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Viscosity approximation process for a sequence of quasinonexpansive mappings

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Abstract

We study the viscosity approximation method due to Moudafi for a fixed point problem of quasinonexpansive mappings in a Hilbert space. First, we establish a strong convergence theorem for a sequence of quasinonexpansive mappings. Then we employ our result to approximate a solution of the variational inequality problem over the common fixed point set of the sequence of quasinonexpansive mappings.

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1 Introduction

Let C be a nonempty closed convex subset of a Hilbert space. This paper is devoted to the study of strong convergence of a sequence $\{x_n\}$ in C defined by an arbitrary point $x_1 \in C$ and

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n) T_n x_n \tag{1.1}$$

for $n \in \mathbb{N}$, where α_n is a real number in $[0,1]$, f_n is a contraction-like mapping on C, and T_n is a quasinonexpansive mapping on C. This iterative method (1.1) is called the viscosity approximation method [1]. In Section 3, we establish that, under some appropriate assumptions, the sequence $\{x_n\}$ converges strongly to a certain common fixed point of $\{T_n\}$ by using the technique developed in [2]. Then, in Section 4, we apply our result to approximate a solution of a variational inequality problem over the common fixed point set of $\{T_n\}$.

The viscosity approximation method (1.1) is based on the study of Moudafi [1], who considered a fixed point problem of a single nonexpansive mapping and proved strong convergence of sequences generated by the method. After that, Xu [3] extended Moudafi's results [1] in the framework of Hilbert spaces and Banach spaces; Suzuki [4] gave simple proofs of Xu's results [3]; Aoyama and Kimura [5] investigated a relationship between viscosity approximation methods and Halpern [6] type iterative methods for a sequence of nonexpansive mappings.

On the other hand, Maingé [7] adopted the viscosity approximation method for a fixed point problem of a single quasinonexpansive mapping; Wongchan and Saejung [8] extended Maingé's result [7]. Our main result (Theorem 3.1) is a generalization of Wongchan and Saejung's result [8] and is closely related to the study in [5]. Moreover, it is also applicable to an approximation method, which is called the hybrid steepest descent method [9, 10], for a variational inequality problem over the common fixed point set of a sequence of quasinonexpansive mappings.

2 Preliminaries

Throughout the present paper, H denotes a real Hilbert space, $\langle\cdot,\cdot\rangle$ the inner product of H, $\|\cdot\|$ the norm of H, C a nonempty closed convex subset of H, I the identity mapping on H, $\mathbb{R}$ the set of real numbers, and $\mathbb{N}$ the set of positive integers. Strong convergence of a sequence $\{x_n\}$ in H to $x \in H$ is denoted by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

Let $T : C \to H$ be a mapping. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. A mapping T is said to be *quasinonexpansive* if $\mathrm{Fix}(T) \neq \emptyset$ and $\|Tx-p\| \leq \|x-p\|$ for all $x \in C$ and $p \in \mathrm{Fix}(T)$; T is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is said to be *strongly quasinonexpansive* if T is quasinonexpansive and $Tx_n - x_n \to 0$ whenever $\{x_n\}$ is a bounded sequence in C and $\|x_n - p\| - \|Tx_n - p\| \to 0$ for some point $p \in \mathrm{Fix}(T)$; T is *demiclosed at* 0 if $Tp = 0$ whenever $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup p$ and $Tx_n \to 0$. We know that if $T : C \to H$ is quasinonexpansive, then $\mathrm{Fix}(T)$ is closed and convex; see [11, Theorem 1].

It is known that, for each $x \in H$, there exists a unique point $x_0 \in C$ such that

$$\|x - x_0\| = \min\bigl\{\|x - y\| : y \in C\bigr\}.$$

Such a point x_0 is denoted by $P_C(x)$ and P_C is called the metric projection of H onto C. It is known that the metric projection P_C is nonexpansive; see [12].

Let $f : C \to C$ be a mapping, F a nonempty subset of C, and θ a real number in $[0,1)$. A mapping f is said to be a *θ-contraction with respect to F* if $\|f(x) - f(z)\| \leq \theta\|x - z\|$ for all $x \in C$ and $z \in F$; f is said to be a *θ-contraction* if f is a θ-contraction with respect to C. By definition, it is easy to check the following results.

Lemma 2.1 *Let F be a nonempty subset of C and $f : C \to C$ a θ-contraction with respect to F, where $0 \leq \theta < 1$. If F is closed and convex, then $P_F \circ f$ is a θ-contraction on F, where P_F is the metric projection of H onto F.*

Lemma 2.2 *Let $f : C \to C$ be a θ-contraction, where $0 \leq \theta < 1$ and $T : C \to C$ a quasinonexpansive mapping. Then $f \circ T$ is a θ-contraction with respect to* $\mathrm{Fix}(T)$.

Let D be a nonempty subset of C. A sequence $\{f_n\}$ of mappings of C into H is said to be *stable on D* [5] if $\{f_n(z) : n \in \mathbb{N}\}$ is a singleton for every $z \in D$. It is clear that if $\{f_n\}$ is stable on D, then $f_n(z) = f_1(z)$ for all $n \in \mathbb{N}$ and $z \in D$.

A function $\tau : \mathbb{N} \to \mathbb{N}$ is said to be *eventually increasing* [2] if $\lim_{n\to\infty} \tau(n) = \infty$ and $\tau(n) \leq \tau(n+1)$ for all $n \in \mathbb{N}$. By definition, we easily obtain the following.

Lemma 2.3 *Let $\tau : \mathbb{N} \to \mathbb{N}$ be an eventually increasing function and $\{\xi_n\}$ a sequence of real numbers such that $\xi_n \to \xi$. Then $\xi_{\tau(n)} \to \xi$.*

The following is a direct consequence of [13, Lemma 3.1].

Lemma 2.4 ([2, Lemma 3.4]) *Let $\{\xi_n\}$ be a sequence of nonnegative real numbers which is not convergent. Then there exist $N \in \mathbb{N}$ and an eventually increasing function $\tau : \mathbb{N} \to \mathbb{N}$ such that $\xi_{\tau(n)} \leq \xi_{\tau(n)+1}$ for all $n \in \mathbb{N}$ and $\xi_n \leq \xi_{\tau(n)+1}$ for all $n \geq N$.*

Under the assumptions of Lemma 2.4, we cannot choose a strictly increasing function τ; see [2, Example 3.3].

Let $\{T_n\}$ be a sequence of mappings of C into H such that $F = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ is nonempty. Then

- $\{T_n\}$ is said to be *strongly quasinonexpansive type* if each T_n is quasinonexpansive and $T_n x_n - x_n \to 0$ whenever $\{x_n\}$ is a bounded sequence in C and

  $$\|x_n - p\| - \|T_n x_n - p\| \to 0$$

  for some point $p \in F$;
- $\{T_n\}$ is said to satisfy the *condition* (Z) [2, 14–16] if every weak cluster point of $\{x_n\}$ belongs to F whenever $\{x_n\}$ is a bounded sequence in C such that $T_n x_n - x_n \to 0$.

Remark 2.5 Since $\beta_n - \alpha_n \to 0$ if and only if $\beta_n^2 - \alpha_n^2 \to 0$ for all bounded sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $[0,\infty)$, $\{T_n\}$ is strongly quasinonexpansive type if and only if it is a strongly relatively nonexpansive sequence in the sense of [2, 17]. See also [18, 19].

We know several examples of strongly quasinonexpansive type sequences satisfying the condition (Z); see [17] and Example 4.5 in Section 4.

The following lemma follows from [2, Lemma 3.5] and Remark 2.5.

Lemma 2.6 *Let $\{T_n\}$ be a sequence of mappings of C into H such that $F = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ is nonempty, $\tau : \mathbb{N} \to \mathbb{N}$ an eventually increasing function, and $\{z_n\}$ a bounded sequence in C such that $\|z_n - p\| - \|T_{\tau(n)} z_n - p\| \to 0$ for some $p \in F$. If $\{T_n\}$ is strongly quasinonexpansive type, then $T_{\tau(n)} z_n - z_n \to 0$.*

In order to prove our main result in Section 3, we need the following lemmas.

Lemma 2.7 ([2, Lemma 3.6]) *Let $\{T_n\}$ be a sequence of mappings of C into H such that $F = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ is nonempty, $\tau : \mathbb{N} \to \mathbb{N}$ an eventually increasing function, and $\{z_n\}$ a bounded sequence in C such that $T_{\tau(n)} z_n - z_n \to 0$. Suppose that $\{T_n\}$ satisfies the condition* (Z). *Then every weak cluster point of $\{z_n\}$ belongs to F.*

Lemma 2.8 ([2, Lemma 3.7]) *Let $\{T_n\}$ be a sequence of mappings of C into H, F a nonempty closed convex subset of H, $\{z_n\}$ a bounded sequence in C such that $T_n z_n - z_n \to 0$, and $u \in H$. Suppose that every weak cluster point of $\{z_n\}$ belongs to F. Then*

$$\limsup_{n\to\infty} \langle T_n z_n - w, u - w \rangle \le 0,$$

where $w = P_F(u)$.

The following lemma is well known; see [20, 21].

Lemma 2.9 *Let $\{\xi_n\}$ be a sequence of nonnegative real numbers, $\{\delta_n\}$ a sequence of real numbers, and $\{\beta_n\}$ a sequence in $[0,1]$. Suppose that $\xi_{n+1} \le (1-\beta_n)\xi_n + \beta_n \delta_n$ for every $n \in \mathbb{N}$, $\limsup_{n\to\infty} \delta_n \le 0$, and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then $\xi_n \to 0$.*

3 Strong convergence of a viscosity approximation process

In this section, we prove the following strong convergence theorem.

Theorem 3.1 *Let H be a real Hilbert space, C a nonempty closed convex subset of H, $\{S_n\}$ a sequence of mappings of C into C such that $F = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$ is nonempty, $\{\alpha_n\}$ a sequence in $(0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $\{f_n\}$ a sequence of mappings of C into C such that each f_n is a θ-contraction with respect to F and $\{f_n\}$ is stable on F, where $0 \le \theta < 1$. Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f_n(x_n) + (1-\alpha_n) S_n x_n \tag{3.1}$$

for $n \in \mathbb{N}$. Suppose that $\{S_n\}$ is strongly quasinonexpansive type and satisfies the condition (Z). Then $\{x_n\}$ converges strongly to $w \in F$, where w is the unique fixed point of a contraction $P_F \circ f_1$.

Note that Lemma 2.1 implies that $P_F \circ f_1$ is a θ-contraction on F and hence it has a unique fixed point on F.

First, we show some lemmas; then we prove Theorem 3.1. In the rest of this section, we set

$$\beta_n = \alpha_n \bigl(1 + (1-2\theta)(1-\alpha_n)\bigr)$$

and

$$\gamma_n = \alpha_n^2 \bigl\| f_n(x_n) - w \bigr\|^2 + 2\alpha_n(1-\alpha_n) \bigl\langle S_n x_n - w, f_1(w) - w \bigr\rangle$$

for $n \in \mathbb{N}$.

Lemma 3.2 *$\{x_n\}$, $\{S_n x_n\}$, and $\{f_n(x_n)\}$ are bounded, and moreover,*

$$\|x_{n+1} - w\| \le \alpha_n \bigl\| f_n(x_n) - w \bigr\| + \|S_n x_n - w\| \tag{3.2}$$

and

$$\|x_{n+1} - w\|^2 \le (1-\beta_n)\|x_n - w\|^2 + \gamma_n \tag{3.3}$$

hold for every $n \in \mathbb{N}$.

Proof Since f_n is a θ-contraction with respect to F, S_n is quasinonexpansive, $w \in F \subset \mathrm{Fix}(S_n)$, and $\{f_n\}$ is stable on F, it follows that

$$\begin{aligned}
&\|x_{n+1} - w\| \\
&\quad \le \alpha_n \bigl\| f_n(x_n) - w \bigr\| + (1-\alpha_n)\|S_n x_n - w\| \\
&\quad \le \alpha_n \bigl(\bigl\| f_n(x_n) - f_n(w) \bigr\| + \bigl\| f_n(w) - w \bigr\| \bigr) + (1-\alpha_n)\|S_n x_n - w\| \\
&\quad \le \bigl(1 - \alpha_n(1-\theta)\bigr)\|x_n - w\| + \alpha_n(1-\theta)\frac{\|f_1(w) - w\|}{1-\theta}
\end{aligned} \tag{3.4}$$

for every $n \in \mathbb{N}$. Thus, by induction on n, we have

$$\|S_n x_n - w\| \leq \|x_n - w\| \leq \max\bigl\{\|x_1 - w\|, \bigl\|f_1(w) - w\bigr\|/(1-\theta)\bigr\}.$$

Therefore, it turns out that $\{x_n\}$ and $\{S_n x_n\}$ are bounded, and moreover, $\{f_n(x_n)\}$ is also bounded.

Equation (3.2) follows from (3.4).

Next, we show (3.3). By assumption, it follows that

$$\begin{aligned}\bigl\langle S_n x_n - w, f_n(x_n) - w\bigr\rangle &\leq \|S_n x_n - w\| \bigl\|f_n(x_n) - f_n(w)\bigr\| + \bigl\langle S_n x_n - w, f_n(w) - w\bigr\rangle \\ &\leq \theta \|x_n - w\|^2 + \bigl\langle S_n x_n - w, f_1(w) - w\bigr\rangle,\end{aligned}$$

and thus

$$\begin{aligned}\|x_{n+1} - w\|^2 &= \alpha_n^2 \bigl\|f_n(x_n) - w\bigr\|^2 + (1-\alpha_n)^2 \|S_n x_n - w\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\bigl\langle S_n x_n - w, f_n(x_n) - w\bigr\rangle \\ &\leq \alpha_n^2 \bigl\|f_n(x_n) - w\bigr\|^2 + \bigl((1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)\theta\bigr)\|x_n - w\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\bigl\langle S_n x_n - w, f_1(w) - w\bigr\rangle \\ &= (1-\beta_n)\|x_n - w\|^2 + \gamma_n \end{aligned} \tag{3.5}$$

for every $n \in \mathbb{N}$. Therefore, (3.3) holds. □

Lemma 3.3 *The following hold*:

- $0 < \beta_n \leq 1$ *for every* $n \in \mathbb{N}$;
- $2\alpha_n(1-\alpha_n)/\beta_n \to 1/(1-\theta)$;
- $\alpha_n^2 \|f_n(x_n) - w\|^2/\beta_n \to 0$;
- $\sum_{n=1}^{\infty} \beta_n = \infty$.

Proof Since $0 < \alpha_n \leq 1$ and $-1 < 1 - 2\theta \leq 1$, we know that

$$0 < \alpha_n^2 = \alpha_n\bigl(1 + (-1)(1-\alpha_n)\bigr) \leq \beta_n \leq \alpha_n\bigl(1 + (1-\alpha_n)\bigr) = \alpha_n(2-\alpha_n) \leq 1.$$

It follows from $\alpha_n \to 0$ that $2\alpha_n(1-\alpha_n)/\beta_n \to 1/(1-\theta)$.

Since $\{f_n(x_n)\}$ is bounded by Lemma 3.2 and

$$\frac{\alpha_n^2}{\beta_n} = \frac{\alpha_n}{1 + (1-2\theta)(1-\alpha_n)} \to 0,$$

it follows that $\alpha_n^2 \|f_n(x_n) - w\|^2/\beta_n \to 0$.

Finally, we prove $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that $1 - 2\theta \geq 0$. Then it is clear that $\beta_n \geq \alpha_n$ for every $n \in \mathbb{N}$. Thus, $\sum_{n=1}^{\infty} \beta_n \geq \sum_{n=1}^{\infty} \alpha_n = \infty$. Next, we suppose that $1 - 2\theta < 0$. Then it is clear that $\beta_n > 2(1-\theta)\alpha_n$ for every $n \in \mathbb{N}$. Thus, $\sum_{n=1}^{\infty} \beta_n \geq 2(1-\theta)\sum_{n=1}^{\infty} \alpha_n = \infty$. This completes the proof. □

Lemma 3.4 $\{\|x_n - w\|\}$ *is convergent*.

Proof We assume, in order to obtain a contraction, that $\{\|x_n - w\|\}$ is not convergent. Then Lemma 2.4 implies that there exist $N \in \mathbb{N}$ and an eventually increasing function $\tau : \mathbb{N} \to \mathbb{N}$ such that

$$\|x_{\tau(n)} - w\| \leq \|x_{\tau(n)+1} - w\| \tag{3.6}$$

for every $n \in \mathbb{N}$ and

$$\|x_n - w\| \leq \|x_{\tau(n)+1} - w\| \tag{3.7}$$

for every $n \geq N$.

We show that $S_{\tau(n)}x_{\tau(n)} - x_{\tau(n)} \to 0$. Since $S_{\tau(n)}$ is quasinonexpansive and $w \in F \subset \operatorname{Fix}(S_{\tau(n)})$, it follows from (3.6), (3.2), and Lemmas 2.3 and 3.2 that

$$\begin{aligned} 0 &\leq \|x_{\tau(n)} - w\| - \|S_{\tau(n)}x_{\tau(n)} - w\| \\ &\leq \|x_{\tau(n)+1} - w\| - \|S_{\tau(n)}x_{\tau(n)} - w\| \\ &\leq \alpha_{\tau(n)} \big\| f_{\tau(n)}(x_{\tau(n)}) - w \big\| \to 0 \end{aligned}$$

as $n \to \infty$. Since $\{x_{\tau(n)}\}$ is bounded and $\{S_n\}$ is strongly quasinonexpansive type, Lemma 2.6 implies that $S_{\tau(n)}x_{\tau(n)} - x_{\tau(n)} \to 0$.

Since $\{S_n\}$ satisfies the condition (Z), it follows from Lemma 2.7 that every weak cluster point of $\{x_{\tau(n)}\}$ belongs to F. Thus Lemma 2.8 shows that

$$\limsup_{n\to\infty} \big\langle S_{\tau(n)}x_{\tau(n)} - w, f_1(w) - w \big\rangle \leq 0.$$

Moreover, Lemmas 2.3 and 3.3 imply that $\alpha_{\tau(n)}^2 \|f_{\tau(n)}(x_{\tau(n)}) - w\|^2/\beta_{\tau(n)} \to 0$ and $2\alpha_{\tau(n)}(1 - \alpha_{\tau(n)})/\beta_{\tau(n)} \to 1/(1-\theta)$. Therefore, we obtain

$$\limsup_{n\to\infty} \frac{\gamma_{\tau(n)}}{\beta_{\tau(n)}} \leq 0. \tag{3.8}$$

On the other hand, from (3.3) and (3.6), we know that

$$\begin{aligned} \|x_{\tau(n)+1} - w\|^2 &\leq (1 - \beta_{\tau(n)})\|x_{\tau(n)} - w\|^2 + \gamma_{\tau(n)} \\ &\leq (1 - \beta_{\tau(n)})\|x_{\tau(n)+1} - w\|^2 + \gamma_{\tau(n)} \end{aligned}$$

for every $n \in \mathbb{N}$. Thus, by $\beta_{\tau(n)} > 0$, this shows that

$$\|x_{\tau(n)+1} - w\|^2 \leq \frac{\gamma_{\tau(n)}}{\beta_{\tau(n)}} \tag{3.9}$$

for every $n \in \mathbb{N}$.

Finally, we obtain a contradiction that $\|x_n - w\| \to 0$. Using (3.7), (3.9), and (3.8), we conclude that

$$\limsup_{n\to\infty} \|x_n - w\|^2 \leq \limsup_{n\to\infty} \|x_{\tau(n)+1} - w\|^2 \leq \limsup_{n\to\infty} \frac{\gamma_{\tau(n)}}{\beta_{\tau(n)}} \leq 0,$$

and hence $\|x_n - w\| \to 0$, which is a contradiction. □

Proof of Theorem 3.1 We first show that $S_n x_n - x_n \to 0$. Since S_n is quasinonexpansive, it follows from (3.2) that

$$0 \le \|x_n - w\| - \|S_n x_n - w\| \le \|x_n - w\| - \|x_{n+1} - w\| + \alpha_n \big\| f_n(x_n) - w \big\|$$

for every $n \in \mathbb{N}$, so that $\|x_n - w\| - \|S_n x_n - w\| \to 0$ by Lemma 3.4, $\alpha_n \to 0$, and Lemma 3.2. Since $\{S_n\}$ is strongly quasinonexpansive type and $\{x_n\}$ is bounded, we conclude that $S_n x_n - x_n \to 0$.

Since $\{S_n\}$ satisfies the condition (Z), Lemma 2.8 implies that

$$\limsup_{n\to\infty} \big\langle S_n x_n - w, f_1(w) - w \big\rangle \le 0.$$

This shows that $\limsup_{n\to\infty} \gamma_n/\beta_n \le 0$ by using Lemmas 3.2 and 3.3. On the other hand, it follows from (3.3) that

$$\|x_{n+1} - w\|^2 \le (1 - \beta_n)\|x_n - w\|^2 + \beta_n \frac{\gamma_n}{\beta_n}$$

for every $n \in \mathbb{N}$. Therefore, noting that $\sum_{n=1}^{\infty} \beta_n = \infty$ and using Lemma 2.9, we conclude that $x_n - w \to 0$. □

A direct consequence of Theorem 3.1 is the following corollary, which is a slight generalization of [8, Theorem 2.3].

Corollary 3.5 *Let H be a real Hilbert space, C a nonempty closed convex subset of H, $S : C \to C$ a strongly quasinonexpansive mapping, $\{\alpha_n\}$ a sequence in $(0,1]$ such that $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$, and $f : C \to C$ a θ-contraction with respect to $F = \mathrm{Fix}(S)$, where $0 \le \theta < 1$. Let $\{x_n\}$ be a sequence defined by $x_1 \in C$ and*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S x_n \tag{3.10}$$

for $n \in \mathbb{N}$. Suppose that $I - S$ is demiclosed at 0. Then $\{x_n\}$ converges strongly to $w \in F$, where w is the unique fixed point of a contraction $P_F \circ f$.

Proof Set $S_n = S$ and $f_n = f$ for $n \in \mathbb{N}$. Then it is clear that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) = \mathrm{Fix}(S)$, $\{S_n\}$ is strongly quasinonexpansive type, $\{S_n\}$ satisfies the condition (Z), and $\{f_n\}$ is stable on C. Thus Theorem 3.1 implies the conclusion. □

4 Application to a variational inequality problem

In this section, applying Theorem 3.1, we study an approximation method for the following variational inequality problem.

Problem 4.1 Let κ and η be positive real numbers such that $\eta^2 < 2\kappa$. Let F be a nonempty closed convex subset of H and $A : H \to H$ a κ-strongly monotone and η-Lipschitz continuous mapping, that is, we assume that $\langle x - y, Ax - Ay \rangle \ge \kappa \|x - y\|^2$ and $\|Ax - Ay\| \le \eta \|x - y\|$ for all $x, y \in H$. Then find $z \in F$ such that

$$\langle y - z, Az \rangle \ge 0 \quad \text{for all } y \in F.$$

The solution set of Problem 4.1 is denoted by $\mathrm{VI}(F, A)$. Under the assumptions of Problem 4.1, it is known that the following hold; see, for example, [22].

- $\kappa \le \eta$, $0 \le 1 - 2\kappa + \eta^2 < 1$ and $I - A$ is a θ-contraction, where $\theta = \sqrt{1 - 2\kappa + \eta^2}$;
- Problem 4.1 has a unique solution and $\mathrm{VI}(F, A) = \mathrm{Fix}(P_F(I - A))$.

Remark 4.2 The assumption that $\eta^2 < 2\kappa$ in Problem 4.1 is not restrictive. Indeed, let F be a nonempty closed convex subset of H and $\tilde{A}$ a $\tilde{\kappa}$-strongly monotone and $\tilde{\eta}$-Lipschitz continuous mapping, where $\tilde{\kappa} > 0$ and $\tilde{\eta} > 0$. Set $A = \mu\tilde{A}$, $\kappa = \mu\tilde{\kappa}$, and $\eta = \mu\tilde{\eta}$, where μ is a positive constant such that $\mu\tilde{\eta}^2 < 2\tilde{\kappa}$. Then it is easy to verify that A is κ-strongly monotone and η-Lipschitz continuous, $\eta^2 < 2\kappa$, and moreover, $\mathrm{VI}(F, A) = \mathrm{VI}(F, \tilde{A})$.

Using Theorem 3.1, we obtain the following convergence theorem for Problem 4.1.

Theorem 4.3 *Let H, κ, η, and A be the same as in Problem* 4.1. *Let $\{S_n\}$ be a sequence of mappings of H into H such that $F = \bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n)$ is nonempty, and $\{\alpha_n\}$ the same as in Theorem* 3.1. *Let $\{x_n\}$ be a sequence defined by $x_1 \in H$ and*

$$x_{n+1} = S_n x_n - \alpha_n A S_n x_n \tag{4.1}$$

for $n \in \mathbb{N}$. Suppose that $\{S_n\}$ is strongly quasinonexpansive type and $\{S_n\}$ satisfies the condition (Z). *Then $\{x_n\}$ converges strongly to the unique solution of Problem* 4.1.

Proof Set $f_n = (I - A)S_n$ for $n \in \mathbb{N}$ and $\theta = \sqrt{1 - 2\kappa + \eta^2}$. Since $I - A$ is a θ-contraction and S_n is quasinonexpansive, Lemma 2.2 implies that each f_n is a θ-contraction with respect to F. It is obvious that $\{f_n\}$ is stable on F. Moreover, it follows from (4.1) that

$$x_{n+1} = \alpha_n f_n(x_n) + (1 - \alpha_n) S_n x_n$$

for $n \in \mathbb{N}$. Thus Theorem 3.1 implies that $\{x_n\}$ converges strongly to $w = (P_F \circ f_1)(w) = P_F(I - A)w$, which is the unique solution of Problem 4.1. □

Remark 4.4 The iteration (4.1) is called the hybrid steepest descent method; see [9, 10] for more details.

We finally construct an example of $\{S_n\}$ in Theorem 4.3 by using the notion of a subgradient projection.

Let $g : H \to \mathbb{R}$ be a continuous and convex function such that

$$C = \{x \in H : g(x) \le 0\}$$

is nonempty and $h : H \to H$ a mapping such that $h(x) \in \partial g(x)$ for all $x \in H$, where ∂g denotes the subdifferential mapping of g defined by

$$\partial g(x) = \{z \in H : g(x) + \langle y - x, z\rangle \le g(y)\ (\forall y \in H)\}$$

for all $x \in H$. Then the subgradient projection $P_{g,h} : H \to H$ with respect to g and h is defined by $P_{g,h}x = P_{L(x)}x$ for all $x \in H$, where $P_{L(x)}$ denotes the metric projection of H onto the set $L(x)$ defined by

$$L(x) = \{y \in H : g(x) + \langle y - x, h(x)\rangle \le 0\}$$

for all $x \in H$. Note that C is a subset of $L(x)$ for all $x \in H$ and that $L(x)$ is a closed half space for all $x \in H \setminus C$. According to [23, Section 7], [24, Proposition 2.3], and [25, Proposition 1.1.11], we know the following:

(S1) $\mathrm{Fix}(P_{g,h}) = C$;

(S2) $\langle z - P_{g,h}x, x - P_{g,h}x\rangle \le 0$ for all $z \in C$ and $x \in H$;

(S3) if $g(V)$ is bounded for each bounded subset V of H, then $I - P_{g,h}$ is demiclosed at 0.

It is known that the metric projection P_D of H onto a nonempty closed convex subset D of H coincides with the subgradient projection $P_{g,h}$ with respect to g and h defined by $g(x) = \inf_{y\in D} \|x - y\|$ for all $x \in H$ and

$$h(x) = \begin{cases} 0 & (x \in D); \\ (x - P_D x)/\|x - P_D x\| & (x \in H \setminus D). \end{cases}$$

The subgradient projection is not necessarily nonexpansive. In fact, if $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ are defined by $g(x) = \max\{x, 2x - 1\}$ for all $x \in \mathbb{R}$ and $h(x) = 1$ if $x < 1$; $h(x) = 2$ if $x \ge 1$, then $P_{g,h}$ is given by

$$P_{g,h}(x) = \begin{cases} x & (x \le 0); \\ 0 & (0 < x < 1); \\ 1/2 & (x \ge 1) \end{cases}$$

and is not nonexpansive.

Using (S1), (S2), and (S3), we show the following.

Example 4.5 Let $g : H \to \mathbb{R}$ be a continuous and convex function such that $C = \{x \in H : g(x) \le 0\}$ is nonempty and $g(V)$ is bounded for each bounded subset V of H, $h : H \to H$ a mapping such that $h(x) \in \partial g(x)$ for all $x \in H$, and $\{S_n\}$ a sequence of mappings of H into H defined by

$$S_n = \beta_n I + (1 - \beta_n)P_{g,h}$$

for all $n \in \mathbb{N}$, where $\{\beta_n\}$ is a sequence of real numbers such that $-1 < \inf_n \beta_n$ and $\sup_n \beta_n < 1$. Then the following hold:

(i) $\mathrm{Fix}(S_n) = C$ for all $n \in \mathbb{N}$;

(ii) $\{S_n\}$ is strongly quasinonexpansive type;

(iii) $\{S_n\}$ satisfies the condition (Z).

Proof Since $\beta_n \ne 1$ for all $n \in \mathbb{N}$, the part (i) obviously follows from (S1).

We first show (ii). By (i), we know that $\bigcap_{n=1}^{\infty} \mathrm{Fix}(S_n) = C$ is nonempty. Let $n \in \mathbb{N}$, $p \in C$, and $x \in H$ be given. Then we have

$$\begin{aligned} \|S_n x - p\|^2 + \|x - S_n x\|^2 - \|x - p\|^2 &= 2\langle S_n x - x, S_n x - p\rangle \\ &= 2(1 - \beta_n)\langle p - S_n x, x - P_{g,h}x\rangle. \end{aligned} \quad (4.2)$$

It follows from (S2) that

$$\langle p - S_n x, x - P_{g,h}x\rangle \leq \langle P_{g,h}x - S_n x, x - P_{g,h}x\rangle. \tag{4.3}$$

On the other hand, we also know that

$$\begin{aligned}
&\langle P_{g,h}x - S_n x, x - P_{g,h}x\rangle \\
&\quad = -\|P_{g,h}x - x\|^2 + \langle x - S_n x, x - P_{g,h}x\rangle \\
&\quad \leq -\left(\|P_{g,h}x - x\| - \frac{1}{2}\|x - S_n x\|\right)^2 + \frac{1}{4}\|x - S_n x\|^2 \leq \frac{1}{4}\|x - S_n x\|^2.
\end{aligned} \tag{4.4}$$

By (4.2), (4.3), and (4.4), each S_n satisfies

$$\|S_n x - p\|^2 + \frac{1}{2}(1 + \beta_n)\|x - S_n x\|^2 \leq \|x - p\|^2 \tag{4.5}$$

for all $p \in C$ and $x \in H$. Since $(1 + \beta_n)/2 > 0$, we know that each S_n is quasinonexpansive.

Let $\{x_n\}$ be a bounded sequence in H such that $\|x_n - p\| - \|S_n x_n - p\| \to 0$ for some $p \in C$. Since $\{S_n x_n\}$ is bounded, it follows from (4.5) that

$$\frac{1}{2}(1 + \beta_n)\|x_n - S_n x_n\|^2 \leq \|x_n - p\|^2 - \|S_n x_n - p\|^2 \to 0$$

and hence $S_n x_n - x_n \to 0$ by $\inf_n(1 + \beta_n) > 0$. Thus $\{S_n\}$ is strongly quasinonexpansive type.

We finally show (iii). Let $\{y_n\}$ be a bounded sequence in H such that $S_n y_n - y_n \to 0$. By the definition of S_n, we have

$$\|P_{g,h}y_n - y_n\| = \frac{1}{1 - \beta_n}\|S_n y_n - y_n\|$$

for all $n \in \mathbb{N}$. Since $\inf_n(1 - \beta_n) > 0$, we obtain $P_{g,h}y_n - y_n \to 0$. Consequently, by (S1) and (S3), we know that $\{S_n\}$ satisfies the condition (Z). □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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