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Some coupled fixed-point theorems in two quasi-partial metric spaces

Feng Gu[1] and Lin Wang[2*]

*Correspondence:
WL64mail@aliyun.com
[2]College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China
Full list of author information is available at the end of the article

Abstract
The purpose of this paper is to prove some new coupled common fixed-point theorems for mappings defined on a set equipped with two quasi-partial metrics. We also provide illustrative examples in support of our new results.

MSC: 47H10; 54H25

Keywords: common coupled fixed point; coupled coincidence point; *w*-compatible mapping pairs; quasi-partial metric space

1 Introduction and preliminaries

In 1994, Matthews [1] introduced the notion of partial metric spaces as follows.

Definition 1.1 [1] A *partial metric* on a nonempty set X is a function $p : X \times X \longrightarrow \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p1) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,

(p2) $p(x,x) \le p(x,y)$,

(p3) $p(x,y) = p(y,x)$,

(p4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X,p) such that X is a nonempty set and p is a partial metric on X.

In [1], Matthews extended the Banach contraction principle from metric spaces to partial metric spaces. Based on the notion of partial metric spaces, several authors (for example, [2–32]) obtained some fixed-point results for mappings satisfying different contractive conditions. Very recently, Haghi *et al.* [33] showed in their interesting paper that some fixed-point theorems in partial metric spaces can be obtained from metric spaces.

Karapınar *et al.* [34] introduced the concept of quasi-partial metric spaces and studied some fixed-point problems on quasi-partial metric spaces. The notion of a quasi-partial metric space is defined as follows.

Definition 1.2 [34] A *quasi-partial metric* on nonempty set X is a function $q : X \times X \to \mathbb{R}^+$ which satisfies:

(QPM_1) If $q(x,x) = q(x,y) = q(y,y)$, then $x = y$,

(QPM_2) $q(x,x) \le q(x,y)$,

(QPM_3) $q(x,x) \leq q(y,x)$, and
(QPM_4) $q(x,y) + q(z,z) \leq q(x,z) + q(z,y)$

for all $x, y, z \in X$.

A *quasi-partial metric space* is a pair (X,q) such that X is a nonempty set and q is a quasi-partial metric on X.

Let q be a quasi-partial metric on set X. Then

$$d_q(x,y) = q(x,y) + q(y,x) - q(x,x) - q(y,y)$$

is a metric on X.

Definition 1.3 [34] Let (X,q) be a quasi-partial metric space. Then

(i) A sequence $\{x_n\}$ *converges* to a point $x \in X$ if and only if

$$q(x,x) = \lim_{n\to\infty} q(x,x_n) = \lim_{n\to\infty} q(x_n,x).$$

(ii) A sequence $\{x_n\}$ is called a *Cauchy sequence* if $\lim_{n,m\to\infty} q(x_n,x_m)$ and $\lim_{n,m\to\infty} q(x_m,x_n)$ exist (and are finite).

(iii) The quasi-partial metric space (X,q) is said to be *complete* if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_q, to a point $x \in X$ such that

$$q(x,x) = \lim_{n,m\to\infty} q(x_n,x_m) = \lim_{n,m\to\infty} q(x_n,x_m).$$

Bhaskar and Lakshmikantham [35] introduced the concept of a coupled fixed point and studied some nice coupled fixed-point theorems. Later, Lakshmikantham and Ćirić [36] introduced the notion of a coupled coincidence point of mappings. For some works on a coupled fixed point, we refer the reader to [37–62].

Definition 1.4 [35] Let X be a nonempty set. We call an element $(x,y) \in X \times X$ a *coupled fixed point* of the mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Definition 1.5 [36] An element $(x,y) \in X \times X$ is called

(i) a *coupled coincidence point* of the mapping $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$; in this case (gx,gy) is called *coupled point of coincidence* of mappings F and g;

(ii) a *common coupled fixed point* of mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = gx = x$ and $F(y,x) = gy = y$;

(iii) a *common coupled fixed point* of mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x,y) = gx = x$ and $F(y,x) = gy = y$.

Abbas *et al.* [37] introduced the concept of w-compatible mappings as follows.

Definition 1.6 [37] Let X be a nonempty set. We say that the mappings $F : X \times X \to X$ and $g : X \to X$ are *w-compatible* if $gF(x,y) = F(gx,gy)$ whenever $gx = F(x,y)$ and $gy = F(y,x)$.

Very recently, Shatanawi and Pitea [38] obtained some common coupled fixed-point results for a pair of mappings in quasi-partial metric space.

Theorem 1.1 (see [38, Theorem 2.1]) *Let (X,q) be a quasi-partial metric space, $g: X \to X$ and $F: X \times X \to X$ be two mappings. Suppose that there exist k_1, k_2, and k_3 in $[0,1)$ with $k_1 + k_2 + k_3 < 1$ such that the condition*

$$\begin{aligned} &q\bigl(F(x,y),F(u,v)\bigr) + q\bigl(F(y,x),F(v,u)\bigr) \\ &\quad \le k_1\bigl[q(gx,gu) + q(gy,gv)\bigr] + k_2\bigl[q\bigl(gx,F(x,y)\bigr) + q\bigl(gy,F(y,x)\bigr)\bigr] \\ &\qquad + k_3\bigl[q\bigl(gu,F(u,v)\bigr) + q\bigl(gv,F(v,u)\bigr)\bigr] \end{aligned} \tag{1.1}$$

holds for all $x,y,u,v \in X$. Also, suppose we have the following hypotheses:

(i) *$F(X \times X) \subset g(X)$.*

(ii) *$g(X)$ is a complete subspace of X with respect to the quasi-partial metric q.*

Then the mappings F and g have a coincidence point (x,y) satisfying $gx = F(x,y)$ and $gy = F(y,x)$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (x,x).

The aim of this article is to prove some new coupled common fixed-point theorems for mappings defined on a set equipped with two quasi-partial metrics.

The following lemma is crucial in our work.

Lemma 1.1 [38] *Let (X,q) be a quasi-partial metric space. Then the following statements hold true*:

(i) *If $q(x,y) = 0$, then $x = y$.*

(ii) *If $x \neq y$, then $q(x,y) > 0$ and $q(y,x) > 0$.*

In this manuscript, we generalize, improve, enrich, and extend the above coupled common fixed-point results. We also state some examples to illustrate our results. This paper can be considered as a continuation of the remarkable works of Aydi [12], Karapınar *et al.* [34], and Shatanawi and Pitea [38].

2 Main results

Now we shall prove our main results.

Theorem 2.1 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x,y) \le q_1(x,y)$, for all $x,y \in X$, and let $F: X \times X \to X$, $g: X \to X$ be two mappings. Suppose that there exist k_1, k_2, k_3, k_4, and k_5 in $[0,1)$ with*

$$k_1 + k_2 + k_3 + 2k_4 + k_5 < 1 \tag{2.1}$$

such that the condition

$$\begin{aligned} &q_1\bigl(F(x,y),F(u,v)\bigr) + q_1\bigl(F(y,x),F(v,u)\bigr) \\ &\quad \le k_1\bigl[q_2(gx,gu) + q_2(gy,gv)\bigr] + k_2\bigl[q_2\bigl(gx,F(x,y)\bigr) + q_2\bigl(gy,F(y,x)\bigr)\bigr] \end{aligned}$$

$$
\begin{aligned}
&+ k_3\big[q_2\big(gu, F(u,v)\big) + q_2\big(gv, F(v,u)\big)\big] + k_4\big[q_2\big(gx, F(u,v)\big) + q_2\big(gy, F(v,u)\big)\big] \\
&+ k_5\big[q_2\big(gu, F(x,y)\big) + q_2\big(gv, F(y,x)\big)\big]
\end{aligned}
\tag{2.2}
$$

holds for all $x, y, u, v \in X$. Also, suppose we have the following hypotheses:

(i) $F(X \times X) \subset g(X)$.

(ii) $g(X)$ *is a complete subspace of X with respect to the quasi-partial metric q_1.*

Then the mappings F and g have a coincidence point (x,y) satisfying $gx = F(x,y) = F(y,x) = gy$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u).

Proof Let $x_0, y_0 \in X$. Since $F(X \times X) \subset g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Similarly, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$ and $gy_2 = F(y_1, x_1)$. Continuing in this way we construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$
gx_{n+1} = F(x_n, y_n) \quad \text{and} \quad gy_{n+1} = F(y_n, x_n), \quad \forall n \geq 0. \tag{2.3}
$$

It follows from (2.2) and (QPM_4) that

$$
\begin{aligned}
&q_1(gx_n, gx_{n+1}) + q_1(gy_n, gy_{n+1}) \\
&\quad = q_1\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) + q_1\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big) \\
&\quad \leq k_1\big[q_2(gx_{n-1}, gx_n) + q_2(gy_{n-1}, gy_n)\big] \\
&\qquad + k_2\big[q_2\big(gx_{n-1}, F(x_{n-1}, y_{n-1}) + q_2\big(gy_{n-1}, F(y_{n-1}, x_{n-1})\big)\big)\big] \\
&\qquad + k_3\big[q_2\big(gx_n, F(x_n, y_n)\big) + q_2\big(gy_n, F(y_n, x_n)\big)\big] \\
&\qquad + k_4\big[q_2\big(gx_{n-1}, F(x_n, y_n)\big) + q_2\big(gy_{n-1}, F(y_n, x_n)\big)\big] \\
&\qquad + k_5\big[q_2\big(gx_n, F(x_{n-1}, y_{n-1})\big) + q_2\big(gy_n, F(y_{n-1}, x_{n-1})\big)\big] \\
&\quad = (k_1 + k_2)\big[q_2(gx_{n-1}, gx_n) + q_2(gy_{n-1}, gy_n)\big] + k_3\big[q_2(gx_n, gx_{n+1}) + q_2(gy_n, gy_{n+1})\big] \\
&\qquad + k_4\big[q_2(gx_{n-1}, gx_{n+1}) + q_2(gy_{n-1}, gy_{n+1})\big] + k_5\big[q_2(gx_n, gx_n) + q_2(gy_n, gy_n)\big] \\
&\quad \leq (k_1 + k_2)\big[q_2(gx_{n-1}, gx_n) + q_2(gy_{n-1}, gy_n)\big] + k_3\big[q_2(gx_n, gx_{n+1}) + q_2(gy_n, gy_{n+1})\big] \\
&\qquad + k_4\big[q_2(gx_{n-1}, gx_n) + q_2(gx_n, gx_{n+1}) - q_2(gx_n, gx_n) + q_2(gy_{n-1}, gy_n) + q_2(gy_n, gy_{n+1}) \\
&\qquad - q_2(gy_n, gy_n)\big] + k_5\big[q_2(gx_n, gx_{n+1}) + q_2(gy_n, gy_{n+1})\big] \\
&\quad \leq (k_1 + k_2 + k_4)\big[q_2(gx_{n-1}, gx_n) + q_2(gy_{n-1}, gy_n)\big] \\
&\qquad + (k_3 + k_4 + k_5)\big[q_2(gx_n, gx_{n+1}) + q_2(gy_n, gy_{n+1})\big] \\
&\quad \leq (k_1 + k_2 + k_4)\big[q_1(gx_{n-1}, gx_n) + q_1(gy_{n-1}, gy_n)\big] \\
&\qquad + (k_3 + k_4 + k_5)\big[q_1(gx_n, gx_{n+1}) + q_1(gy_n, gy_{n+1})\big],
\end{aligned}
$$

which implies that

$$q_1(gx_n, gx_{n+1}) + q_1(gy_n, gy_{n+1}) \le \frac{k_1 + k_2 + k_4}{1 - k_3 - k_4 - k_5}\bigl[q_1(gx_{n-1}, gx_n) + q_1(gy_{n-1}, gy_n)\bigr]. \tag{2.4}$$

Put $k = \frac{k_1+k_2+k_4}{1-k_3-k_4-k_5}$. Obviously, $0 \le k < 1$. By repetition of the above inequality (2.4) n times, we get

$$q_1(gx_n, gx_{n+1}) + q_1(gy_n, gy_{n+1}) \le k^n\bigl[q_1(gx_0, gx_1) + q_1(gy_0, gy_1)\bigr]. \tag{2.5}$$

Next, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$.

In fact, for each $n, m \in \mathbb{N}$, $m > n$, from (QPM_4) and (2.5) we have

$$\begin{aligned} q_1(gx_n, gx_m) + q_1(gy_n, gy_m) &\le \sum_{i=n}^{m-1}\bigl[q_1(gx_i, gx_{i+1}) + q_1(gy_i, gy_{i+1})\bigr] \\ &\le \sum_{i=n}^{m-1} k^i\bigl[q_1(gx_0, gx_1) + q_1(gy_0, gy_1)\bigr] \\ &\le \frac{k^n}{1-k}\bigl[q_1(gx_0, gx_1) + q_1(gy_0, gy_1)\bigr]. \end{aligned} \tag{2.6}$$

This implies that

$$\lim_{n,m\to\infty}\bigl[q_1(gx_n, gx_m) + q_1(gy_n, gy_m)\bigr] = 0,$$

and so

$$\lim_{n,m\to\infty} q_1(gx_n, gx_m) = 0 \quad \text{and} \quad \lim_{n,m\to\infty} q_1(gy_n, gy_m) = 0. \tag{2.7}$$

By similar arguments as above, we can show that

$$\lim_{n,m\to\infty} q_1(gx_m, gx_n) = 0 \quad \text{and} \quad \lim_{n,m\to\infty} q_1(gy_m, gy_n) = 0. \tag{2.8}$$

Hence $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in (gX, q_1). Since (gX, q_1) is complete, there exist $gx, gy \in g(X)$ such that $\{gx_n\}$ and $\{gy_n\}$ converge to gx and gy with respect to τ_{q_1}, that is,

$$\begin{aligned} q_1(gx, gx) &= \lim_{n\to\infty} q_1(gx, gx_n) = \lim_{n\to\infty} q_1(gx_n, gx) \\ &= \lim_{n,m\to\infty} q_1(gx_m, gx_n) = \lim_{n,m\to\infty} q_1(gx_n, gx_m) \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} q_1(gy, gy) &= \lim_{n\to\infty} q_1(gy, gy_n) = \lim_{n\to\infty} q_1(gy_n, gy) \\ &= \lim_{n,m\to\infty} q_1(gy_m, gy_n) = \lim_{n,m\to\infty} q_1(gy_n, gy_m). \end{aligned} \tag{2.10}$$

Combining (2.7)-(2.10), we have

$$\begin{aligned} q_1(gx,gx) &= \lim_{n\to\infty} q_1(gx,gx_n) = \lim_{n\to\infty} q_1(gx_n,gx) \\ &= \lim_{n,m\to\infty} q_1(gx_m,gx_n) = \lim_{n,m\to\infty} q_1(gx_n,gx_m) = 0 \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} q_1(gy,gy) &= \lim_{n\to\infty} q_1(gy,gy_n) = \lim_{n\to\infty} q_1(gy_n,gy) \\ &= \lim_{n,m\to\infty} q_1(gy_m,gy_n) = \lim_{n,m\to\infty} q_1(gy_n,gy_m) = 0. \end{aligned} \tag{2.12}$$

By (QPM_4) we obtain

$$\begin{aligned} q_1\bigl(gx_{n+1},F(x,y)\bigr) &\le q_1(gx_{n+1},gx) + q_1\bigl(gx,F(x,y)\bigr) - q_1(gx,gx) \\ &\le q_1(gx_{n+1},gx) + q_1\bigl(gx,F(x,y)\bigr) \\ &\le q_1(gx_{n+1},gx) + q_1(gx,gx_{n+1}) + q_1\bigl(gx_{n+1},F(x,y)\bigr) - q_1(gx_{n+1},gx_{n+1}) \\ &\le q_1(gx_{n+1},gx) + q_1(gx,gx_{n+1}) + q_1\bigl(gx_{n+1},F(x,y)\bigr). \end{aligned}$$

Letting $n\to\infty$ in the above inequalities and using (2.11), we have

$$\lim_{n\to\infty} q_1\bigl(gx_{n+1},F(x,y)\bigr) \le q_1\bigl(gx,F(x,y)\bigr) \le \lim_{n\to\infty} q_1\bigl(gx_{n+1},F(x,y)\bigr).$$

That is,

$$\lim_{n\to\infty} q_1\bigl(gx_{n+1},F(x,y)\bigr) = q_1\bigl(gx,F(x,y)\bigr). \tag{2.13}$$

Similarly, using (2.12) we have

$$\lim_{n\to\infty} q_1\bigl(gy_{n+1},F(y,x)\bigr) = q_1\bigl(gy,F(y,x)\bigr). \tag{2.14}$$

Now we prove that $F(x,y) = gx$ and $F(y,x) = gy$. In fact, it follows from (2.2) and (2.3) that

$$\begin{aligned} &q_1\bigl(gx_{n+1},F(x,y)\bigr) + q_1\bigl(gy_{n+1},F(y,x)\bigr) \\ &\quad = q_1\bigl(F(x_n,y_n),F(x,y)\bigr) + q_1\bigl(F(y_n,x_n)\bigr) \\ &\quad \le k_1\bigl[q_2(gx_n,gx) + q_2(gy_n,gy)\bigr] + k_2\bigl[q_2\bigl(gx_n,F(x_n,y_n)\bigr) + q_2\bigl(gy_n,F(y_n,x_n)\bigr)\bigr] \\ &\qquad + k_3\bigl[q_2\bigl(gx,F(x,y)\bigr) + q_2\bigl(gy,F(y,x)\bigr)\bigr] + k_4\bigl[q_2\bigl(gx_n,F(x,y)\bigr) + q_2\bigl(gy_n,F(y,x)\bigr)\bigr] \\ &\qquad + k_5\bigl[q_2\bigl(gx,F(x_n,y_n)\bigr) + q_2\bigl(gy,F(y_n,x_n)\bigr)\bigr] \\ &\quad = k_1\bigl[q_2(gx_n,gx) + q_2(gy_n,gy)\bigr] + k_2\bigl[q_2(gx_n,gx_{n+1}) + q_2(gy_n,gy_{n+1})\bigr] \\ &\qquad + k_3\bigl[q_2\bigl(gx,F(x,y)\bigr) + q_2\bigl(gy,F(y,x)\bigr)\bigr] + k_4\bigl[q_2\bigl(gx_n,F(x,y)\bigr) + q_2\bigl(gy_n,F(y,x)\bigr)\bigr] \\ &\qquad + k_5\bigl[q_2(gx,gx_{n+1}) + q_2(gy,gy_{n+1})\bigr] \\ &\quad \le k_1\bigl[q_1(gx_n,gx) + q_1(gy_n,gy)\bigr] + k_2\bigl[q_1(gx_n,gx_{n+1}) + q_1(gy_n,gy_{n+1})\bigr] \end{aligned}$$

$$\begin{aligned}
&+ k_3\bigl[q_1\bigl(gx, F(x,y)\bigr) + q_1\bigl(gy, F(y,x)\bigr)\bigr] + k_4\bigl[q_1\bigl(gx_n, F(x,y)\bigr) + q_1\bigl(gy_n, F(y,x)\bigr)\bigr] \\
&+ k_5\bigl[q_1(gx, gx_{n+1}) + q_1(gy, gy_{n+1})\bigr].
\end{aligned}$$

Letting $n \to \infty$ in the above inequality, using (2.11)-(2.14), we obtain

$$q_1\bigl(gx, F(x,y)\bigr) + q_1\bigl(gy, F(y,x)\bigr) \le (k_3 + k_4)\bigl[q_1\bigl(gx, F(x,y)\bigr) + q_1\bigl(gy, F(y,x)\bigr)\bigr]. \tag{2.15}$$

By (2.1) we have $k_3 + k_4 < 1$. Hence, it follows from (2.15) that $q_1(gx, F(x,y)) = q_1(gy, F(y,x)) = 0$. By Lemma 1.1, we get $F(x,y) = gx$ and $F(y,x) = gy$. Hence, (gx, gy) is a coupled point of coincidence of mappings F and g.

Next, we will show that the coupled point of coincidence is unique. Suppose that $(x^*, y^*) \in X \times X$ with $F(x^*, y^*) = gx^*$ and $F(y^*, x^*) = gy^*$. Using (2.2), (2.11), (2.12), and (QPM_3), we obtain

$$\begin{aligned}
&q_1\bigl(gx, gx^*\bigr) + q_1\bigl(gy, gy^*\bigr) \\
&\quad = q_1\bigl(F(x,y), F\bigl(x^*, y^*\bigr)\bigr) + q_1\bigl(F(y,x), F\bigl(y^*, x^*\bigr)\bigr) \\
&\quad \le k_1\bigl[q_2\bigl(gx, gx^*\bigr) + q_2\bigl(gy, gy^*\bigr)\bigr] + k_2\bigl[q_2\bigl(gx, F(x,y)\bigr) + q_2\bigl(gy, F(y,x)\bigr)\bigr] \\
&\qquad + k_3\bigl[q_2\bigl(gx^*, F\bigl(x^*, y^*\bigr)\bigr) + q_2\bigl(gy^*, F\bigl(y^*, x^*\bigr)\bigr)\bigr] \\
&\qquad + k_4\bigl[q_2\bigl(gx, F\bigl(x^*, y^*\bigr)\bigr) + q_2\bigl(gy, F\bigl(y^*, x^*\bigr)\bigr)\bigr] \\
&\qquad + k_5\bigl[q_2\bigl(gx^*, F(x,y)\bigr) + q_2\bigl(gy^*, F(y,x)\bigr)\bigr] \\
&\quad = k_1\bigl[q_2\bigl(gx, gx^*\bigr) + q_2\bigl(gy, gy^*\bigr)\bigr] + k_2\bigl[q_2(gx, gx) + q_2(gy, gy)\bigr] \\
&\qquad + k_3\bigl[q_2\bigl(gx^*, gx^*\bigr) + q_2\bigl(gy^*, gy^*\bigr)\bigr] + k_4\bigl[q_2\bigl(gx, gx^*\bigr) + q_2\bigl(gy, gy^*\bigr)\bigr] \\
&\qquad + k_5\bigl[q_2\bigl(gx^*, gx\bigr) + q_2\bigl(gy^*, gy\bigr)\bigr] \\
&\quad \le (k_1 + k_4)\bigl[q_1\bigl(gx, gx^*\bigr) + q_1\bigl(gy, gy^*\bigr)\bigr] + k_2\bigl[q_1(gx, gx) + q_1(gy, gy)\bigr] \\
&\qquad + k_3\bigl[q_1\bigl(gx^*, gx^*\bigr) + q_1\bigl(gy^*, gy^*\bigr)\bigr] + k_5\bigl[q_1\bigl(gx^*, gx\bigr) + q_1\bigl(gy^*, gy\bigr)\bigr] \\
&\quad \le (k_1 + k_3 + k_4)\bigl[q_1\bigl(gx, gx^*\bigr) + q_1\bigl(gy, gy^*\bigr)\bigr] \\
&\qquad + k_5\bigl[q_1\bigl(gx^*, gx\bigr) + q_1\bigl(gy^*, gy\bigr)\bigr].
\end{aligned}$$

This implies that

$$q_1\bigl(gx, gx^*\bigr) + q_1\bigl(gy, gy^*\bigr) \le \frac{k_5}{1 - k_1 - k_3 - k_4} \cdot \bigl[q_1\bigl(gx^*, gx\bigr) + q_1\bigl(gy^*, gy\bigr)\bigr]. \tag{2.16}$$

Similarly, we have

$$q_1\bigl(gx^*, gx\bigr) + q_1\bigl(gy^*, gy\bigr) \le \frac{k_5}{1 - k_1 - k_3 - k_4} \cdot \bigl[q_1\bigl(gx, gx^*\bigr) + q_1\bigl(gy, gy^*\bigr)\bigr]. \tag{2.17}$$

Substituting (2.17) into (2.16), we obtain

$$q_1\bigl(gx, gx^*\bigr) + q_1\bigl(gy, gy^*\bigr) \le \left(\frac{k_5}{1 - k_1 - k_3 - k_4}\right)^2 \cdot \bigl[q_1\bigl(gx, gx^*\bigr) + q_1\bigl(gy, gy^*\bigr)\bigr]. \tag{2.18}$$

Since $\frac{k_5}{1-k_1-k_3-k_4} < 1$, from (2.18), we must have $q_1(gx, gx^*) = q_1(gy, gy^*) = 0$. By Lemma 1.1, we get $gx = gx^*$ and $gy = gy^*$, which implies the uniqueness of the coupled point of coincidence of F and g, that is, (gx, gy).

Next, we will show that $gx = gy$. In fact, from (2.2), (2.11), and (2.12) we have

$$\begin{aligned}
& q_1(gx, gy) + q_1(gy, gx) \\
&\quad = q_1\big(F(x,y), F(y,x)\big) + q_1\big(F(y,x), F(x,y)\big) \\
&\quad \le k_1\big[q_2(gx, gy) + q_2(gy, gx)\big] + k_2\big[q_2\big(gx, F(x,y)\big) + q_2\big(gy, F(y,x)\big)\big] \\
&\qquad + k_3\big[q_2\big(gy, F(y,x)\big) + q_2\big(gx, F(x,y)\big)\big] + k_4\big[q_2\big(gx, F(y,x)\big) + q_2\big(gy, F(x,y)\big)\big] \\
&\qquad + k_5\big[q_2\big(gy, F(x,y)\big) + q_2\big(gx, F(y,x)\big)\big] \\
&\quad = k_1\big[q_2(gx, gy) + q_2(gy, gx)\big] + k_2\big[q_2(gx, gx) + q_2(gy, gy)\big] \\
&\qquad + k_3\big[q_2(gy, gy) + q_2(gx, gx)\big] + k_4\big[q_2(gx, gy) + q_2(gy, gx)\big] \\
&\qquad + k_5\big[q_2(gy, gx) + q_2(gx, gy)\big] \\
&\quad \le k_1\big[q_1(gx, gy) + q_1(gy, gx)\big] + k_2\big[q_1(gx, gx) + q_1(gy, gy)\big] \\
&\qquad + k_3\big[q_1(gy, gy) + q_1(gx, gx)\big] + k_4\big[q_1(gx, gy) + q_1(gy, gx)\big] \\
&\qquad + k_5\big[q_1(gy, gx) + q_1(gx, gy)\big] \\
&\quad = (k_1 + k_4 + k_5)\big[q_1(gx, gy) + q_1(gy, gx)\big]. \qquad (2.19)
\end{aligned}$$

Since $k_1 + k_4 + k_5 < 1$, we have $q_1(gx, gy) = q_1(gy, gx) = 0$. By Lemma 1.1, we get $gx = gy$.

Finally, assume that g and F are w-compatible. Let $u = gx$, then we have $u = gx = F(x,y) = gy = F(y,x)$, so that

$$gu = ggx = g\big(F(x,y)\big) = F(gx, gy) = F(u,u). \qquad (2.20)$$

Consequently, (u,u) is a coupled coincidence point of F and g, and therefore (gu, gu) is a coupled point of coincidence of F and g, and by its uniqueness, we get $gu = gx$. Thus, we obtain $F(u,u) = gu = u$. Therefore, (u,u) is the unique common coupled fixed point of F and g. This completes the proof of Theorem 2.1. □

In Theorem 2.1, if we take $q_1(x,y) = q_2(x,y)$ for all $x, y \in X$, then we get the following.

Corollary 2.1 *Let (X, q) be a quasi-partial metric space, $F : X \times X \to X$ and $g : X \to X$ be two mappings. Suppose that there exist k_1, k_2, k_3, k_4 and k_5 in $[0,1)$ with $k_1 + k_2 + k_3 + 2k_4 + k_5 < 1$ such that the condition*

$$\begin{aligned}
& q\big(F(x,y), F(u,v)\big) + q\big(F(y,x), F(v,u)\big) \\
&\quad \le k_1\big[q(gx, gu) + q(gy, gv)\big] + k_2\big[q\big(gx, F(x,y)\big) + q\big(gy, F(y,x)\big)\big] \\
&\qquad + k_3\big[q\big(gu, F(u,v)\big) + q\big(gv, F(v,u)\big)\big] + k_4\big[q\big(gx, F(u,v)\big) + q\big(gy, F(v,u)\big)\big] \\
&\qquad + k_5\big[q\big(gu, F(x,y)\big) + q\big(gv, F(y,x)\big)\big] \qquad (2.21)
\end{aligned}$$

holds for all $x, y, u, v \in X$. Also, suppose we have the following hypotheses:

(i) $F(X \times X) \subset g(X)$.
(ii) $g(X)$ *is a complete subspace of* X *with respect to the quasi-partial metric* q.
Then the mappings F *and* g *have a coincidence point* (x, y) *satisfying* $gx = F(x, y) = F(y, x) = gy$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u, u).

Remark 2.1 Corollary 2.1 improve and extend Theorem 2.1 of Shatanawi and Pitea [38]; the contractive condition defined by (1.1) is replaced by the new contractive condition defined by (2.23).

Corollary 2.2 *Let* q_1 *and* q_2 *be two quasi-metrics on* X *such that* $q_2(x, y) \leq q_1(x, y)$, *for all* $x, y \in X$, *and* $F : X \times X \to X$, $g : X \to X$ *be two mappings. Suppose that there exist* $a_i \in [0, 1)$ $(i = 1, 2, 3, \ldots, 10)$ *with*

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + 2(a_7 + a_8) + a_9 + a_{10} < 1 \tag{2.22}$$

such that the condition

$$\begin{aligned} &q_1\bigl(F(x, y), F(u, v)\bigr) \\ &\quad \leq a_1 q_2(gx, gu) + a_2 q_2(gy, gv) + a_3 q_2\bigl(gx, F(x, y)\bigr) + a_4 q_2\bigl(gy, F(y, x)\bigr) \\ &\qquad + a_5 q_2\bigl(gu, F(u, v)\bigr) + a_6 q_2\bigl(gv, F(v, u)\bigr) + a_7 q_2\bigl(gx, F(u, v)\bigr) + a_8 q_2\bigl(gy, F(v, u)\bigr) \\ &\qquad + a_9 q_2\bigl(gu, F(x, y)\bigr) + a_{10} q_2\bigl(gv, F(y, x)\bigr) \end{aligned} \tag{2.23}$$

holds for all $x, y, u, v \in X$. *Also, suppose we have the following hypotheses*:
(i) $F(X \times X) \subset g(X)$.
(ii) $g(X)$ *is a complete subspace of* X *with respect to the quasi-partial metric* q_1.
Then the mappings F *and* g *have a coincidence point* (x, y) *satisfying* $gx = F(x, y) = F(y, x) = gy$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u, u).

Proof Given $x, y, u, v \in X$. It follows from (2.23) that

$$\begin{aligned} &q_1\bigl(F(x, y), F(u, v)\bigr) \\ &\quad \leq a_1 q_2(gx, gu) + a_2 q_2(gy, gv) + a_3 q_2\bigl(gx, F(x, y)\bigr) + a_4 q_2\bigl(gy, F(y, x)\bigr) \\ &\qquad + a_5 q_2\bigl(gu, F(u, v)\bigr) + a_6 q_2\bigl(gv, F(v, u)\bigr) + a_7 q_2\bigl(gx, F(u, v)\bigr) + a_8 q_2\bigl(gy, F(v, u)\bigr) \\ &\qquad + a_9 q_2\bigl(gu, F(x, y)\bigr) + a_{10} q_2\bigl(gv, F(y, x)\bigr) \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &q_1\bigl(F(y, x), F(v, u)\bigr) \\ &\quad \leq a_1 q_2(gy, gv) + a_2 q_2(gx, gu) + a_3 q_2\bigl(gy, F(y, x)\bigr) + a_4 q_2\bigl(gx, F(x, y)\bigr) \end{aligned}$$

$$\begin{aligned}&+ a_5 q_2\bigl(gv, F(v,u)\bigr) + a_6 q_2\bigl(gu, F(u,v)\bigr)\\&+ a_7 q_2\bigl(gy, F(v,u)\bigr) + a_8 q_2\bigl(gx, F(u,v)\bigr)\\&+ a_9 q_2\bigl(gv, F(y,x)\bigr) + a_{10} q_2\bigl(gu, F(x,y)\bigr).\end{aligned}\tag{2.25}$$

Adding inequality (2.24) to inequality (2.25), we get

$$\begin{aligned}&q_1\bigl(q_1\bigl(F(x,y),F(u,v)\bigr) + F(y,x),F(v,u)\bigr)\\&\quad\le (a_1+a_2)\bigl[q_2(gx,gu) + q_2(gy,gv)\bigr] + (a_3+a_4)\bigl[q_2\bigl(gx,F(x,y)\bigr) + q_2\bigl(gy,F(y,x)\bigr)\bigr]\\&\qquad+ (a_5+a_6)\bigl[q_2\bigl(gu,F(u,v)\bigr) + q_2\bigl(gv,F(v,u)\bigr)\bigr]\\&\qquad+ (a_7+a_8)\bigl[q_2\bigl(gx,F(u,v)\bigr) + q_2\bigl(gy,F(v,u)\bigr)\bigr]\\&\qquad+ (a_9+a_{10})\bigl[q_2\bigl(gu,F(x,y)\bigr) + q_2\bigl(gv,F(y,x)\bigr)\bigr].\end{aligned}\tag{2.26}$$

Therefore, the result follows from Theorem 2.1. □

Remark 2.2 If we take $q_1(x,y) = q_2(x,y)$ for all $x,y \in X$ and $a_7 = a_8 = a_9 = a_{10} = 0$, then Corollary 2.2 is reduced to Corollary 2.1 of Shatanawi and Pitea [38].

Corollary 2.3 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x,y) \le q_1(x,y)$, for all $x,y \in X$, and $F: X \times X \to X$, $g: X \to X$ be two mappings. Suppose that there exists $k \in [0,1)$ such that the condition*

$$q_1\bigl(F(x,y),F(u,v)\bigr) + q\bigl(F(y,x),F(v,u)\bigr) \le k\bigl[q_2(gx,gu) + q_2(gy,gv)\bigr]\tag{2.27}$$

holds for all $x,y,u,v \in X$. Also, suppose we have the following hypotheses:

(i) $F(X \times X) \subset g(X)$.

(ii) $g(X)$ *is a complete subspace of X with respect to the quasi-partial metric q_1.*

Then the mappings F and g have a coincidence point (x,y) satisfying $gx = F(x,y) = F(y,x) = gy$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u,u).

Remark 2.3 If we take $q_1(x,y) = q_2(x,y)$ for all $x,y \in X$, then Corollary 2.3 is reduced to Corollary 2.2 of Shatanawi and Pitea [38].

Corollary 2.4 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x,y) \le q_1(x,y)$, for all $x,y \in X$, and $F: X \times X \to X$, $g: X \to X$ be two mappings. Suppose that there exists $k \in [0,1)$ such that the condition*

$$q_1\bigl(F(x,y),F(u,v)\bigr) + q\bigl(F(y,x),F(v,u)\bigr) \le k\bigl[q_2\bigl(gx,F(x,y)\bigr) + q_2\bigl(gy,F(y,x)\bigr)\bigr]\tag{2.28}$$

holds for all $x,y,u,v \in X$. Also, suppose we have the following hypotheses:

(i) $F(X \times X) \subset g(X)$.

(ii) $g(X)$ *is a complete subspace of X with respect to the quasi-partial metric q_1.*

Then the mappings F and g have a coincidence point (x, y) satisfying $gx = F(x, y) = F(y, x) = gy$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u, u).

Remark 2.4 If we take $q_1(x, y) = q_2(x, y)$ for all $x, y \in X$, then Corollary 2.4 is reduced to Corollary 2.3 of Shatanawi and Pitea [38].

Corollary 2.5 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x, y) \leq q_1(x, y)$, for all $x, y \in X$, and $F : X \times X \to X$, $g : X \to X$ be two mappings. Suppose that there exists $k \in [0, 1)$ such that the condition*

$$q_1\big(F(x,y),F(u,v)\big) + q\big(F(y,x),F(v,u)\big) \leq k\big[q_2\big(gu,F(u,v)\big) + q_2\big(gv,F(v,u)\big)\big] \tag{2.29}$$

holds for all $x, y, u, v \in X$. Also, suppose we have the following hypotheses:

(i) *$F(X \times X) \subset g(X)$.*

(ii) *$g(X)$ is a complete subspace of X with respect to the quasi-partial metric q_1.*

Then the mappings F and g have a coincidence point (x, y) satisfying $gx = F(x, y) = F(y, x) = gy$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u, u).

Remark 2.5 If we take $q_1(x, y) = q_2(x, y)$ for all $x, y \in X$, then Corollary 2.5 is reduced to Corollary 2.4 of Shatanawi and Pitea [38].

Corollary 2.6 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x, y) \leq q_1(x, y)$, for all $x, y \in X$, and $F : X \times X \to X$, $g : X \to X$ be two mappings. Suppose that there exists $k \in [0, \frac{1}{2})$ such that the condition*

$$q_1\big(F(x,y),F(u,v)\big) + q\big(F(y,x),F(v,u)\big) \leq k\big[q_2\big(gx,F(u,v)\big) + q_2\big(gy,F(v,u)\big)\big] \tag{2.30}$$

holds for all $x, y, u, v \in X$. Also, suppose we have the following hypotheses:

(i) *$F(X \times X) \subset g(X)$.*

(ii) *$g(X)$ is a complete subspace of X with respect to the quasi-partial metric q_1.*

Then the mappings F and g have a coincidence point (x, y) satisfying $gx = F(x, y) = F(y, x) = gy$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u, u).

Corollary 2.7 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x, y) \leq q_1(x, y)$, for all $x, y \in X$, and $F : X \times X \to X$, $g : X \to X$ be two mappings. Suppose that there exists $k \in [0, 1)$ such that the condition*

$$q_1\big(F(x,y),F(u,v)\big) + q\big(F(y,x),F(v,u)\big) \leq k\big[q_2\big(gu,F(x,y)\big) + q_2\big(gv,F(y,x)\big)\big] \tag{2.31}$$

holds for all $x, y, u, v \in X$. Also, suppose we have the following hypotheses:

(i) $F(X \times X) \subset g(X)$.

(ii) $g(X)$ *is a complete subspace of X with respect to the quasi-partial metric q_1.*

Then the mappings F and g have a coincidence point (x, y) satisfying $gx = F(x, y) = F(y, x) = gy$.

Moreover, if F and g are w-compatible, then F and g have a unique common coupled fixed point of the form (u, u).

Let $g = I_X$ (the identity mapping) in Theorem 2.1 and Corollaries 2.1-2.7. Then we have the following results.

Corollary 2.8 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x, y) \le q_1(x, y)$, for all $x, y \in X$, and $F : X \times X \to X$ be a mapping. Suppose that there exist k_1, k_2, k_3, k_4, and k_5 in $[0, 1)$ with $k_1 + k_2 + k_3 + 2k_4 + k_5 < 1$ such that the condition*

$$\begin{aligned} &q_1\bigl(F(x,y),F(u,v)\bigr) + q_1\bigl(F(y,x),F(v,u)\bigr) \\ &\quad\le k_1\bigl[q_2(x,u) + q_2(y,v)\bigr] + k_2\bigl[q_2\bigl(x,F(x,y)\bigr) + q_2\bigl(y,F(y,x)\bigr)\bigr] \\ &\qquad + k_3\bigl[q_2\bigl(u,F(u,v)\bigr) + q_2\bigl(v,F(v,u)\bigr)\bigr] + k_4\bigl[q_2\bigl(x,F(u,v)\bigr) + q_2\bigl(y,F(v,u)\bigr)\bigr] \\ &\qquad + k_5\bigl[q_2\bigl(u,F(x,y)\bigr) + q_2\bigl(v,F(y,x)\bigr)\bigr] \end{aligned} \tag{2.32}$$

holds for all $x, y, u, v \in X$. If (X, q_1) is a complete quasi-partial metric space, then the mapping F has a unique coupled fixed point of the form (u, u).

Corollary 2.9 *Let (X, q) be a complete quasi-partial metric space, $F : X \times X \to X$ be a mapping. Suppose that there exist k_1, k_2, k_3, k_4, and k_5 in $[0, 1)$ with $k_1 + k_2 + k_3 + 2k_4 + k_5 < 1$ such that the condition*

$$\begin{aligned} &q\bigl(F(x,y),F(u,v)\bigr) + q\bigl(F(y,x),F(v,u)\bigr) \\ &\quad\le k_1\bigl[q(x,u) + q(y,v)\bigr] + k_2\bigl[q\bigl(x,F(x,y)\bigr) + q\bigl(y,F(y,x)\bigr)\bigr] \\ &\qquad + k_3\bigl[q\bigl(u,F(u,v)\bigr) + q\bigl(v,F(v,u)\bigr)\bigr] + k_4\bigl[q\bigl(x,F(u,v)\bigr) + q\bigl(y,F(v,u)\bigr)\bigr] \\ &\qquad + k_5\bigl[q\bigl(u,F(x,y)\bigr) + q\bigl(v,F(y,x)\bigr)\bigr] \end{aligned} \tag{2.33}$$

holds for all $x, y, u, v \in X$. Then F has a unique coupled fixed point of the form (u, u).

Remark 2.6 Corollary 2.9 improve and extend Corollary 2.5 of Shatanawi and Pitea [38], the contractive condition is replaced by the new contractive condition defined by (2.35).

Corollary 2.10 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x, y) \le q_1(x, y)$, for all $x, y \in X$, and $F : X \times X \to X$ be a mapping. Suppose that there exist $a_i \in [0, 1)$ ($i = 1, 2, 3, \dots, 10$) with*

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + 2(a_7 + a_8) + a_9 + a_{10} < 1 \tag{2.34}$$

such that the condition

$$\begin{aligned} & q_1\bigl(F(x,y),F(u,v)\bigr) \\ &\quad \le a_1q_2(x,u)+a_2q_2(y,v)+a_3q_2\bigl(x,F(x,y)\bigr)+a_4q_2\bigl(y,F(y,x)\bigr) \\ &\qquad + a_5q_2\bigl(u,F(u,v)\bigr)+a_6q_2\bigl(v,F(v,u)\bigr)+a_7q_2\bigl(x,F(u,v)\bigr)+a_8q_2\bigl(y,F(v,u)\bigr) \\ &\qquad + a_9q_2\bigl(u,F(x,y)\bigr)+a_{10}q_2\bigl(v,F(y,x)\bigr) \end{aligned} \tag{2.35}$$

holds for all $x,y,u,v \in X$. If (X,q_1) is a complete quasi-partial metric space. Then the mapping F has a unique coupled fixed point of the form (u,u).

Remark 2.7

(1) If we take $q_1(x,y)=q_2(x,y)$ for all $x,y\in X$ and $a_7=a_8=a_9=a_{10}=0$, then Corollary 2.10 is reduced to Corollary 2.6 of Shatanawi and Pitea [38].
(2) If we take $q_1(x,y)=q_2(x,y)$ for all $x,y\in X$ and $a_i=0$ $(i=3,4,5,\dots,10)$, then Corollary 2.10 extends Theorem 2.1 of Aydi [12] on the class of quasi-partial metric spaces.
(3) If we take $q_1(x,y)=q_2(x,y)$ for all $x,y\in X$, $a_1=a_2$ and $a_i=0$ $(i=3,4,5,\dots,10)$, then Corollary 2.10 extends the Corollary 2.2 of Aydi [12] on the class of quasi-partial metric spaces.
(4) If we take $q_1(x,y)=q_2(x,y)$ for all $x,y\in X$ and $a_i=0$ $(i=1,2,4,6,7,8,9,10)$, then Corollary 2.10 extends Theorem 2.4 of Aydi [12] on the class of quasi-partial metric spaces.
(5) If we take $q_1(x,y)=q_2(x,y)$ for all $x,y\in X$ and $a_i=0$ $(i=1,2,3,4,5,6,8,10)$, then Corollary 2.10 extends Theorem 2.5 of Aydi [12] on the class of quasi-partial metric spaces.
(6) If we take $q_1(x,y)=q_2(x,y)$ for all $x,y\in X$, $a_3=a_9$ and $a_i=0$ $(i=1,2,4,5,6,7,8,10)$, then Corollary 2.10 extends Corollary 2.6 of Aydi [12] on the class of quasi-partial metric spaces.
(7) If we take $q_1(x,y)=q_2(x,y)$ for all $x,y\in X$, $a_7=a_9$ and $a_i=0$ $(i=1,2,3,4,5,6,8,10)$, then Corollary 2.10 extends Corollary 2.7 of Aydi [12] on the class of quasi-partial metric spaces.

Corollary 2.11 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x,y)\le q_1(x,y)$, for all $x,y\in X$, and $F:X\times X\to X$ be a mapping. Suppose that there exists $k\in[0,1)$ such that the condition*

$$q_1\bigl(F(x,y),F(u,v)\bigr)+q\bigl(F(y,x),F(v,u)\bigr)\le k\bigl[q_2(x,u)+q_2(y,v)\bigr] \tag{2.36}$$

holds for all $x,y,u,v\in X$. If (X,q_1) is a complete quasi-partial metric space. Then the mapping F has a unique coupled fixed point of the form (u,u).

Remark 2.8 If we take $q_1(x,y)=q_2(x,y)$ for all $x,y\in X$, then Corollary 2.11 is reduced to Corollary 2.7 of Shatanawi and Pitea [38].

Corollary 2.12 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x,y)\le q_1(x,y)$, for all $x,y\in X$, and $F:X\times X\to X$ be a mapping. Suppose that there exists $k\in[0,1)$ such that the* *condition*

$$q_1\big(F(x,y),F(u,v)\big)+q\big(F(y,x),F(v,u)\big)\le k\big[q_2\big(x,F(x,y)\big)+q_2\big(y,F(y,x)\big)\big] \tag{2.37}$$

holds for all $x, y, u, v \in X$. If (X, q_1) is a complete quasi-partial metric space, then the mapping F has a unique coupled fixed point of the form (u, u).

Remark 2.9 If we take $q_1(x, y) = q_2(x, y)$ for all $x, y \in X$, then Corollary 2.12 is reduced to Corollary 2.8 of Shatanawi and Pitea [38].

Corollary 2.13 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x, y) \le q_1(x, y)$, for all $x, y \in X$, and $F : X \times X \to X$ be a mapping. Suppose that there exists $k \in [0, 1)$ such that the condition*

$$q_1\big(F(x,y),F(u,v)\big)+q\big(F(y,x),F(v,u)\big)\le k\big[q_2\big(u,F(u,v)\big)+q_2\big(v,F(v,u)\big)\big] \tag{2.38}$$

holds for all $x, y, u, v \in X$. If (X, q_1) is a complete quasi-partial metric space, then the mapping F has a unique coupled fixed point of the form (u, u).

Remark 2.10 If we take $q_1(x, y) = q_2(x, y)$ for all $x, y \in X$, then Corollary 2.13 is reduced to Corollary 2.9 of Shatanawi and Pitea [38].

Corollary 2.14 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x, y) \le q_1(x, y)$, for all $x, y \in X$, and $F : X \times X \to X$ be a mapping. Suppose that there exists $k \in [0, \frac{1}{2})$ such that the condition*

$$q_1\big(F(x,y),F(u,v)\big)+q\big(F(y,x),F(v,u)\big)\le k\big[q_2\big(x,F(u,v)\big)+q_2\big(y,F(v,u)\big)\big] \tag{2.39}$$

holds for all $x, y, u, v \in X$. If (X, q_1) is a complete quasi-partial metric space, then the mapping F has a unique coupled fixed point of the form (u, u).

Corollary 2.15 *Let q_1 and q_2 be two quasi-metrics on X such that $q_2(x, y) \le q_1(x, y)$, for all $x, y \in X$, and $F : X \times X \to X$ be a mapping. Suppose that there exists $k \in [0, 1)$ such that the condition*

$$q_1\big(F(x,y),F(u,v)\big)+q\big(F(y,x),F(v,u)\big)\le k\big[q_2\big(u,F(x,y)\big)+q_2\big(v,F(y,x)\big)\big] \tag{2.40}$$

holds for all $x, y, u, v \in X$. If (X, q_1) is a complete quasi-partial metric space, then the mapping F has a unique coupled fixed point of the form (u, u).

Now, we introduce an example to support our results.

Example 2.1 Let $X = [0, 1]$, and two quasi-partial metrics q_1, q_2 on X be given as

$$q_1(x,y)=|x-y|+x \quad \text{and} \quad q_2(x,y)=\frac{1}{2}\big(|x-y|+x\big)$$

for all $x, y \in X$. Also, define $F : X \times X \to X$ and $g : X \to X$ as

$$F(x,y) = \frac{x+y}{16} \quad \text{and} \quad gx = \frac{x}{2}$$

for all $x, y \in X$. Then

(1) (X, q_1) is a complete quasi-partial metric space.
(2) $F(X \times X) \subset X$.
(3) F and g is w-compatible.
(4) For any $x, y, u, v \in X$, we have

$$q_1\big(F(x,y), F(u,v)\big) + q_1\big(F(y,x) + F(v,u)\big) \leq \frac{1}{2}\big(q_2(gx,gu) + q_2(gy,gv)\big).$$

Proof The proofs of (1), (2), and (3) are clear. Next we show that (4). In fact, for $x, y, u, v \in X$, we have

$$\begin{aligned}
&q_1\big(F(x,y), F(u,v)\big) + q_1\big(F(y,x) + F(v,u)\big) \\
&\quad = q_1\left(\frac{x+y}{16}, \frac{u+v}{16}\right) + q_1\left(\frac{y+x}{16}, \frac{v+u}{16}\right) \\
&\quad = \frac{1}{8}\big(\big|x+y-(u+v)\big| + (x+y)\big) \\
&\quad = \frac{1}{4}\left(\left|\frac{1}{2}(x+y) - \frac{1}{2}(u+v)\right| + \frac{1}{2}(x+y)\right) \\
&\quad \leq \frac{1}{4}\left(\left|\frac{1}{2}x - \frac{1}{2}u\right| + \frac{1}{2}x + \left|\frac{1}{2}y - \frac{1}{2}v\right| + \frac{1}{2}y\right) \\
&\quad = \frac{1}{2}\big(q_2(gx,gu) + q_2(gy,gv)\big).
\end{aligned}$$

Thus, F and g satisfy all the hypotheses of Corollary 2.3. So, F and g have a unique common coupled fixed point. Here $(0,0)$ is the unique common coupled fixed point of F and g. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this work. Both authors read and approved the final manuscript.

Author details
[1]Institute of Applied Mathematics and Department of Mathematics, Hangzhou Normal University, Hangzhou, Zhejiang 310036, China. [2]College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, China.

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