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Convergence of a regularization algorithm for nonexpansive and monotone operators in Hilbert spaces

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Abstract

Variational inequality, fixed point and generalized equilibrium problems are investigated via a regularization algorithm. It is proved that the sequence generated in the regularization algorithm converges strongly to a common solution of the three problems in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding ones announced by many authors.

Keywords: equilibrium problem; fixed point; variational inequality; zero point

1 Introduction and preliminaries

In this paper, we always assume that H is a real Hilbert space with inner product $\langle x,y\rangle$ and induced norm $\|x\| = \sqrt{\langle x,x\rangle}$ for $x,y \in H$. Let C be a nonempty, closed, and convex subset of H.

Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x,y \in C.$$

Recall that A is said to be strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x,y \in C.$$

For such a case, A is also said to be α-strongly monotone. Recall that A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x,y \in C.$$

For such a case, A is also said to be α-inverse-strongly monotone.

Recall that the classical variational inequality is to find an $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

In this paper, we always use $VI(C,A)$ to denote the solution set of (1.1) and use P_C denote the metric projection from H onto C. It is well known that $x \in C$ is a solution of (1.1) iff x is a fixed point of the mapping $P_C(I - rA)$, where $r > 0$ is a constant, I stands for the identity mapping. If A is strongly monotone and Lipschitz continuous, the existence and uniqueness of solutions of equilibrium (1.1) is guaranteed by the Banach contraction principle.

Recall that a set-valued mapping $M : H \rightrightarrows H$ is said to be monotone iff, for all $x, y \in H$, $f \in Mx$, and $g \in My$ imply $\langle x - y, f - g\rangle > 0$. M is maximal iff the graph Graph(M) of R is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for all $(y, g) \in \mathrm{Graph}(M)$ implies $f \in Rx$.

Let $S : C \to C$ be a mapping. $F(S)$ stands for the fixed point set of S; that is, $F(S) := \{x \in C : x = Sx\}$.

Recall that S is said to be contractive iff there exists a constant $\alpha \in (0, 1)$ such that

$$\|Sx - Sy\| \leq \alpha\|x - y\|, \quad \forall x, y \in C.$$

For such a case, S is also said to be α-contractive. We know that the mapping enjoys a unique fixed point and Picard's algorithm can be employed to approximate its unique fixed point.

Recall that S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

If C is a closed, bounded and convex subset of H, then $F(S)$ is not empty; see [1].

Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings and $\{\gamma_i\}$ be a nonnegative real sequence with $0 \leq \gamma_i < 1$, $\forall i \geq 1$. For $n \geq 1$, define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&U_{n,n+1} = I, \\
&U_{n,n} = \gamma_n S_n U_{n,n+1} + (1 - \gamma_n)I, \\
&U_{n,n-1} = \gamma_{n-1} S_{n-1} U_{n,n} + (1 - \gamma_{n-1})I, \\
&\vdots \\
&U_{n,k} = \gamma_k S_k U_{n,k+1} + (1 - \gamma_k)I, \\
&U_{n,k-1} = \gamma_{k-1} S_{k-1} U_{n,k} + (1 - \gamma_{k-1})I, \\
&\vdots \\
&U_{n,2} = \gamma_2 S_2 U_{n,3} + (1 - \gamma_2)I, \\
&W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1 - \gamma_1)I.
\end{aligned} \tag{1.2}$$

Such a mapping W_n is nonexpansive from C to C and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$; see [2] and the references therein.

Let $T : C \to H$ be a monotone mapping and let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. We consider the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x,y) + \langle Tx, y-x\rangle \geq 0, \quad \forall y \in C. \tag{1.3}$$

In this paper, the set of such $x \in C$ is denoted by $EP(F,T)$, *i.e.*,

$$\mathrm{GEP}(F,A) = \bigl\{x \in C : F(x,y) + \langle Tx, y-x\rangle \geq 0,\ \forall y \in C\bigr\}.$$

If $T \equiv 0$, the zero mapping, then the problem (1.3) is reduced to the following equilibrium problem [3]:

$$\text{Find } x \in C \text{ such that } F(x,y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

In this paper, the set of such an $x \in C$ is denoted by $EP(F)$.

If $F \equiv 0$, then the problem (1.3) is reduced to the classical variational inequality (1.1).

To study equilibrium problems (1.3) and (1.4), we may assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x,y \in C$;

(A3) for each $x,y,z \in C$,

$$\limsup_{t\downarrow 0} F\bigl(tz + (1-t)x, y\bigr) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and weakly lower semi-continuous.

Many important problems have reformulations which require finding solutions of equilibriums (1.3) and (1.4), for instance, image recovery, inverse problems, network allocation, transportation problems and optimization problems; see [3–11] and the references therein. For solving solutions of equilibriums (1.3) and (1.4), regularization methods recently have been extensively studied; see [11–28] and the references therein.

In this paper, motivated and inspired by the research going on in this direction, we study the variational inequality (1.1), and the fixed point and equilibrium problem (1.3) based on a regularization algorithm. It is proved that the sequence generated in the regularization algorithm converges strongly to a common solutions of the three problems in the framework of Hilbert spaces. The results presented in this paper improve and extend the corresponding results in Chang *et al.* [11], Takahashi and Takahashi [13] and Hao [29].

The following lemmas play an important role in our paper.

Lemma 1.1 [3] *Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying* (A1)-(A4). *Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0, \quad \forall y \in C.$$

Define a mapping $T_r : H \to C$ as follows:

$$T_r x = \biggl\{z \in C : F(z,y) + \frac{1}{r}\langle y-z, z-x\rangle \geq 0,\ \forall y \in C\biggr\}, \quad x \in H,$$

then the following conclusions hold:

(1) T_r *is single-valued;*
(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(F)$;
(4) $EP(F)$ *is closed and convex.*

Lemma 1.2 [30] *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*
(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 1.3 [2] *Let* $\{S_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with a nonempty common fixed point set and let* $\{\gamma_i\}$ *be a real sequence such that* $0 < \gamma_i \leq l < 1$, *where* l *is some real number,* $\forall i \geq 1$. *Then*
(1) W_n *is nonexpansive and* $F(W_n) = \bigcap_{i=1}^{\infty} F(S_i)$, *for each* $n \geq 1$;
(2) *for each* $x \in C$ *and for each positive integer* k, *the limit* $\lim_{n\to\infty} U_{n,k}$ *exists.*
(3) *the mapping* $W : C \to C$ *defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x, \quad x \in C, \tag{1.5}$$

is a nonexpansive mapping satisfying $F(W) = \bigcap_{i=1}^{\infty} F(S_i)$ *and it is called the* W*-mapping generated by* $S_1, S_2, \ldots$ *and* $\gamma_1, \gamma_2, \ldots$.

Lemma 1.4 [31] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in* H *and let* $\{\beta_n\}$ *be a sequence in* $(0,1)$ *with* $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. *Suppose that* $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all* $n \geq 0$ *and*

$$\limsup_{n\to\infty}\big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5 [11] *Let* $\{S_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with a nonempty common fixed point set and let* $\{\gamma_i\}$ *be a real sequence such that* $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$. *If* K *is any bounded subset of* C, *then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

Throughout this paper, we always assume that $0 < \gamma_i \leq l < 1$, $\forall i \geq 1$.

Lemma 1.6 [10] *Let $A : C \to H$ a Lipschitz monotone mapping and let $N_C x$ be the normal cone to C at $x \in C$; that is, $N_C x = \{y \in H : \langle x - u, y\rangle, \forall u \in C\}$. Define*

$$Dx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset & x \notin C. \end{cases}$$

Then D is maximal monotone and $0 \in Dx$ if and only if $x \in VI(C, A)$.

2 Main results

Theorem 2.1 *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and let $f : C \to C$ be a κ-contraction. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let $T : C \to H$ be a τ-inverse-strongly monotone mapping. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Sigma = \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{GEP}(F, T) \cap VI(C, A) \cap VI(C, B)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences in $(0, 1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{r_n\}$, $\{s_n\}$, and $\{\lambda_n\}$ be positive number sequences. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = P_C(u_n - s_n B u_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n P_C(y_n - r_n A y_n) + \gamma_n x_n, \quad \forall n \geq 1, \end{cases} \tag{2.1}$$

where $\{u_n\}$ is such that $F(u_n, y) + \langle Tx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0$, $\forall y \in C$, and $\{W_n\}$ is the sequence generated in (1.5). *Assume that the following restrictions hold*:

(a) $0 < a \leq \lambda_n \leq b < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,
(b) $0 < a' \leq r_n \leq b' < 2\alpha$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,
(c) $0 < a'' \leq s_n \leq b'' < 2\beta$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$,
(d) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(e) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

where a, a', a'', b, b', and b'' are real constants. Then $\{x_n\}$ converges strongly to $\bar{x} \in \Sigma$, which solves uniquely the following variational inequality:

$$\langle \bar{x} - f(\bar{x}), \bar{x} - x \rangle \leq 0, \quad \forall x \in \Sigma.$$

Proof Since A is inverse-strongly monotone, we see from restriction (b) that

$$\begin{aligned} &\|(I - r_n A)x - (I - r_n A)y\|^2 \\ &\quad = \|x - y\|^2 - 2r_n\langle x - y, Ax - Ay\rangle + r_n^2\|Ax - Ay\|^2 \\ &\quad \leq \|x - y\|^2 - 2r_n\alpha\|Ax - Ay\|^2 + r_n^2\|Ax - Ay\|^2 \\ &\quad = \|x - y\|^2 + r_n(r_n - 2\alpha)\|Ax - Ay\|^2 \\ &\quad \leq \|x - y\|^2, \quad \forall x, y \in C. \end{aligned}$$

This shows that $I - r_n A$ is nonexpansive. In the same way, we find that $I - s_n B$ and $I - \lambda_n T$ are nonexpansive. Note that u_n can be re-written as $u_n = T_{\lambda_n}(I - \lambda_n T)x_n$. Let $x^* \in \Sigma$. It follows that

$$\|u_n - x^*\| \le \|(I - \lambda_n T)x_n - (I - \lambda_n T)x^*\| \le \|x_n - x^*\|.$$

Putting $z_n = P_C(y_n - r_n)Ay_n$, we see that $\|z_n - x^*\| \le \|y_n - x^*\| \le \|x_n - x^*\|$.

$$\begin{aligned}\|x_{n+1} - x^*\| &\le \alpha_n \|f(y_n) - x^*\| + \beta_n \|W_n z_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &\le \alpha_n \kappa \|y_n - x^*\| + \alpha_n \|f(x^*) - x^*\| + \beta_n \|z_n - x^*\| + \gamma_n \|x_n - x^*\| \\ &\le \bigl(1 - \alpha_n(1-\kappa)\bigr)\|x_n - x^*\| + \alpha_n(1-\kappa)\frac{\|f(x^*) - x^*\|}{1-\kappa}.\end{aligned}$$

This implies that

$$\|x_n - x^*\| \le \max\left\{\|x_1 - x^*\|, \frac{\|f(x^*) - x^*\|}{1-\kappa}\right\} < \infty.$$

This yields the result that the sequence $\{x_n\}$ is bounded, and so are $\{y_n\}$, $\{z_n\}$, and $\{u_n\}$. Without loss of generality, we can assume that there exists a bounded set $K \subset C$ such that $x_n, y_n, z_n, u_n \in K$. Since $u_n = T_{\lambda_n}(I - \lambda_n)x_n$, we find that

$$F(u_{n+1}, y) + \frac{1}{\lambda_{n+1}}\langle y - u_{n+1}, u_{n+1} - (I - r_{n+1}T)x_{n+1}\rangle \ge 0, \quad \forall y \in C, \tag{2.2}$$

and

$$F(u_n, y) + \frac{1}{\lambda_n}\langle y - u_n, u_n - (I - \lambda_n T)x_n\rangle \ge 0, \quad \forall y \in C. \tag{2.3}$$

Let $y = u_n$ in (2.2) and $y = u_{n+1}$ in (2.3). By adding up these two inequalities, we obtain

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - (I - \lambda_n A_3)x_n - \frac{\lambda_n}{\lambda_{n+1}}\bigl(u_{n+1} - (I - \lambda_{n+1}A_3)x_{n+1}\bigr)\right\rangle \ge 0.$$

This implies that

$$\begin{aligned}\|u_{n+1} - u_n\|^2 &\le \Bigl\langle u_{n+1} - u_n, (I - \lambda_{n+1}T)x_{n+1} - (I - \lambda_n T)x_n \\ &\quad + \left(1 - \frac{\lambda_n}{\lambda_{n+1}}\right)\bigl(u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\bigr)\Bigr\rangle \\ &\le \|u_{n+1} - u_n\|\Bigl(\|(I - \lambda_{n+1}T)x_{n+1} - (I - \lambda_n T)x_n\| \\ &\quad + \left|1 - \frac{\lambda_n}{\lambda_{n+1}}\right|\|u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\|\Bigr).\end{aligned}$$

It follows that

$$\begin{aligned}\|u_{n+1} - u_n\| &\le \|(I - \lambda_{n+1}T)x_{n+1} - (I - \lambda_n T)x_n\| \\ &\quad + \frac{|\lambda_{n+1} - \lambda_n|}{\lambda_{n+1}}\|u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\| \\ &\le \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|M_1,\end{aligned} \tag{2.4}$$

where M_1 is an appropriate constant such that

$$M_1 = \sup_{n\geq 1}\left\{ \|Tx_n\| + \frac{\|u_{n+1} - (I - \lambda_{n+1}T)x_{n+1}\|}{a} \right\}.$$

It follows from (2.4) that

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \big\|P_C(u_{n+1} - s_{n+1}Bu_{n+1}) - P_C(u_n - s_{n+1}Bu_n)\big\| \\
&\quad + \big\|P_C(u_n - s_{n+1}Bu_n) - P_C(u_n - s_nBu_n)\big\| \\
&\leq \|u_{n+1} - u_n\| + |s_{n+1} - s_n|\|Bu_n\| \\
&\leq \|x_{n+1} - x_n\| + |\lambda_{n+1} - \lambda_n|M_1 + |s_{n+1} - s_n|\|Bu_n\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|z_{n+1} - z_n\| &\leq \big\|P_C(y_{n+1} - r_{n+1}Ay_{n+1}) - P_C(y_n - r_{n+1}Ay_n)\big\| \\
&\quad + \big\|P_C(y_n - r_{n+1}Ay_n) - P_C(y_n - r_nAy_n)\big\| \\
&\leq \|y_{n+1} - y_n\| + |r_{n+1} - r_n|\|Ay_n\| \\
&\leq \|x_{n+1} - x_n\| + M_2\big(|\lambda_{n+1} - \lambda_n| + |s_{n+1} - s_n| + |r_{n+1} - r_n|\big),
\end{aligned} \tag{2.5}$$

where $M_2 = \max\{M_1, \sup_{n\geq 1}\{Ay_n\}, \sup_{n\geq 1}\{Bu_n\}\}$. This implies from (2.5) that

$$\begin{aligned}
&\|W_{n+1}z_{n+1} - W_nz_n\| \\
&\quad\leq \|W_{n+1}z_{n+1} - Wz_{n+1}\| + \|Wz_{n+1} - Wz_n\| + \|Wz_n - W_nz_n\| \\
&\quad\leq \sup_{x\in K}\big\{\|W_{n+1}x - Wx\| + \|Wx - W_nx\|\big\} + \|x_{n+1} - x_n\| \\
&\quad\quad + M_2\big(|\lambda_{n+1} - \lambda_n| + |s_{n+1} - s_n| + |r_{n+1} - r_n|\big),
\end{aligned} \tag{2.6}$$

where K is the bounded subset of C defined above. Let $x_{n+1} = (1 - \gamma_n)q_n + \gamma_nx_n$. It follows that

$$\begin{aligned}
q_{n+1} - q_n &= \frac{\alpha_{n+1}f(y_{n+1}) + \beta_{n+1}W_{n+1}z_{n+1}}{1 - \gamma_{n+1}} - \frac{\alpha_nf(y_n) + \beta_nW_nz_n}{1 - \gamma_n} \\
&= \frac{\alpha_{n+1}}{1 - \gamma_{n+1}}f(y_{n+1}) + \frac{1 - \alpha_{n+1} - \gamma_{n+1}}{1 - \gamma_{n+1}}W_{n+1}z_{n+1} \\
&\quad - \left(\frac{\alpha_n}{1 - \gamma_n}f(y_n) + \frac{1 - \alpha_n - \gamma_n}{1 - \gamma_n}W_nz_n\right) \\
&= \frac{\alpha_{n+1}}{1 - \gamma_{n+1}}\big(f(y_{n+1}) - W_{n+1}z_{n+1}\big) - \frac{\alpha_n}{1 - \gamma_n}\big(f(y_n) - W_nz_n\big) \\
&\quad + W_{n+1}z_{n+1} - W_nz_n.
\end{aligned}$$

By use of (2.6), we find that

$$\begin{aligned}\|q_{n+1}-q_n\| &\le \frac{\alpha_{n+1}}{1-\gamma_{n+1}}\big\|f(y_{n+1})-W_{n+1}z_{n+1}\big\| + \frac{\alpha_n}{1-\gamma_n}\big\|f(y_n)-W_nz_n\big\| \\ &\quad + \|W_{n+1}z_{n+1}-W_nz_n\| \\ &\le \frac{\alpha_{n+1}}{1-\gamma_{n+1}}\big\|f(y_{n+1})-W_{n+1}z_{n+1}\big\| + \frac{\alpha_n}{1-\gamma_n}\big\|f(y_n)-W_nz_n\big\| \\ &\quad + \sup_{x\in K}\big\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\big\} + \|x_{n+1}-x_n\| \\ &\quad + M_2\big(|\lambda_{n+1}-\lambda_n|+|s_{n+1}-s_n|+|r_{n+1}-r_n|\big).\end{aligned}$$

This implies that

$$\begin{aligned}&\|q_{n+1}-q_n\|-\|x_{n+1}-x_n\| \\ &\quad\le \frac{\alpha_{n+1}}{1-\gamma_{n+1}}\big\|f(y_{n+1})-W_{n+1}z_{n+1}\big\| + \frac{\alpha_n}{1-\gamma_n}\big\|f(y_n)-W_nz_n\big\| \\ &\qquad + \sup_{x\in K}\big\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\big\} \\ &\qquad + M_2\big(|\lambda_{n+1}-\lambda_n|+|s_{n+1}-s_n|+|r_{n+1}-r_n|\big).\end{aligned}$$

It follows from restrictions (a)-(e) that

$$\limsup_{n\to\infty}\big(\|q_{n+1}-q_n\|-\|x_{n+1}-x_n\|\big)\le 0.$$

This implies from Lemma 1.4 that $\lim_{n\to\infty}\|q_n-x_n\|=0$. It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.7}$$

Since A is inverse-strongly monotone, we find that

$$\begin{aligned}\big\|z_n-x^*\big\|^2 &\le \big\|(I-r_nA)y_n-(I-r_nA)x^*\big\|^2 \\ &\le \big\|y_n-x^*\big\|^2-2r_n\alpha\big\|Ay_n-Ax^*\big\|^2+r_n^2\big\|Ay_n-Ax^*\big\|^2 \\ &\le \big\|x_n-x^*\big\|^2+r_n(r_n-2\alpha)\big\|Ay_n-Ax^*\big\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\big\|x_{n+1}-x^*\big\|^2 &\le \alpha_n\big\|f(y_n)-x^*\big\|^2+\beta_n\big\|W_nz_n-x^*\big\|^2+\gamma_n\big\|x_n-x^*\big\|^2 \\ &\le \alpha_n\big\|f(y_n)-x^*\big\|^2+\beta_n\big\|z_n-x^*\big\|^2+\gamma_n\big\|x_n-x^*\big\|^2 \\ &\le \alpha_n\big\|f(y_n)-x^*\big\|^2+r_n(r_n-2\alpha)\beta_n\big\|Ay_n-Ax^*\big\|^2+\big\|x_n-x^*\big\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}&r_n(2\alpha-r_n)\beta_n\big\|Ay_n-Ax^*\big\|^2 \\ &\quad\le \alpha_n\big\|f(y_n)-x^*\big\|^2+\big(\big\|x_n-x^*\big\|+\big\|x_{n+1}-x^*\big\|\big)\|x_n-x_{n+1}\|.\end{aligned}$$

By use of the restrictions (a), (d), and (e), we obtain from (2.7)

$$\lim_{n\to\infty} \|Ay_n - Ax^*\| = 0. \tag{2.8}$$

Since the metric projection is firmly nonexpansive, we find that

$$\begin{aligned}
\|z_n - x^*\|^2 &\le \langle (I - r_nA)y_n - (I - r_nA)x^*, z_n - x^*\rangle \\
&= \frac{1}{2}\Bigl\{ \|(I - r_nA)y_n - (I - r_nA)x^*\|^2 + \|z_n - x^*\|^2 \\
&\quad - \|(I - r_nA)y_n - (I - r_nA)x^* - (z_n - x^*)\|^2 \Bigr\} \\
&\le \frac{1}{2}\Bigl\{ \|y_n - x^*\|^2 + \|z_n - x^*\|^2 - \|y_n - z_n - r_n(Ay_n - Ax^*)\|^2 \Bigr\} \\
&\le \frac{1}{2}\Bigl\{ \|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2r_n\|y_n - z_n\|\|Ay_n - Ax^*\| \Bigr\}.
\end{aligned}$$

Hence, we have

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|y_n - z_n\|^2 + 2r_n\|y_n - z_n\|\|Ay_n - Ax^*\|.$$

This further implies that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n\|f(y_n) - x^*\|^2 + \beta_n\|W_nz_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 \\
&\le \alpha_n\|f(y_n) - x^*\|^2 + \beta_n\|z_n - x^*\|^2 + \gamma_n\|x_n - x^*\|^2 \\
&\le \alpha_n\|f(y_n) - x^*\|^2 - \beta_n\|y_n - z_n\|^2 + 2r_n\|y_n - z_n\|\|Ay_n - Ax^*\| \\
&\quad + \|x_n - x^*\|^2,
\end{aligned}$$

which yields

$$\begin{aligned}
\beta_n\|y_n - z_n\|^2 &\le \alpha_n\|f(y_n) - x^*\|^2 + 2r_n\|y_n - z_n\|\|Ay_n - Ax^*\| \\
&\quad + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\le \alpha_n\|f(y_n) - x^*\|^2 + 2r_n\|y_n - z_n\|\|Ay_n - Ax^*\| \\
&\quad + (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_{n+1} - x_n\|.
\end{aligned}$$

By use of restrictions (b), (d), and (e), we find from (2.7) that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{2.9}$$

Since B is inverse-strongly monotone, we find that

$$\begin{aligned}
\|y_n - x^*\|^2 &\le \|(I - s_nB)u_n - (I - s_nB)x^*\|^2 \\
&\le \|u_n - x^*\|^2 - 2s_n\beta\|Bu_n - Bx^*\|^2 + s_n^2\|Bu_n - Bx^*\|^2 \\
&\le \|x_n - x^*\|^2 + s_n(s_n - 2\beta)\|Bu_n - Bx^*\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(y_n)-x^*\|^2+\beta_n\|W_nz_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\ &\le \alpha_n\|f(y_n)-x^*\|^2+\beta_n\|y_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\ &\le \alpha_n\|f(y_n)-x^*\|^2+s_n(s_n-2\beta)\beta_n\|Bu_n-Bx^*\|^2+\|x_n-x^*\|^2.\end{aligned}$$

Hence, we have

$$\begin{aligned}&s_n(2\beta-s_n)\beta_n\|Bu_n-Bx^*\|^2\\ &\quad\le \alpha_n\|f(y_n)-x^*\|^2+\bigl(\|x_n-x^*\|+\|x_{n+1}-x^*\|\bigr)\|x_n-x_{n+1}\|.\end{aligned}$$

By use of the restrictions (c), (d), and (e), we obtain from (2.7)

$$\lim_{n\to\infty}\|Bu_n-Bx^*\|=0. \tag{2.10}$$

Since the metric projection is firmly nonexpansive, we find that

$$\begin{aligned}\|y_n-x^*\|^2 &\le \bigl\langle (I-s_nB)u_n-(I-s_nB)x^*, y_n-x^*\bigr\rangle\\ &= \frac{1}{2}\Bigl\{\|(I-s_nB)u_n-(I-s_nB)x^*\|^2+\|y_n-x^*\|^2\\ &\quad -\|(I-s_nB)y_n-(I-s_nB)x^*-(y_n-x^*)\|^2\Bigr\}\\ &\le \frac{1}{2}\Bigl\{\|u_n-x^*\|^2+\|y_n-x^*\|^2-\|u_n-y_n-s_n(Bu_n-Bx^*)\|^2\Bigr\}\\ &\le \frac{1}{2}\Bigl\{\|x_n-x^*\|^2+\|y_n-x^*\|^2-\|u_n-y_n\|^2\\ &\quad +2s_n\|u_n-y_n\|\|Bu_n-Bx^*\|\Bigr\}.\end{aligned}$$

Hence, we have

$$\|y_n-x^*\|^2\le \|x_n-x^*\|^2-\|u_n-y_n\|^2+2s_n\|u_n-y_n\|\|Bu_n-Bx^*\|.$$

This further implies that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(y_n)-x^*\|^2+\beta_n\|W_nz_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\ &\le \alpha_n\|f(y_n)-x^*\|^2+\beta_n\|y_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\ &\le \alpha_n\|f(y_n)-x^*\|^2-\beta_n\|u_n-y_n\|^2+2s_n\|u_n-y_n\|\|Bu_n-Bx^*\|\\ &\quad +\|x_n-x^*\|^2,\end{aligned}$$

which yields

$$\begin{aligned}\beta_n\|u_n-y_n\|^2 &\le \alpha_n\|f(y_n)-x^*\|^2+2s_n\|u_n-y_n\|\|Bu_n-Bx^*\|\\&\quad+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\&\le \alpha_n\|f(y_n)-x^*\|^2+2s_n\|u_n-y_n\|\|Bu_n-Bx^*\|\\&\quad+\big(\|x_n-x^*\|+\|x_{n+1}-x^*\|\big)\|x_{n+1}-x_n\|.\end{aligned}$$

By use of restrictions (c), (d), and (e), we find from (2.7) that

$$\lim_{n\to\infty}\|y_n-u_n\|=0. \tag{2.11}$$

Since T is inverse-strongly monotone, we find that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(y_n)-x^*\|^2+\beta_n\|u_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\&\le \alpha_n\|f(y_n)-x^*\|^2+\beta_n\|x_n-x^*-\lambda_n(Tx_n-Tx^*)\|^2+\gamma_n\|x_n-x^*\|^2\\&\le \alpha_n\|f(y_n)-x^*\|^2+\|x_n-x^*\|^2-\lambda_n\beta_n(2\tau-\lambda_n)\|Tx_n-Tx^*\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}&\lambda_n\beta_n(2\tau-\lambda_n)\|Tx_n-Tx^*\|^2\\&\quad\le \alpha_n\|f(y_n)-x^*\|^2+\big(\|x_n-x^*\|+\|x_{n+1}-x^*\|\big)\|x_n-x_{n+1}\|.\end{aligned}$$

In view of the restrictions (a), (d), and (e), we see from (2.7) that

$$\lim_{n\to\infty}\|Tx_n-Tx^*\|=0. \tag{2.12}$$

Since T_{λ_n} is firmly nonexpansive, we find that

$$\begin{aligned}\|u_n-x^*\|^2 &= \|T_{\lambda_n}(I-\lambda_nT)x_n-T_{\lambda_n}(I-\lambda_nT)x^*\|^2\\&\le \langle (I-\lambda_nT)x_n-(I-\lambda_nT)x^*,u_n-x^*\rangle\\&\le \frac{1}{2}\big(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n\|^2\\&\quad+2\lambda_n\|Tx_n-Tx^*\|\|x_n-u_n\|\big).\end{aligned}$$

This in turn implies that

$$\|u_n-x^*\|^2\le\|x_n-x^*\|^2-\|x_n-u_n\|^2+2\lambda_n\|Tx_n-Tx^*\|\|x_n-u_n\|.$$

It follows that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \alpha_n\|f(y_n)-x^*\|^2+\beta_n\|u_n-x^*\|^2+\gamma_n\|x_n-x^*\|^2\\&\le \alpha_n\|f(y_n)-x^*\|^2-\beta_n\|x_n-u_n\|^2\\&\quad+2\lambda_n\|Tx_n-Tx^*\|\|x_n-u_n\|+\|x_n-x^*\|^2.\end{aligned}$$

By use of restrictions (a), (d), and (e), we see from (2.7) and (2.12) that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.13}$$

Next, we prove that

$$\limsup_{n\to\infty} \langle f(\bar{x}) - \bar{x}, x_n - \bar{x} \rangle \le 0,$$

where $\bar{x} = P_\Sigma f(\bar{x})$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (f - I)\bar{x}, x_n - \bar{x} \rangle = \lim_{i\to\infty} \langle (f - I)\bar{x}, x_{n_i} - \bar{x} \rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. Since

$$\beta_n \| W_n z_n - x_n \| \le \|x_n - x_{n+1}\| + \alpha_n \left\| f(y_n) - x_n \right\|.$$

In view of the restrictions (d) and (e), we obtain from (2.7)

$$\lim_{n\to\infty} \| W_n z_n - x_n \| = 0. \tag{2.14}$$

Note that

$$\| W_n z_n - z_n \| \le \| W_n z_n - x_n \| + \|x_n - u_n\| + \|u_n - y_n\| + \|y_n - z_n\|.$$

In view of (2.8), (2.11), (2.13), and (2.14), we find that

$$\lim_{n\to\infty} \| W_n z_n - z_n \| = 0. \tag{2.15}$$

Suppose the contrary, $w \notin \bigcap_{i=1}^{\infty} F(S_i)$, *i.e.*, $Ww \neq w$. Since $y_{n_i} \rightharpoonup w$, we find from Opial's condition [32] that

$$\begin{aligned} \liminf_{i\to\infty} \|z_{n_i} - w\| &< \liminf_{i\to\infty} \|z_{n_i} - Ww\| \\ &\le \liminf_{i\to\infty} \left\{ \|z_{n_i} - Wz_{n_i}\| + \| Wz_{n_i} - Ww\| \right\} \\ &\le \liminf_{i\to\infty} \left\{ \|z_{n_i} - Wz_{n_i}\| + \|z_{n_i} - w\| \right\}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \| Wz_n - z_n \| &\le \| Wz_n - W_n z_n \| + \| W_n z_n - y_n \| \\ &\le \sup_{x\in K} \| Wx - W_n x\| + \| W_n z_n - z_n \|. \end{aligned}$$

In view of Lemma 1.5, we obtain from (2.15) $\lim_{n\to\infty} \| Wz_n - z_n \| = 0$. It follows that $\liminf_{i\to\infty} \|z_{n_i} - w\| < \liminf_{i\to\infty} \|z_{n_i} - w\|$. Thus one derives a contradiction. Thus, we have $w \in \bigcap_{i=1}^{\infty} F(S_i)$.

Next, we show that $\bar{x} \in VI(C,A)$. Let T be the maximal monotone mapping defined by

$$Dx = \begin{cases} Bx + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x,y) \in \mathrm{Graph}(D)$, we have $y - Bx \in N_C x$. Since $y_n \in C$, by the definition of N_C, we have $\langle x - y_n, y - Bx\rangle \geq 0$. Since $y_n = P_C(u_n - s_n Bu_n)$, we see that $\langle x - y_n, \frac{y_n - u_n}{s_n} + Bu_n\rangle \geq 0$. It follows that

$$\begin{aligned}
\langle x - y_{n_i}, y\rangle &\geq \langle x - y_{n_i}, Bx\rangle \\
&\geq \langle x - y_{n_i}, Bx\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{s_{n_i}} + Bu_{n_i}\right\rangle \\
&= \langle x - y_{n_i}, Bx - By_{n_i}\rangle + \langle x - y_{n_i}, By_{n_i} - Bu_{n_i}\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{s_{n_i}}\right\rangle \\
&\geq \langle x - y_{n_i}, By_{n_i} - Bu_{n_i}\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - u_{n_i}}{s_{n_i}}\right\rangle.
\end{aligned}$$

Since B is Lipschitz continuous, we see that $\langle x - w, y\rangle \geq 0$. Notice that D is maximal monotone and hence $0 \in Tw$. This shows that $w \in VI(C,A)$. In the same way, we find that $w \in VI(C,B)$.

Next, we show that $w \in \mathrm{GEP}(F,T)$. Since $u_n = T_{\lambda_n}(I - \lambda_n T)x_n$, for any $y \in C$, we find from (A2) that

$$\langle Tx_{n_i}, y - u_{n_i}\rangle + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle \geq F(y, u_{n_i}), \quad \forall y \in C. \tag{2.16}$$

Putting $y_t = ty + (1-t)w$ for any $t \in (0,1]$ and $y \in C$, we see that $y_t \in C$. It follows from (2.16) that

$$\begin{aligned}
&\langle y_t - u_{n_i}, Ty_t\rangle \\
&\quad\geq \langle y_t - u_{n_i}, Ty_t\rangle - \langle Tx_{n_i}, y_t - u_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(y_t, u_{n_i}) \\
&\quad= \langle y_t - u_{n_i}, Ty_t - Tu_{n_i}\rangle + \langle y_t - u_{n_i}, Tu_{n_i} - Tx_{n_i}\rangle - \left\langle y_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{\lambda_{n_i}}\right\rangle + F(y_t, u_{n_i}).
\end{aligned}$$

In view of the monotonicity of T, and the restriction (a), we obtain from (A4)

$$\langle y_t - w, Ty_t\rangle \geq F(y_t, q). \tag{2.17}$$

From (A1) and (A4), we see that

$$\begin{aligned}
0 = F(y_t, y_t) &\leq tF(y_t, y) + (1-t)F(y_t, w) \\
&\leq tF(y_t, y) + (1-t)\langle y_t - w, Ty_t\rangle \\
&= tF(y_t, y) + (1-t)t\langle y - w, Ty_t\rangle.
\end{aligned}$$

It follows from (A3) that $w \in \mathrm{GEP}(F,T)$. This proves that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, x_n - \bar{x}\rangle \leq 0$.

Finally, we show that $x_n \to \bar{x}$, as $n \to \infty$. Note that

$$\begin{aligned}
&\|x_{n+1} - \bar{x}\|^2 \\
&\quad = \alpha_n \langle f(y_n) - \bar{x}, x_{n+1} - \bar{x} \rangle + \beta_n \langle W_n z_n - \bar{x}, x_{n+1} - \bar{x} \rangle + \gamma_n \langle x_n - \bar{x}, x_{n+1} - \bar{x} \rangle \\
&\quad \le \alpha_n \kappa \|y_n - \bar{x}\| \|x_{n+1} - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x} \rangle + \beta_n \|z_n - \bar{x}\| \|x_{n+1} - \bar{x}\| \\
&\qquad + \gamma_n \|x_n - \bar{x}\| \|x_{n+1} - \bar{x}\| \\
&\quad \le \bigl(1 - \alpha_n(1-\kappa)\bigr) \|x_n - \bar{x}\| \|x_{n+1} - \bar{x}\| + \alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x} \rangle.
\end{aligned}$$

This implies that

$$\|x_{n+1} - \bar{x}\|^2 \le \bigl(1 - \alpha_n(1-\kappa)\bigr) \|x_n - \bar{x}\|^2 + 2\alpha_n \langle f(\bar{x}) - \bar{x}, x_{n+1} - \bar{x} \rangle.$$

From the restriction (d), we obtain from Lemma 1.2 $\lim_{n\to\infty} \|x_n - \bar{x}\| = 0$. This completes the proof. □

Corollary 2.2 *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and let $f : C \to C$ be a κ-contraction. Let $T : C \to H$ be a τ-inverse-strongly monotone mapping. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Sigma = \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{GEP}(F,T)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\lambda_n\}$ be a positive number sequence. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(y_n) + \beta_n W_n u_n + \gamma_n x_n, \quad \forall n \ge 1,$$

where $\{u_n\}$ is such that $F(u_n, y) + \langle Tx_n, y - u_n\rangle + \frac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \ge 0$, $\forall y \in C$, and $\{W_n\}$ is the sequence generated in (1.5). *Assume that the following restrictions hold*:

(a) $0 < a \le \lambda_n \le b < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

(b) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(c) $0 < \liminf_{n\to\infty} \gamma_n \le \limsup_{n\to\infty} \gamma_n < 1$,

where a and b are real constants. Then $\{x_n\}$ converges strongly to $\bar{x} \in \Sigma$, which solves uniquely the following variational inequality:

$$\langle \bar{x} - f(\bar{x}), \bar{x} - x \rangle \le 0, \quad \forall x \in \Sigma.$$

Corollary 2.3 *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies* (A1)-(A4) *and let $f : C \to C$ be a κ-contraction. Let $B : C \to H$ be a β-inverse-strongly monotone mapping. Let $T : C \to H$ be a τ-inverse-strongly monotone mapping. Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings. Assume that $\Sigma = \bigcap_{i=1}^{\infty} F(S_i) \cap \mathrm{GEP}(F,T) \cap VI(C,B)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{s_n\}$ and $\{\lambda_n\}$ be positive number sequences. Let $x_1 \in C$ and let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = P_C(u_n - s_n B u_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n y_n + \gamma_n x_n, \quad \forall n \ge 1, \end{cases}$$

where $\{u_n\}$ *is such that* $F(u_n,y) + \langle Tx_n, y-u_n\rangle + \frac{1}{\lambda_n}\langle y-u_n, u_n-x_n\rangle \geq 0$, $\forall y \in C$, *and* $\{W_n\}$ *is the sequence generated in* (1.5). *Assume that the following restrictions hold*:

(a) $0 < a \leq \lambda_n \leq b < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,
(b) $0 < a'' \leq s_n \leq b'' < 2\beta$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$,
(c) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(d) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

where a, a', b, *and* b' *are real constants. Then* $\{x_n\}$ *converges strongly to* $\bar{x} \in \Sigma$, *which solves uniquely the following variational inequality*:

$$\langle \bar{x} - f(\bar{x}), \bar{x} - x\rangle \leq 0, \quad \forall x \in \Sigma.$$

Corollary 2.4 *Let* C *be a nonempty closed convex subset of* H. *Let* F *be a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* (A1)-(A4) *and let* $f: C \to C$ *be a* κ*-contraction. Let* $A: C \to H$ *be an* α*-inverse-strongly monotone mapping and let* $B: C \to H$ *be a* β*-inverse-strongly monotone mapping. Let* $\{S_i: C \to C\}$ *be a family of infinitely nonexpansive mappings. Assume that* $\Sigma = \bigcap_{i=1}^{\infty} F(S_i) \cap EP(F) \cap VI(C,A) \cap VI(C,B)$ *is not empty. Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$. *Let* $\{r_n\}$, $\{s_n\}$, *and* $\{\lambda_n\}$ *be positive number sequences. Let* $x_1 \in C$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} y_n = P_C(u_n - s_n B u_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n P_C(y_n - r_n A y_n) + \gamma_n x_n, \quad \forall n \geq 1, \end{cases}$$

where $\{u_n\}$ *is such that* $F(u_n,y) + \frac{1}{\lambda_n}\langle y-u_n, u_n-x_n\rangle \geq 0$, $\forall y \in C$, *and* $\{W_n\}$ *is the sequence generated in* (1.5). *Assume that the following restrictions hold*:

(a) $0 < a \leq \lambda_n \leq b < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,
(b) $0 < a' \leq r_n \leq b' < 2\alpha$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,
(c) $0 < a'' \leq s_n \leq b'' < 2\beta$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$,
(d) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(e) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

where a, a', a'', b, b', *and* b'' *are real constants. Then* $\{x_n\}$ *converges strongly to* $\bar{x} \in \Sigma$, *which solves uniquely the following variational inequality*:

$$\langle \bar{x} - f(\bar{x}), \bar{x} - x\rangle \leq 0, \quad \forall x \in \Sigma.$$

Proof In Theorem 2.1, put $T = 0$. Then, for all $\tau \in (0,\infty)$, we have

$$\langle x, y, Tx - Ty\rangle \geq \tau \|Tx - Ty\|^2, \quad \forall x, y \in C.$$

Taking $a, b \in (0,\infty)$ with $0 < a < b < \infty$ and choosing a sequence $\{\lambda_n\}$ of real numbers with $a \leq \lambda_n \leq b$, we obtain the desired result by Theorem 2.1. □

Corollary 2.5 *Let* C *be a nonempty closed convex subset of* H. *Let* $f: C \to C$ *be a* κ*-contraction and let* $T: C \to H$ *be a* τ*-inverse-strongly monotone mapping. Let* $A: C \to H$ *be an* α*-inverse-strongly monotone mapping and let* $B: C \to H$ *be a* β*-inverse-strongly monotone mapping. Let* $\{S_i: C \to C\}$ *be a family of infinitely nonexpansive mappings. Assume that* $\Sigma = \bigcap_{i=1}^{\infty} F(S_i) \cap VI(C,T) \cap VI(C,A) \cap VI(C,B)$ *is not empty. Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and*

$\{\gamma_n\}$ *be sequences in* $(0,1)$ *such that* $\alpha_n + \beta_n + \gamma_n = 1$. *Let* $\{r_n\}$, $\{s_n\}$, *and* $\{\lambda_n\}$ *be positive number sequences*. *Let* $x_1 \in C$ *and let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} u_n = P_C(x_n - \lambda_n T x_n), \\ y_n = P_C(u_n - s_n B u_n), \\ x_{n+1} = \alpha_n f(y_n) + \beta_n W_n P_C(y_n - r_n A y_n) + \gamma_n x_n, \quad \forall n \geq 1, \end{cases}$$

where $\{W_n\}$ *is the sequence generated in* (1.5). *Assume that the following restrictions hold*:

(a) $0 < a \leq \lambda_n \leq b < 2\tau$ *and* $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$,

(b) $0 < a' \leq r_n \leq b' < 2\alpha$ *and* $\lim_{n\to\infty} |r_{n+1} - r_n| = 0$,

(c) $0 < a'' \leq s_n \leq b'' < 2\beta$ *and* $\lim_{n\to\infty} |s_{n+1} - s_n| = 0$,

(d) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(e) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$,

where a, a', a'', b, b', *and* b'' *are real constants*. *Then* $\{x_n\}$ *converges strongly to* $\bar{x} \in \Sigma$, *which solves uniquely the following variational inequality*:

$$\langle \bar{x} - f(\bar{x}), \bar{x} - x \rangle \leq 0, \quad \forall x \in \Sigma.$$

Proof Putting $F = 0$, we find that

$$\langle Tx_n, y - u_n \rangle + \frac{1}{\lambda_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

is equivalent to

$$\langle y - u_n, x_n - \lambda_n T x_n - u_n \rangle \leq 0, \quad \forall y \in C,$$

that is, $u_n = P_C(x_n - \lambda_n T x_n)$. This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to this manuscript. Both authors read and approved the final manuscript.

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