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Correction: Nonlinear quasi-contractions in non-normal cone metric spaces

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Abstract
In the note we correct some errors that appeared in the article (Jiang and Li in Fixed Point Theory Appl. 2014:165, 2014).
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Correction

Upon critical examination of the main results and their proofs in [1], we note some critical errors under the conditions of the main theorem and its proof in our article [1].

In this note, we would like to supplement some essential conditions, which will ensure that the mapping B is well defined, to achieve our claim.

The following theorem is a slight modification of [1, Theorem 1].

Theorem 1 *Let (X,d) be a complete cone metric space over a solid cone P of a Banach space $(E,\|\cdot\|)$ and $T: X\to X$ a quasi-contraction (i.e., there exists a mapping $A: P\to P$ such that*

$$d(Tx,Ty)\preceq Au,\quad \forall x,y\in X, \tag{1}$$

where $u\in\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}$). Assume that $A: P\to P$ is a nondecreasing, continuous and subadditive (i.e., $A(u+v)\preceq Au+Av$ for each $u,v\in P$) mapping with $A\theta=\theta$ such that

$$\sum_{i=0}^{\infty}\|A^iu\|<\infty,\quad \forall u\in P. \tag{2}$$

If B is continuous at θ, where $Bu=\sum_{i=0}^{\infty}A^iu$ for each $u\in P$, then T has a unique fixed point $x^*\in X$, and for each $x_0\in X$, the Picard iterative sequence $\{x_n\}$ converges to x^*, where $x_n=T^nx_0$ for each n.

Remark 1 In the case that the normed vector space $(E,\|\cdot\|)$ is complete, if (2) holds then the mapping B is well defined. In fact, fix $u\in P$, let $s_n=\sum_{i=0}^{n}A^iu$ and $S_n=\sum_{i=0}^{n}\|A^iu\|$. By (2), we get $\lim_{n\to\infty}S_n=\sum_{i=0}^{\infty}\|A^iu\|$ and hence $\{S_n\}$ is a Cauchy sequence of reals. Note that $\|s_m-s_n\|=\|\sum_{i=n+1}^{m}A^iu\|\le\sum_{i=n+1}^{m}\|A^iu\|=S_m-S_n$ for each $m>n$, then $\{s_n\}$ is a Cauchy sequence in E. Moreover, by the completeness of E, $\{s_n\}$ is convergent. This implies that $Bu = \lim_{n\to\infty} \sum_{i=0}^{n} A^i u$ for each u, *i.e.*, B is well defined. However, in [1, Theorem 1] the normed vector space E is not assumed to be complete, then $\{s_n\}$ may not be convergent, and consequently, B may be not meaningful.

Remark 2 (i) In [1] the authors claim that (see (4) in [1])

$$BA = AB, \tag{3}$$

which plays an important role in the proof of [1, Theorem 1]. However, if A is a nonlinear mapping, the above claim may not hold. For example, let $E = P = [0, a]$ and $A(t) = t^2$ for each $t \in P$, where $0 < a < 1$. It is clear that $A : P \to P$ is nonlinear. Note that $A^0(t) = t$ and $A^i(t) = t^{2^i}$ $(i = 1, 2, 3, \ldots)$, then $\sum_{i=0}^{\infty} A^i(t) = t + \sum_{i=1}^{\infty} t^{2^i} \le t + \sum_{i=1}^{\infty} t^{2i} = t + \frac{t^2}{1-t^2}$ for each $t \in [0, a]$, and hence $\sum_{i=0}^{\infty} A^i(t)$ is convergent for each $t \in [0, a]$. This implies that the function $B(t)$ is well defined, where $B(t) = \sum_{i=0}^{\infty} A^i(t)$ for each $t \in [0, a]$. For each $t \in [0, a]$, we have $AB(t) = \sqrt{t + \sum_{i=1}^{\infty} t^{2^i}}$ and $BA(t) = \sum_{i=1}^{\infty} t^{2^i}$. Suppose that there exists $t_0 \in (0, \frac{\sqrt{2}}{2}]$ such that $AB(t_0) = BA(t_0)$, and set $b = BA(t_0)$. Then we have $0 < b \le \frac{t_0^2}{1-t_0^2}$ and $t_0 + b = b^2$. Solve the equation $t_0 + b = b^2$, then $b = \frac{1+\sqrt{1+4t_0}}{2}$ by $b > 0$. Thus we get $1 < \frac{1+\sqrt{1+4t_0}}{2} = b \le \frac{t_0^2}{1-t_0^2} \le 1$, a contradiction. Hence $BA(t) \ne AB(t)$ for each $t \in (0, \frac{\sqrt{2}}{2}]$. This shows that (3) does not hold.

(ii) Note that A is not confined to a linear mapping in [1, Theorem 1], then from (i) we know that (3) may not hold, and consequently, the proof of [1, Theorem 1] is not finished yet. In order to complete its proof, we add the continuity of A to Theorem 1.

(iii) Suppose that E is a Banach space and A is a continuous and subadditive mapping such that (2) is satisfied, then by Remark 1 we get

$$\begin{aligned} ABu &= A\left(\lim_{n\to\infty} \sum_{i=0}^{n} A^i u\right) = \lim_{n\to\infty} A\left(\sum_{i=0}^{n} A^i u\right) \\ &\preceq \lim_{n\to\infty} \left(\sum_{i=0}^{n} A^{i+1} u\right) = \sum_{i=1}^{\infty} A^i u = BAu, \quad \forall u \in P. \end{aligned} \tag{4}$$

In what follows, we shall complete the proof of Theorem 1 by using (4) instead of (3). Since there are too many changes required for the proof of [1, Theorem 1], we present the full proof of Theorem 1 as follows.

Proof of Theorem 1 It follows from (2) and Remark 1 that the mapping B is well defined. Clearly, B is a nondecreasing and subadditive mappings with $B(P) \subset P$ and $B\theta = \theta$ since A is nondecreasing and subadditive, $A(P) \subset (P)$ and $A\theta = \theta$. We claim that for each $n \ge 1$,

$$d(x_i, x_j) \preceq BAd(x_0, x_1), \quad \forall 1 \le i, j \le n. \tag{5}$$

In the following we shall show this claim by induction.

If $n = 1$, then $i = j = 1$, and so the claim is trivial.

Assume that (5) holds for n. To prove (5) holds for $n + 1$, it suffices to show

$$d(x_{i_0}, x_{n+1}) \preceq BAd(x_0, x_1), \quad \forall 1 \le i_0 \le n. \tag{6}$$

By (1),

$$d(x_{i_0},x_{n+1}) \preceq Au, \tag{7}$$

where

$$u \in \bigl\{d(x_{i_0-1},x_n), d(x_{i_0-1},x_{i_0}), d(x_n,x_{n+1}), d(x_{i_0-1},x_{n+1}), d(x_n,x_{i_0})\bigr\}.$$

Consider the case that $i_0 = 1$.

If $u = d(x_0,x_n)$, then by the triangle inequality, the nondecreasing property, subadditivity of A, the definition of B, (4), (5), and (7)

$$\begin{aligned} d(x_{i_0},x_{n+1}) &\preceq Ad(x_0,x_n) \preceq A\bigl[d(x_0,x_1) + d(x_1,x_n)\bigr] \\ &\preceq A\bigl[d(x_0,x_1) + BAd(x_0,x_1)\bigr] \preceq Ad(x_0,x_1) + ABAd(x_0,x_1) \\ &\preceq Ad(x_0,x_1) + BA^2d(x_0,x_1) = Ad(x_0,x_1) + \sum_{i=2}^{\infty} A^i d(x_0,x_1) \\ &= \sum_{i=1}^{\infty} A^i d(x_0,x_1) = BAd(x_0,x_1), \end{aligned}$$

i.e., (6) holds.

If $u = d(x_0,x_1)$, then by the definition of B and (7)

$$d(x_{i_0},x_{n+1}) \preceq Ad(x_0,x_1) \preceq \sum_{i=1}^{\infty} A^i d(x_0,x_1) = BAd(x_0,x_1),$$

i.e., (6) holds.

If $u = d(x_0,x_{n+1})$, then by the triangle inequality, the nondecreasing property and subadditivity of A, and (7)

$$\begin{aligned} d(x_{i_0},x_{n+1}) &\preceq Ad(x_0,x_{n+1}) \preceq A\bigl[d(x_0,x_1) + d(x_{i_0},x_{n+1})\bigr] \\ &\preceq Ad(x_0,x_1) + Ad(x_{i_0},x_{n+1}). \end{aligned}$$

Acting on the above inequality with B, by the nondecreasing property and subadditivity of B

$$\begin{aligned} Bd(x_{i_0},x_{n+1}) &\preceq B\bigl[Ad(x_0,x_1) + Ad(x_{i_0},x_{n+1})\bigr] \\ &\preceq BAd(x_0,x_1) + BAd(x_{i_0},x_{n+1}), \end{aligned}$$

which together with the definition of B implies that

$$d(x_{i_0},x_{n+1}) = Bd(x_{i_0},x_{n+1}) - BAd(x_{i_0},x_{n+1}) \preceq BAd(x_{i_0},x_{n+1}),$$

i.e., (6) holds.

If $u = d(x_n, x_{i_0})$, then by the definition and the nondecreasing property of A, (4), (5), and (7)

$$\begin{aligned} d(x_{i_0}, x_{n+1}) &\preceq Ad(x_{i_0}, x_n) \preceq ABAd(x_0, x_1) \preceq BA^2 d(x_0, x_1) \\ &= \sum_{i=2}^{\infty} A^i d(x_0, x_1) \preceq \sum_{i=1}^{\infty} A^i d(x_0, x_1) = BAd(x_0, x_1), \end{aligned}$$

i.e., (6) holds.

If $u = d(x_n, x_{n+1})$, we set $i_1 = n - 1$, and then by (7)

$$d(x_{i_0}, x_{n+1}) \preceq Ad(x_{i_1}, x_{n+1}). \tag{8}$$

Consider the case that $2 \le i_0 \le n$.

If $u = d(x_{i_0-1}, x_n)$, or $u = d(x_{i_0-1}, x_{i_0})$, or $d(x_n, x_{i_0})$, then by the definition and the nondecreasing property of A, (4), (5), and (7)

$$\begin{aligned} d(x_{i_0}, x_{n+1}) &\preceq Au \preceq ABAd(x_0, x_1) \preceq BA^2 d(x_0, x_1) \\ &= \sum_{i=2}^{\infty} A^i d(x_0, x_1) \preceq \sum_{i=1}^{\infty} A^i d(x_0, x_1) = BAd(x_0, x_1), \end{aligned}$$

i.e., (6) holds.

If $u = d(x_n, x_{n+1})$, or $u = d(x_{i_0-1}, x_{n+1})$, we set $i_1 = n$, or $i_1 = i_0 - 1 \ge 1$, respectively, and then (8) follows.

From the above discussions of both cases, we find the result that either (6) holds, and so the proof of our claim is complete, or there exists $i_1 \in \{1, 2, \dots, n\}$ such that (8) holds. For the latter situation, continuing in a similar way, it will be found as a result that either

$$d(x_{i_1}, x_{n+1}) \preceq BAd(x_0, x_1),$$

which together with the definition and the nondecreasing property of A, (4), and (8), forces that

$$\begin{aligned} d(x_{i_0}, x_{n+1}) &\preceq ABAd(x_0, x_1) \preceq BA^2 d(x_0, x_1) \\ &= \sum_{i=2}^{\infty} A^i d(x_0, x_1) \preceq \sum_{i=1}^{\infty} A^i d(x_0, x_1) = BAd(x_0, x_1), \end{aligned}$$

i.e., (6) holds, and so the proof of our claim is complete; or there exists $i_2 \in \{1, 2, \dots, n\}$ such that

$$d(x_{i_1}, x_{n+1}) \preceq Ad(x_{i_2}, x_{n+1}).$$

If the above procedure ends by the kth step with $k \le n-1$, that is, there exist $k+1$ integers $i_0, i_1, \dots, i_k \in \{1, 2, \dots, n\}$ such that

$$d(x_{i_0}, x_{n+1}) \preceq Ad(x_{i_1}, x_{n+1}),$$

$$d(x_{i_1}, x_{n+1}) \preceq Ad(x_{i_2}, x_{n+1}), \quad \dots,$$

$$d(x_{i_{k-1}}, x_{n+1}) \preceq Ad(x_{i_k}, x_{n+1}),$$
$$d(x_{i_k}, x_{n+1}) \preceq BAd(x_0, x_1),$$

then by the nondecreasing property of A and (4)

$$\begin{aligned} d(x_{i_0}, x_{n+1}) &\preceq A^k BAd(x_0, x_1) \preceq BA^{k+1} d(x_0, x_1) = \sum_{i=k+1}^{\infty} A^i d(x_0, x_1) \\ &\preceq \sum_{i=1}^{\infty} A^i d(x_0, x_1) = BAd(x_0, x_1), \end{aligned}$$

i.e. (6) holds, and so the proof of our claim is complete.

If the above procedure continues more than n steps, then there exist $n + 1$ integers $i_0, i_1, i_n \in \{1, 2, \ldots, n\}$ such that

$$\begin{aligned} &d(x_{i_0}, x_{n+1}) \preceq Ad(x_{i_1}, x_{n+1}), \\ &d(x_{i_1}, x_{n+1}) \preceq Ad(x_{i_2}, x_{n+1}), \quad \ldots, \\ &d(x_{i_{n-1}}, x_{n+1}) \preceq Ad(x_{i_n}, x_{n+1}). \end{aligned} \tag{9}$$

It is clear that $i_0, i_1, i_n \in \{1, 2, \ldots, n\}$ implies there exist two integers $k, l \in \{0, 1, 2, \ldots, n\}$ with $k < l$ such that $i_k = i_l$, then by the nondecreasing property of A and (9)

$$d(x_{i_k}, x_{n+1}) \preceq A^{l-k} d(x_{i_l}, x_{n+1}) = A^{l-k} d(x_{i_k}, x_{n+1}). \tag{10}$$

Acting on (10) with B, by the nondecreasing property of B we get

$$Bd(x_{i_k}, x_{n+1}) \preceq BA^{l-k} d(x_{i_k}, x_{n+1}),$$

which together with the definition of B implies that

$$d(x_{i_k}, x_{n+1}) \preceq \sum_{j=0}^{l-k-1} A^j d(x_{i_k}, x_{n+1}) = Bd(x_{i_k}, x_{n+1}) - BA^{l-k} d(x_{i_k}, x_{n+1}) \preceq \theta \preceq BAd(x_0, x_1),$$

i.e., (6) holds. The proof of our claim is complete.

Note that B and A are nondecreasing and continuous at θ, $B\theta = \theta$ and $A\theta = \theta$, then it follows from Lemma 3 of [1] that for each $\{u_n\} \in P$,

$$Bu_n \xrightarrow{w} \theta, \qquad Au_n \xrightarrow{w} \theta, \qquad BAu_n \xrightarrow{w} \theta, \tag{11}$$

provided that $u_n \xrightarrow{w} \theta$. By (2), we get

$$\lim_{n\to\infty} \|A^n u\| = 0, \quad \forall u \in P. \tag{12}$$

Then in analogy to the proof of [1, Theorem 1], by (5), (11), (12) we can show that

$$d(x_m, x_n) \xrightarrow{w} \theta \quad (n > m \to \infty), \tag{13}$$

and there exists some $x^* \in X$ such that

$$d(x_n, x^*) \xrightarrow{w} \theta \quad (n \to \infty). \tag{14}$$

By (1),

$$d(Tx^*, x^*) \preceq d(x_{n+1}, Tx^*) + d(x_{n+1}, x^*) \preceq Au + d(x_{n+1}, x^*), \quad \forall n, \tag{15}$$

where $u \in \{d(x_n, x^*), d(x_n, x_{n+1}), d(x^*, Tx^*), d(x_n, Tx^*), d(x^*, x_{n+1})\}$.

If $u = d(x_n, x^*)$, or $u = d(x_n, x_{n+1})$, or $u = d(x^*, x_{n+1})$, then it follows from (11), (13), and (14) that $Au \xrightarrow{w} \theta$ and hence $d(Tx^*, x^*) = \theta$ by (15).

If $u = d(x^*, Tx^*)$, then by (15)

$$d(x^*, Tx^*) \preceq Ad(x^*, Tx^*) + d(x_{n+1}, x^*), \quad \forall n.$$

Acting on the above inequality with B, by the nondecreasing and subadditivity of B we get

$$Bd(x^*, Tx^*) \preceq B\big[Ad(x^*, Tx^*) + d(x_{n+1}, x^*)\big] \preceq BAd(x^*, Tx^*) + Bd(x_{n+1}, x^*), \quad \forall n,$$

which together with the definition of B implies that

$$d(x^*, Tx^*) = Bd(x^*, Tx^*) - BAd(x^*, Tx^*) \preceq Bd(x_{n+1}, x^*), \quad \forall n,$$

and hence $d(x^*, Tx^*) = \theta$ since $Bd(x_{n+1}, x^*) \xrightarrow{w} \theta$ by (11) and (14).

If $u = d(x_n, Tx^*)$, then, by the triangle inequality, the nondecreasing property, and subadditivity of A and (15), we get

$$\begin{aligned} d(Tx^*, x^*) &\preceq d(x_{n+1}, x^*) + Ad(x_n, Tx^*) \\ &\preceq d(x_{n+1}, x^*) + A\big[d(x_n, x^*) + d(x^*, Tx^*)\big] \\ &\preceq d(x_{n+1}, x^*) + Ad(x_n, x^*) + Ad(x^*, Tx^*), \quad \forall n. \end{aligned}$$

Acting on the above inequality with B, then by the nondecreasing property and subadditivity of B

$$\begin{aligned} Bd(x^*, Tx^*) &\preceq B\big[d(x_{n+1}, x^*) + Ad(x_n, x^*) + Ad(x^*, Tx^*)\big] \\ &\preceq Bd(x_{n+1}, x^*) + BAd(x_n, x^*) + BAd(x^*, Tx^*), \quad \forall n, \end{aligned}$$

which together with the definition of B implies that

$$d(x^*, Tx^*) = Bd(x^*, Tx^*) - BAd(x^*, Tx^*) \preceq Bd(x_{n+1}, x^*) + BAd(x_n, x^*), \quad \forall n,$$

and hence $d(x^*, Tx^*) = \theta$ since $Bd(x_{n+1}, x^*) \xrightarrow{w} \theta$ and $BAd(x_n, x^*) \xrightarrow{w} \theta$ by (11) and (14). This shows that x^* is a fixed point of T.

If x is another fixed point of T, then by (1)

$$d(x, x^*) = d(Tx, Tx^*) \preceq Au,$$

where $u \in \{d(x,x^*), d(x,Tx), d(x^*,Tx^*), d(x,Tx^*), d(x^*,Tx)\}$. If $u = d(x,Tx)$, or $u = d(x^*,Tx^*)$, then $u = \theta$, and hence $d(x,x^*) = \theta$ since $A\theta = \theta$. If $u = d(x,x^*)$, or $u = d(x,Tx^*)$ or $u = d(x^*,Tx)$, then we must have $u = d(x,x^*)$, and hence $d(x,x^*) \preceq Ad(x,x^*)$. Acting on it with B, by the nondecreasing property of B we get $Bd(x,x^*) \preceq BAd(x,x^*)$. Moreover, by the definition of B, we have $d(x,x^*) = Bd(x,x^*) - BAd(x,x^*) \preceq \theta$ and hence $d(x,x^*) = \theta$. This shows x^* is the unique fixed point of T. The proof is complete. $\square$

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