \in \mathcal{N}^*$.

Step 3. Claim that the sequence $\{x^n\}$ is bounded and there exists the limit $\lim_{n\to\infty}\|x^n - x^0\| = c$.

From $x^n = \mathrm{Pr}_{C_n\cap D_n} x^0$, it follows that

$$\langle x^0 - x^n, x^n - y\rangle \ge 0 \quad \forall y \in C_n \cap D_n. \tag{2.3}$$

Then, using Step 2, we have $F \cap \mathrm{Sol}(f,C) \subseteq C_n \cap D_n$ and

$$\langle x^0 - x^n, x^n - w\rangle \ge 0 \quad \forall w \in F \cap \mathrm{Sol}(f,C). \tag{2.4}$$

Combining this and assumption (A_5), the projection $\mathrm{Pr}_{F\cap\mathrm{Sol}(f,C)} x^0$ is well defined and there exits a unique point p such that $p = \mathrm{Pr}_{F\cap\mathrm{Sol}(f,C)} x^0$. So, we have

$$\begin{aligned}0 &\le \langle x^0 - x^n, x^n - p\rangle = \langle x^0 - x^n, x^n - x^0 + x^0 - p\rangle \\ &\le -\|x^0 - x^n\|^2 + \|x^0 - x^n\|\|x^0 - p\|,\end{aligned}$$

and hence

$$\|x^0 - x^n\| \le \|x^0 - p\|.$$

Then the sequence $\{x^n\}$ is bounded. So, the sequences $\{y^n\}$, $\{z^n\}$, $\{y^{n,i}\}$, $\{T_i y^{n,i}\}$ also are bounded. Since $x^{n+1} \in C_{n+1} \cap D_{n+1} \subset C_n \cap D_n$ and (2.3), we have

$$\begin{aligned}0 &\le \langle x^0 - x^n, x^n - x^{n+1}\rangle = \langle x^0 - x^n, x^n - x^0 + x^0 - x^{n+1}\rangle \\ &\le -\|x^0 - x^n\|^2 + \|x^0 - x^n\|\|x^0 - x^{n+1}\|,\end{aligned}$$

and hence $\|x^0 - x^n\| \leq \|x^0 - x^{n+1}\|$. This together with the boundedness of $\{x^n\}$ implies that the limit $\lim_{n\to\infty} \|x^n - x^0\| = c$ exists.

Step 4. We claim that $\lim_{n\to\infty} x^n = q \in C$.

Since $C_m \cap D_m \subseteq C_n \cap D_n$, $x^m = \Pr_{C_m\cap D_m} x^0 \in C_n \cap D_n$ for any positive integer $m \geq n$ and (2.3), we have

$$\langle x^0 - x^n, x^n - x^{n+m}\rangle \geq 0.$$

Then

$$\begin{aligned}\|x^n - x^{n+m}\|^2 &= \|x^n - x^0 + x^0 - x^{n+m}\|^2 \\ &= \|x^n - x^0\|^2 + \|x^0 - x^{n+m}\|^2 - 2\langle x^0 - x^n, x^0 - x^{n+m}\rangle \\ &\leq \|x^0 - x^{n+m}\|^2 - \|x^n - x^0\|^2 - 2\langle x^0 - x^n, x^n - x^{n+m}\rangle \\ &\leq \|x^0 - x^{n+m}\|^2 - \|x^n - x^0\|^2. \end{aligned} \tag{2.5}$$

Passing the limit in (2.5) as $n \to \infty$, we get $\lim_{n\to\infty} \|x^n - x^{n+m}\| = 0\ \forall m \in \mathcal{N}^*$. Hence, $\{x^n\}$ is a Cauchy sequence in a real Hilbert space $\mathcal{H}$ and so $\lim_{n\to\infty} x^n = q \in C$.

Step 5. We claim that $q = \Pr_{F\cap \mathrm{Sol}(f,C)} x^0$, where $q = \lim_{n\to\infty} x^n$.

First we show that $q \in F \cap \mathrm{Sol}(f, C)$. Since $x^{n+1} = \Pr_{C_{n+1}\cap D_{n+1}} x^0$, we have $x^{n+1} \in D_{n+1}$. Then $x^{n+1} \in D_{n+1,i}$ and

$$\|y^{n,i} - x^{n+1}\| \leq \|x^n - x^{n+1}\|,$$

which yields that

$$\begin{aligned}\|x^n - y^{n,i}\| &\leq \|x^n - x^{n+1}\| + \|x^{n+1} - y^{n,i}\| \\ &\leq 2\|x^n - x^{n+1}\|.\end{aligned}$$

Combining this and $\lim_{n\to\infty} \|x^n - x^m\| = 0$ for all $m \in \mathcal{N}^*$, we get

$$\lim_{n\to\infty} \|x^n - y^{n,i}\| = 0. \tag{2.6}$$

For each $x^* \in \mathrm{Sol}(f, C) \cap F$, by (2.2) we have

$$\begin{aligned}(1 - 2bc_1)\|x^n - y^n\|^2 &\leq (1 - 2\lambda_n c_1)\|x^n - y^n\|^2 \\ &\leq \|x^n - x^*\|^2 - \|y^{n,i} - x^*\|^2 \\ &= (\|x^n - x^*\| + \|y^{n,i} - x^*\|)(\|x^n - x^*\| - \|y^{n,i} - x^*\|) \\ &\leq (\|x^n - x^*\| + \|y^{n,i} - x^*\|)(\|x^n - y^{n,i}\|).\end{aligned}$$

Using this, the boundedness of sequences $\{x^n\}$, $\{y^{n,i}\}$ and (2.6), we obtain

$$\lim_{n\to\infty} \|x^n - y^n\| = 0. \tag{2.7}$$

By a similar way, we also have $\lim_{n\to\infty}\|z^n - y^n\| = 0$. Then it follows from the inequality

$$\|x^n - z^n\| \le \|x^n - y^n\| + \|y^n - z^n\|$$

that

$$\lim_{n\to\infty}\|x^n - z^n\| = 0. \tag{2.8}$$

On the other hand, we have

$$\|y^{n,i} - z^n\| \le \|y^{n,i} - x^n\| + \|x^n - z^n\|.$$

Combining this, (2.6) and (2.8), we obtain $\lim_{n\to\infty}\|y^{n,i} - z^n\| = 0$. By the definition of the sequence $\{y^{n,i}\}$, we have

$$\|y^{n,i} - z^n\| = \alpha_{n,i}\|T_i z^n - z^n\|,$$

and hence

$$\lim_{n\to\infty}\|T_i z^n - z^n\| = 0,$$

which yields that

$$\begin{aligned}\|T_i x^n - x^n\| &\le \|T_i x^n - T_i z^n\| + \|T_i z^n - z^n\| + \|x^n - z^n\| \\ &\le 2\|x^n - z^n\| + \|T_i z^n - z^n\| \\ &\to 0 \quad \text{as } n \to \infty\end{aligned}$$

and

$$\lim_{n\to\infty}\|T_i x^n - x^n\| = 0.$$

It follows from Step 4 that $\lim_{n\to\infty} T_i x^n = q$. Hence $q \in F$.

Now we show that $q \in \mathrm{Sol}(f, C)$. By Step 5, we have $y^n \to q$ as $n \to \infty$.

Since y^n is the unique solution of the strongly convex problem

$$\min\left\{\frac{1}{2}\|y - x^n\|^2 + \lambda_n f(x^n, y) : y \in C\right\},$$

we get

$$0 \in \partial_2\left(\lambda_n f(x^n, y) + \frac{1}{2}\|y - x^n\|^2\right)(y^n) + N_C(y^n).$$

From this it follows that

$$0 = \lambda_n w + y^n - x^n + \bar{w},$$

where $w \in \partial_2 f(x^n, \cdot)(y^n)$ and $\bar{w} \in N_C(y^n)$. By the definition of the normal cone N_C, we have

$$\langle y^n - x^n, y - y^n \rangle \geq \lambda_n \langle w, y^n - y \rangle \quad \forall y \in C. \tag{2.9}$$

On the other hand, since $f(x^n, \cdot)$ is subdifferentiable on C, by the well-known Moreau-Rockafellar theorem, there exists $w \in \partial_2 f(x^n, \cdot)(y^n)$ such that

$$f(x^n, y) - f(x^n, y^n) \geq \langle w, y - y^n \rangle \quad \forall y \in C.$$

Combining this with (2.9), we have

$$\lambda_n \big(f(x^n, y) - f(x^n, y^n) \big) \geq \langle y^n - x^n, y^n - y \rangle \quad \forall y \in C.$$

Then, using $\{\lambda_n\} \subset [a, b] \subset (0, \frac{1}{L})$, (2.7), $x^n \to q$, $y^n \to q$ as $n \to \infty$ and the upper semicontinuity of f, we have

$$f(q, y) \geq 0 \quad \forall q \in C.$$

This means that $q \in \mathrm{Sol}(f, C)$. By taking the limit in (2.4), we have

$$\langle x^0 - q, q - w \rangle \geq 0 \quad \forall w \in F \cap \mathrm{Sol}(f, C),$$

which implies that $q = \mathrm{Pr}_{F \cap \mathrm{Sol}(f,C)} x^0$. Thus, the subsequences $\{x^n\}$, $\{y^n\}$, $\{z^n\}$ strongly converge to the same point $q = \mathrm{Pr}_{F \cap \mathrm{Sol}(f,C)} x^0$. This completes the proof. □

Now, notice that $\forall w \in F$

$$\begin{aligned} \|z^n - T_i z^n\|^2 &= \|z^n - w + w - T_i z^n\|^2 \\ &= \|z^n - w\|^2 + \|w - T_i z^n\|^2 + 2\langle z^n - w, w - T_i z^n \rangle \\ &\leq 2\|z^n - w\|^2 + 2\langle z^n - w, w - z^n + z^n - T_i z^n \rangle \\ &= 2\|z^n - w\|^2 - 2\|z^n - w\|^2 + 2\langle z^n - w, z^n - T_i z^n \rangle \\ &= 2\langle z^n - w, z^n - T_i z^n \rangle. \end{aligned}$$

Hence

$$\begin{aligned} \|y^{n,i} - w\|^2 &= \|(1 - \alpha_{n,i})(z^n - w) + \alpha_{n,i}(T_i z^n - w)\|^2 \\ &= (1 - \alpha_{n,i})\|z^n - w\|^2 + \alpha_{n,i}\|T_i z^n - w\|^2 - \alpha_{n,i}(1 - \alpha_{n,i})\|T_i z^n - z^n\|^2 \\ &= (1 - \alpha_{n,i})\|z^n - w\|^2 + \alpha_{n,i}\|T_i z^n - z^n + z^n - w\|^2 \\ &\quad - \alpha_{n,i}(1 - \alpha_{n,i})\|T_i z^n - z^n\|^2 \\ &= (1 - \alpha_{n,i})\|z^n - w\|^2 + \alpha_{n,i}\|T_i z^n - z^n\|^2 + \alpha_{n,i}\|z^n - w\|^2 \\ &\quad + 2\alpha_{n,i}\langle T_i z^n - z^n, z^n - w \rangle - \alpha_{n,i}(1 - \alpha_{n,i})\|T_i z^n - z^n\|^2 \\ &\leq \|z^n - w\|^2 + 2\alpha_{n,i}\langle z^n - w, z^n - T_i z^n \rangle + 2\alpha_{n,i}\langle T_i z^n - z^n, z^n - w \rangle \end{aligned}$$

$$\begin{aligned} &- \alpha_{n,i}(1-\alpha_{n,i})\left\| T_i z^n - z^n \right\|^2 \\ &= \left\| z^n - w \right\|^2 - \alpha_{n,i}(1-\alpha_{n,i})\left\| T_i z^n - z^n \right\|^2. \end{aligned} \tag{2.10}$$

From (2.10) and using the methods in Theorem 2.1, we can get the following convergence result.

Theorem 2.2 *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Suppose that assumptions* (A_1)-(A_5) *are satisfied and $\{T_i\}_{i\in\Gamma}$ is a family of nonexpansive mappings from C into itself and a nonempty common fixed points set F. Let $\{x^n\}$ be a sequence generated by the following scheme*:

$$\begin{cases} x^0 \in \mathcal{H} \text{ chosen arbitrarily}, \\ C_{1,i} = D_{1,i} = C, C_1 = \bigcap_{i\in\Gamma} C_{1,i}, D_1 = \bigcap_{i\in\Gamma} D_{1,i}, \\ x^1 = \mathrm{Pr}_{C_1\cap D_1} x^0, \\ y^n = \operatorname{argmin}\{\lambda_n f(x^n, y) + \frac{1}{2}\|y - x^n\|^2 : y \in C\}, \\ z^n = \operatorname{argmin}\{\lambda_n f(y^n, y) + \frac{1}{2}\|z - x^n\|^2 : z \in C\}, \\ y^{n,i} = (1-\alpha_{n,i})z^n + \alpha_{n,i} T_i z^n, \\ C_{n+1,i} = \{z \in C_{n,i} : \|y^{n,i} - z\|^2 \le \|z^n - z\|^2 - \alpha_{n,i}(1-\alpha_{n,i})\|z^n - T_i z^n\|^2\}, \\ C_{n+1} = \bigcap_{i\in\Gamma} C_{n+1,i}, \\ D_{n+1,i} = \{z \in D_{n,i} : \|y^{n,i} - z\| \le \|x^n - z\|\}, \\ D_{n+1} = \bigcap_{i\in\Gamma} D_{n+1,i}, \\ x^{n+1} = \mathrm{Pr}_{C_{n+1}\cap D_{n+1}} x^0, \\ 0 < \liminf \alpha_{n,i} \le \limsup \alpha_{n,i} < 1, \\ \{\lambda_n\} \subset [a,b] \text{ for some } a, b \in (0, \frac{1}{L}), \text{ where } L = \max\{2c_1, 2c_2\}. \end{cases}$$

Then the sequences $\{x^n\}$, $\{y^n\}$ and $\{z^n\}$ converge strongly to the same point $\mathrm{Pr}_{F\cap \mathrm{Sol}(f,C)} x^0$.

Competing interests
The author declares that he has no competing interests.

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