Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Results on n-tupled fixed points in metric spaces with uniform normal structure

Ahmed H Soliman*

*Correspondence: ahsolimanm@gmail.com
Permanent address: Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut, 71524, Egypt
Present address: Department of Mathematics, Faculty of Science, King Khaled University, Abha, 9004, Saudi Arabia

Abstract

In this paper, we introduce the concept of new notions related to n-tupled fixed point and prove some related results for an asymptotically regular one-parameter semigroup $\Im = \{F(t) : t \in G, \text{where } G \text{ is an unbounded subset of } [0,\infty)\}$ of Lipschitzian self-mappings on $\prod_{i=1}^{n} X$ in the case when (X, d) is a complete bounded metric space with uniform normal structure. Our results extend the results due to Yao and Zeng (J. Nonlinear Convex Anal. 8(1):153-163, 2007) and Soliman (Fixed Point Theory Appl. 2013:346, 2013; J. Adv. Math. Stud. 7(2):2-14, 2014).

Keywords: coupled fixed point; tripled fixed point; n-fixed point; asymptotically regular semigroup; uniform normal structure; convexity structure

1 Introduction

The Banach contraction principle is the most natural and significant result of fixed point theory. In complete metric spaces it continues to be an indispensable and effective tool in theory and applications, which guarantees the existence and uniqueness of fixed points of contraction self-mappings besides offering a constructive procedure to compute the fixed point of the underlying mapping. There already exists an extensive literature on this topic. Keeping in view the relevance of this paper, we merely refer to [1–5]. In 1987, the idea of coupled fixed point was initiated by Guo and Lakshmikantham [6]; it was also followed by Bhaskar and Lakshmikantham [7] wherein authors proved some interesting coupled fixed point theorems for mappings satisfying the mixed monotone property. Many authors obtained important coupled, tripled and n-tupled fixed point theorems (see [7–16]). In this continuation, Lakshmikantham and Ćirić [13] introduced coupled common fixed point theorems for nonlinear ϕ-contraction mappings in partially ordered complete metric spaces which indeed generalize the corresponding fixed point theorems contained in Bhaskar and Lakshmikantham [7]. In 2010, Samet and Vetro [17] introduced the concept of fixed point of n-tupled fixed point (where $n = 2, 3, 4, \dots$) for nonlinear mappings in complete metric spaces. They obtained the existence and uniqueness theorems for contractive type mappings. Their results generalized and extended coupled fixed point theorems established by Bhaskar and Lakshmikantham [7]. Recently, Imdad *et al.* [18] introduced a generalization of n-tupled fixed point and n-tupled coincidence point by considering n even besides using the idea of mixed g-monotone property on $\prod_{i=1}^{n} X$ and proved an n-tupled (where n is even) coincidence point theorem for nonlinear ϕ-contraction mappings satisfying the mixed g-monotone property. For more information about n-tupled fixed points, see [10, 17–19].

On the other hand, normal structure is one of the most important aspects of metric fixed point theory. It was introduced by Brodskii and Milman in [20]. They found the first application of normal structure to fixed point theory. In 1965, Kirk [21] introduced the following theorem: Every nonexpansive self-mapping on a weakly compact convex subset of a Banach space with normal structure has a fixed point. In 1969, Kijima and Takahashi [22] established the metric space version of Kirk's theorem [21]. Subsequently, many authors successfully generalized certain fixed point theorems and structure properties from Banach spaces to metric spaces. For example, Khamsi [23] defined normal and uniform normal structure for metric spaces and proved that if (X,d) is a complete bounded metric space with uniform normal structure, then it has the fixed point property for nonexpansive mappings and a kind of intersection property which extends a result of Maluta [24] to metric spaces. In 1995, Lim and Xu [25] proved a fixed point theorem for uniformly Lipschitzian mappings in metric spaces with both property (P) and uniform normal structure, which extended the result of Khamsi [23]. This is the metric space version of Casini and Maluta's theorem [2]. In 2007, Yao and Zeng [26] established a fixed point theorem for an asymptotically regular one-parameter semigroup of uniformly k-Lipschitzian mappings with property $(*)$ in a complete bounded metric space with uniform normal structure, which extended the results of Lim and Xu [25]. Recently, the idea of coupled and tripled fixed point results in a complete bounded metric space X with uniform normal structure was initiated by Soliman [27, 28]. He proved that every asymptotically regular one-parameter semigroup $\Im = \{F(t) : t \in G,\}$ of Lipschitzian mappings on $X \times X$ has a coupled fixed point and on $X \times X \times X$ has a tripled fixed point.

In the present paper, we prove an n-tupled fixed point theorem for asymptotically regular Lipschitzian one-parameter semigroups $\Im = \{F(t) : t \in G\}$ on $\prod_{i=1}^{n} X$, where X is a complete bounded metric space with uniform normal structure. Also, some corollaries of our main theorem are presented.

2 Preliminaries

Definition 2.1 [7] An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F : X \times X \to X$ if

$$F(x,y) = x \quad \text{and} \quad F(y,x) = y.$$

Definition 2.2 [7] Let $(X,\leq)$ be a partially ordered set and $F : X \times X \to X$. We say that F has the mixed monotone property if $F(x,y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x,y \in X$, $x_1,x_2 \in X$, $x_1 \leq x_2 \Rightarrow F(x_1,y) \leq F(x_2,y)$ and $y_1,y_2 \in X$, $y_1 \leq y_2 \Rightarrow F(x,y_1) \geq F(x,y_2)$.

Theorem 2.1 [7] *Let $(X,\leq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a constant $k \in [0,1)$ with*

$$d\big(F(x,y),F(u,v)\big) \leq \frac{k}{2}\big[d(x,u) + d(y,v)\big] \quad \forall x \geq u, y \leq v.$$

If there exist $x_0, y_0 \in X$ such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Definition 2.3 [17] An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of the mapping $F : X \times X \times X \to X$ if

$$F(x, y, z) = x, \qquad F(y, z, x) = y \quad \text{and} \quad F(z, x, y) = z.$$

Definition 2.4 [10] Let X be a nonempty set. An element $(x^1, x^2, x^3, \dots, x^r) \in \prod_{i=1}^{r} X$ is called an r-tupled fixed point of the mapping $F : \prod_{i=1}^{r} X \to X$ if

$$\begin{aligned}
&x^1 = F\bigl(x^1, x^2, x^3, \dots, x^r\bigr), \\
&x^2 = F\bigl(x^2, x^3, \dots, x^r, x^1\bigr), \\
&x^3 = F\bigl(x^3, \dots, x^r, x^1, x^2\bigr), \\
&\vdots \\
&x^r = F\bigl(x^r, x^1, x^2, \dots, x^{r-1}\bigr).
\end{aligned}$$

Definition 2.5 [23] Suppose that (X, d) is a metric space, and let μ denote a nonempty family of subsets of X. Then μ defines a convexity structure on X if it is stable under intersection.

Definition 2.6 [23] Let μ be a convexity structure on a metric (X, d). Then μ has *property* (R) if any decreasing sequence $\{C_n\}$ of nonempty bounded closed subsets of X with $C_n \in \mu$ has a nonempty intersection.

Definition 2.7 [5] A subset of X is said to be admissible if it is an intersection of closed balls.

Remark 2.1 Let $A(X)$ be a family of all admissible subsets of X. Then we note that $A(X)$ defines a convexity structure on X.

In this paper any other convexity structure μ on X is always assumed to contain $A(X)$.

Let M be a bounded subset of X. Following Lim and Xu [25], we shall adopt the following notations:

$B(x, r)$ is the closed ball centered at x with radius r,
$r(x, M) = \sup\{d(x, y) : y \in M\}$ for $x \in X$,
$\delta(M) = \sup\{r(x, M) : x \in M\}$,
$R(M) = \inf\{r(x, M) : x \in M\}$.

For a bounded subset A of X, we define the admissible hull of A, denoted by $\operatorname{ad}(A)$, as the intersection of all those admissible subsets of X which contain A, *i.e.*,

$$\operatorname{ad}(A) = \bigcap \{B : A \subseteq B \subseteq X \text{ with } B \text{ admissible}\}.$$

Proposition 2.1 [25] *For a point $x \in X$ and a bounded subset A of X, we have*

$$r\bigl(x, \operatorname{ad}(A)\bigr) = r(x, A).$$

Definition 2.8 [23] A metric space (X,d) is said to have normal (resp. uniform normal) structure if there exists a convexity structure μ on X such that $R(A) < \delta(A)$ (resp. $R(A) \leq c \cdot \delta(A)$ for some constant $c \in (0,1)$) for all $A \in \mu$ which is bounded and consists of more than one point. In this case μ is said to be normal (resp. uniformly normal) in X.

We define the normal structure coefficient $N(X)$ of X (with respect to a given convexity structure μ) as the number

$$\sup\left\{\frac{R(A)}{\delta(A)}\right\},$$

where the supremum is taken over all bounded $A \in F$ with $\delta(A) > 0$. X then has uniform normal structure if and only if $N(X) < 1$.

Khamsi proved the following result that will be very useful in the proof of our main theorem.

Proposition 2.2 [23] *Let X be a complete bounded metric space and μ be a convexity structure of X with uniform normal structure. Then μ has property* (R).

Definition 2.9 [26] Let (X,d) be a metric space and $\Im = \{F(t) : t \in G\}$ be a semigroup on $\prod_{i=1}^{r} X$. Let us write the set

$$w(\infty) = \bigl\{\{t_n\} : \{t_n\} \subset G \text{ and } t_n \to \infty\bigr\}.$$

Lemma 2.1 [26] *If $\{t_n\} \in \omega(\infty)$, then $\{t_{n+1} - t_n\} \in \omega(\infty)$.*

Definition 2.10 [25] A metric space (X,d) is said to have property (P) if given any two bounded sequences $\{x_n\}$ and $\{z_n\}$ in X, one can find some $z \in \bigcap_{n=1}^{\infty} \operatorname{ad}\{z_j : j \geq n\}$ such that

$$\limsup_{n\to\infty} d(z,x_n) \leq \limsup_{j\to\infty} \limsup_{n\to\infty} d(z_j,x_n).$$

3 Main results

Let G be a subsemigroup of $[0,\infty)$ with addition '+' such that

$$t - s \in G \quad \forall t,s \in G \text{ with } t \geq s.$$

This condition is satisfied if $G = [0,\infty)$ or $G = Z^+$, the set of nonnegative integers. Let $\Im = \{F(t) : t \in G\}$ be a family of self-mappings on $\prod_{i=1}^{r} X$. Then $\Im$ is called a (one-parameter) semigroup on $\prod_{i=1}^{r} X$ if the following conditions are satisfied:

(i) $F(0)(x^1,x^2,x^3,\ldots,x^r) = x^1$, $F(0)(x^2,x^3,\ldots,x^r,x^1) = x^2, \ldots, F(0)(x^r,x^1,x^2,\ldots,x^{r-1}) = x^r$ $\forall x^1,x^2,x^3,\ldots,x^r \in X$;

(ii) $F(s)(F(t)(x^1,x^2,x^3,\ldots,x^r), F(t)(x^2,x^3,\ldots,x^r,x^1),\ldots,F(t)(x^r,x^1,x^2,\ldots,x^{r-1})) = F(s+t)(x^1,x^2,x^3,\ldots,x^r)$ $\forall s,t \in G$ and $x^1,x^2,x^3,\ldots,x^r \in X$;

(iii) $\forall x^1,x^2,x^3,\ldots,x^r \in X$, the self-mappings $t \mapsto F(t)(x^1,x^2,x^3,\ldots,x^r)$, $t \mapsto F(t)(x^2,x^3,\ldots,x^r,x^1),\ldots,$ $t \mapsto F(t)(x^r,x^1,x^2,\ldots,x^{r-1})$ from G into X are continuous when G has the relative topology of $[0,\infty)$.

Definition 3.1 A semigroup $\Im = \{F(t) : t \in G\}$ on $\prod_{i=1}^r X$ is said to be asymptotically regular at a point $(x^1, x^2, x^3, \dots, x^r) \in \prod_{i=1}^r X$ if

$$\lim_{t\to\infty} d\big(F(t+h)\big(x^1, x^2, x^3, \dots, x^r\big), F(t)\big(x^1, x^2, x^3, \dots, x^r\big)\big) = 0 \quad \forall h \in G. \tag{1}$$

If $\Im$ is asymptotically regular at each $(x^1, x^2, x^3, \dots, x^r) \in \prod_{i=1}^r X$, then $\Im$ is called an asymptotically regular semigroup on $\prod_{i=1}^r X$.

A semigroup $\{F(n) : n \in N$ (the set of all natural numbers)$\}$ on $\prod_{i=1}^r X$ is called simplest asymptotically regular at a point $(x^1, x^2, x^3, \dots, x^r) \in \prod_{i=1}^r X$ if

$$\lim_{n\to\infty} d\big(F^{n+1}\big(x^1, x^2, x^3, \dots, x^r\big), F^n\big(x^1, x^2, x^3, \dots, x^r\big)\big) = 0 \quad \forall x^1, x^2, x^3, \dots, x^r \in X.$$

Definition 3.2 A semigroup $\Im = \{F(t) : t \in G\}$ on $\prod_{i=1}^r X$ is called a uniformly Lipschitzian semigroup if

$$\sup\big\{k(t) : t \in G\big\} = k < \infty,$$

where

$$\begin{aligned} k(t) = r\sup\Bigg\{ & \frac{d(F(t)(x^1, x^2, x^3, \dots, x^r), F(t)(y^1, y^2, y^3, \dots, y^r))}{[d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^r, y^r)] \neq 0} : \\ & x^1, x^2, \dots, x^r, y^1, y^2, \dots, y^r \in X\Bigg\}. \end{aligned} \tag{2}$$

The simplest uniformly Lipschitzian semigroup is a semigroup of iterates of a mapping $F : \prod_{i=1}^r X \to X$ with

$$\begin{aligned} &\sup\{k_n : n \in N\} = k < \infty, \\ &k_n = r\sup\left\{\frac{d(F^n(x^1, x^2, x^3, \dots, x^r), F^n(y^1, y^2, y^3, \dots, y^r))}{[d(x^1, y^1) + d(x^2, y^2) + \cdots + d(x^r, y^r)] \neq 0} : x^1, x^2, \dots, x^r, y^1, y^2, \dots, y^r \in X\right\}. \end{aligned}$$

$\forall x^1, x^2, \dots, x^r, y^1, y^2, \dots, y^r \in X$, where $F^n(x^1, x^2, x^3, \dots, x^r) = F^{n-1}(F(x^1, x^2, x^3, \dots, x^r), F(x^2, x^3, \dots, x^r, x^1), \dots, F(x^r, x^1, x^2, \dots, x^{r-1}))$.

Definition 3.3 A mapping $F(t) : \prod_{i=1}^r X \to X$ has an r-tupled fixed point $(x^1, x^2, x^3, \dots, x^r) \in \prod_{i=1}^r X$ if $x^1 = F(t)(x^1, x^2, x^3, \dots, x^r)$, $x^2 = F(t)(x^2, x^3, \dots, x^r, x^1)$, $x^3 = F(t)(x^3, \dots, x^r, x^1, x^2), \dots, x^r = F(t)(x^r, x^1, x^2, \dots, x^{r-1})$.

Definition 3.4 Let (X, d) be a complete bounded metric space and $\Im = \{F(t) : t \in G\}$ be a semigroup on $\prod_{i=1}^r X$. Then $\Im$ has property $(*)$ if for each $x \in X$ and each $\{t_n\} \in w(\infty)$, the following conditions are satisfied:

(a) the sequences $\{F(t_n)(x^1, x^2, x^3, \dots, x^r)\}, \dots, \{F(t_n)(x^r, x^1, x^2, \dots, x^{r-1})\}$ are bounded;

(b) for any sequence $\{s_n^1\}$ in $\mathrm{ad}\{F(t_n)(x^1, x^2, x^3, \dots, x^r) : n \geq 1\}$, there exists some $s^1 \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{s_j^1 : j \geq n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty} d\big(s^1, F(t_n)\big(x^1, x^2, x^3, \dots, x^r\big)\big)\\&\quad\leq \limsup_{j\to\infty}\limsup_{n\to\infty} d\big(s_j^1, F(t_n)\big(x^1, x^2, x^3, \dots, x^r\big)\big),\end{aligned}$$

$\vdots$

for any sequence $\{s_n^r\}$ in $\mathrm{ad}\{F(t_n)(x^r, x^1, x^2, \dots, x^{r-1}) : n \geq 1\}$, there exists some $s^r \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{s_j^r : j \geq n\}$ such that

$$\limsup_{n\to\infty} d\big(s^r, F(t_n)\big(x^r, x^1, x^2, \dots, x^{r-1}\big)\big) \leq \limsup_{j\to\infty}\limsup_{n\to\infty} d\big(s_j^r, F(t_n)\big(x^r, x^1, x^2, \dots, x^{r-1}\big)\big).$$

Remark 3.1 If X is a complete bounded metric space with property (P), then each semigroup $\Im = \{F(t) : t \in G\}$ on $\prod_{i=1}^r X$ has property $(*)$.

Lemma 3.1 *Let (X, d) be a complete bounded metric space with uniform normal structure, and let $\Im = \{F(t) : t \in G\}$ be a semigroup on $\prod_{i=1}^r X$ with property $(*)$. Then, for each $x \in X$, each $\{t_n\} \in \omega(\infty)$ and for any constant $\widetilde{N}(X) < c$, the normal structure coefficient with respect to the given convexity structure μ, there exist some $a^1 \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{a_j^1 : j \geq n\}, \dots, a^r \in \bigcap_{n=1}^{\infty} \mathrm{ad}\{a_j^r : j \geq n\}$ satisfying the following properties*:

(I)

$$\limsup_{n\to\infty} d\big(a^1, F(t_n)\big(x^1, x^2, x^3, \dots, x^r\big)\big) \leq c \cdot A_1\big(\big\{F(t_n)\big(x^1, x^2, x^3, \dots, x^r\big)\big\}\big),$$

$\vdots$

$$\limsup_{n\to\infty} d\big(a^r, F(t_n)\big(x^r, x^1, x^2, \dots, x^{r-1}\big)\big) \leq c \cdot A_r\big(\big\{F(t_n)\big(x^r, x^1, x^2, \dots, x^{r-1}\big)\big\}\big),$$

where

$$\begin{aligned}&A_1\big(\big\{F(t_n)\big(x^1, x^2, x^3, \dots, x^r\big)\big\}\big)\\&\quad= \limsup_{n\to\infty}\big\{d\big(F(t_i)\big(x^1, x^2, x^3, \dots, x^r\big), F(t_j)\big(x^1, x^2, x^3, \dots, x^r\big)\big) : i, j \geq n\big\},\end{aligned}$$

$\vdots$

$$\begin{aligned}&A_r\big(\big\{F(t_n)\big(x^r, x^1, x^2, \dots, x^{r-1}\big)\big\}\big)\\&\quad= \limsup_{n\to\infty}\big\{d\big(F(t_i)\big(x^r, x^1, x^2, \dots, x^{r-1}\big), F(t_j)\big(x^r, x^1, x^2, \dots, x^{r-1}\big)\big) : i, j \geq n\big\};\end{aligned}$$

(II)

$$d\big(a^1, w\big) \leq \limsup_{n\to\infty} d\big(F(t_n)\big(x^1, x^2, x^3, \dots, x^r\big), w\big),$$

$\vdots$

$$d\big(a^r, w\big) \leq \limsup_{n\to\infty} d\big(F(t_n)\big(x^r, x^1, x^2, \dots, x^{r-1}\big), w\big) \quad \textit{for all } w \in X.$$

Proof For each integer $n \geq 1$, let $A_n^1 = \{F(t_j)(x^1, x^2, x^3, \ldots, x^r) : j \geq n\}$, $A_n^2 = \{F(t_j)(x^2, x^3, x^4, \ldots, x^r, x^1) : j \geq n\}, \ldots, A_n^r = \{F(t_j)(x^r, x^1, x^2, \ldots, x^{r-1}) : j \geq n\}$. Then $\{A_n^1\}, \{A_n^2\}, \ldots, \{A_n^r\}$ are decreasing sequences of admissible subsets of X hence $A^1 := \bigcap_{n=1}^{\infty} A_n^1 \neq \phi$, $A^2 := \bigcap_{n=1}^{\infty} A_n^2 \neq \phi, \ldots, A^r := \bigcap_{n=1}^{\infty} A_n^r \neq \phi$ by Proposition 2.2. From Proposition 2.1, it is not difficult to see that $\delta(A_n^1) = \delta(\{F(t_i)(x^1, x^2, x^3, \ldots, x^r) : i \geq n\})$, $\delta(A_n^2) = \delta(\{F(t_i)(x^2, x^3, x^4, \ldots, x^r, x^1) : i \geq n\}), \ldots, \delta(A_n^r) = \delta(\{F(t_i)(x^r, x^1, x^2, \ldots, x^{r-1}) : i \geq n\})$. Indeed, observe that

$$\begin{aligned}
\delta(A_n^1) &= \sup\{r(w, A_n^1) : w \in A_n^1\} \\
&= \sup_{w \in A_n^1} \sup_{j \geq n} d(w, F(t_j)(x^1, x^2, x^3, \ldots, x^r)) \\
&= \sup_{j \geq n} \sup_{w \in A_n^1} d(w, F(t_j)(x^1, x^2, x^3, \ldots, x^r)) \\
&= \sup_{j \geq n} r(F(t_j)(x^1, x^2, x^3, \ldots, x^r), A_n^1) \\
&= \sup_{j \geq n} \sup_{i \geq n} d(F(t_j)(x^1, x^2, x^3, \ldots, x^r), F(t_i)(x^1, x^2, x^3, \ldots, x^r)) \\
&= \delta(\{F(t_i)(x^1, x^2, x^3, \ldots, x^r) : i \geq n\}).
\end{aligned}$$

Similarly, one can obtain

$$\begin{aligned}
&\delta(A_n^2) = \delta(\{F(t_i)(x^2, x^3, x^4, \ldots, x^r, x^1) : i \geq n\}), \\
&\delta(A_n^3) = \delta(\{F(t_i)(x^3, x^4, \ldots, x^r, x^1, x^2) : i \geq n\}), \\
&\vdots \\
&\delta(A_n^r) = \delta(\{F(t_i)(x^r, x^1, x^2, \ldots, x^{r-1}) : i \geq n\}).
\end{aligned}$$

On the other hand, for any $a^1 \in A^1$ and any $w \in X$, we have

$$\sup_{j \geq n} d(w, F(t_j)(x^1, x^2, x^3, \ldots, x^r)) = r(w, A_n^1) \geq r(w, A^1) \geq d(w, a^1).$$

Therefore,

$$d(w, a^1) \leq \limsup_{n \to \infty} d(w, F(t_n)(x^1, x^2, x^3, \ldots, x^r)).$$

Also, one can deduce that for any $a^2 \in A^2, \ldots, a^r \in A^r$ and any $w \in X$, we have

$$\begin{aligned}
&d(w, a^2) \leq \limsup_{n \to \infty} d(w, F(t_n)(x^2, x^3, x^4, \ldots, x^r, x^1)), \\
&\vdots \\
&d(w, a^r) \leq \limsup_{n \to \infty} d(w, F(t_n)(x^r, x^1, x^2, \ldots, x^{r-1})),
\end{aligned}$$

from which (II) follows.

We now suppose that for each $n \geq 1$, there exist $a_n^1 \in A_n^1$, $a_n^2 \in A_n^2, \dots, a_n^r \in A_n^r$ such that

$$r(a_n^1, A_n^1) \leq c \cdot \delta(\{F(t_j)(x^1, x^2, x^3, \dots, x^r) : j \geq n\}), \tag{3}$$

$$r(a_n^2, A_n^2) \leq c \cdot \delta(\{F(t_j)(x^2, x^3, x^4, \dots, x^r, x^1) : j \geq n\}), \tag{4}$$

$$\vdots$$

$$r(a_n^2, A_n^r) \leq c \cdot \delta(\{F(t_j)(x^r, x^1, x^2, \dots, x^{r-1}) : j \geq n\}). \tag{5}$$

Indeed, if $\delta(\{F(t_j)(x^1,x^2,x^3,\dots,x^r) : j \geq n\}) = 0$, then $\delta(A_n^1) = \delta(\{F(t_j)(x^1,x^2,x^3,\dots,x^r) : j \geq n\})$, we conclude that (3) holds. Without loss of generality, we may assume that $\delta(\{F(t_j)(x^1,x^2,x^3,\dots,x^r) : j \geq 0\}) > 0$. Then, for $N(X) < c$, we choose $\epsilon > 0$ so small that it satisfies the following:

$$\begin{aligned} &N(X)\delta(\{F(t_j)(x^1, x^2, x^3, \dots, x^r) : j \geq n\}) + \epsilon \\ &\quad \leq c \cdot \delta(\{F(t_j)(x^1, x^2, x^3, \dots, x^r z) : j \geq n\}). \end{aligned} \tag{6}$$

From the definition of $R(A_n^1)$, one can find $a_n^1 \in A_n^1$ such that

$$\begin{aligned} r(a_n^1, A_n^1) &< R(A_n^1) + \epsilon \leq N(X)\delta(A_n^1) + \epsilon \\ &= N(X)\delta(\{F(t_j)(x^1, x^2, x^3, \dots, x^r) : j \geq n\}) + \epsilon \\ &\leq c \cdot \delta(\{F(t_j)(x^1, x^2, x^3, \dots, x^r) : j \geq n\}), \end{aligned}$$

which implies that (3) holds. Obviously, it follows from (3) that for each $n \geq 1$,

$$\limsup_{j \to \infty} r(a_n^1, x_j) \leq c \cdot \delta(\{F(t_j)(x^1, x^2, x^3, \dots, x^r) : j \geq n\}),$$

which implies

$$\limsup_{n \to \infty} \limsup_{j \to \infty} r(a_n^1, F(t_j(x^1, x^2, x^3, \dots, x^r))) \leq c \cdot A^1(\{F(t_n)(x^1, x^2, x^3, \dots, x^r)\}), \tag{7}$$

where $A^1(\{F(t_n)(x^1,x^2,x^3,\dots,x^r)\}) = \{d(F(t_j)(x^1,x^2,x^3,\dots,x^r), F(t_i)(x^1,x^2,x^3,\dots,x^r)) : i,j \geq n\}$. Noticing

$$a_n^1 \in A_n^1 \subset \operatorname{ad}\{F(t_j)(x^1, x^2, x^3, \dots, x^r) : j \geq n\} \quad \text{for each } n \geq 1,$$

we know that property (∗) yields a point $a^1 \in \bigcap_{n=1}^{\infty} \operatorname{ad}\{a_j^1 : j \geq n\}$ such that

$$\limsup_{j \to \infty} d(a^1, F(t_j)(x^1, x^2, x^3, \dots, x^r)) \leq \limsup_{n \to \infty} \limsup_{j \to \infty} r(a_n^1, F(t_j)(x^1, x^2, x^3, \dots, x^r)). \tag{8}$$

Since $\{a_j^1 : j \geq n\} \subset A_n^1$, $a^1 \in A^1 = \bigcap_{n=1}^{\infty} \operatorname{ad}\{F(t_j)(x^1,x^2,x^3,\dots,x^r) : j \geq n\}$ and satisfies

$$\limsup_{j \to \infty} d(a^1, F(t_j)(x^1, x^2, x^3, \dots, x^r)) \leq c \cdot A^1(\{F(t_j)(x^1, x^2, x^3, \dots, x^r)\}), \quad \text{by (7)},$$

similarly one can obtain that

$$\limsup_{j\to\infty} d\bigl(a^2, F(t_j)\bigl(x^2, x^3, \ldots, x^r, x^1\bigr)\bigr) \le c \cdot A^2\bigl(\bigl\{F(t_j)\bigl(x^2, x^3, \ldots, x^r, x^1\bigr)\bigr\}\bigr),$$

$$\vdots$$

$$\limsup_{j\to\infty} d\bigl(a^r, F(t_j)\bigl(x^r, x^1, x^2, \ldots, x^{r-1}\bigr)\bigr) \le c \cdot A^r\bigl(\bigl\{F(t_j)\bigl(x^r, x^1, x^2, \ldots, x^{r-1}\bigr)\bigr\}\bigr),$$

where $a^2 \in A^2 = \bigcap_{n=1}^{\infty} \operatorname{ad}\{F(t_j)(x^2, x^3, \ldots, x^r, x^1) : j \ge n\}, \ldots, a^r \in A^r = \bigcap_{n=1}^{\infty} \operatorname{ad}\{F(t_j)(x^r, x^1, x^2, \ldots, x^{r-1}) : j \ge n\}$.

Therefore (I) holds. □

We are ready to prove our main theorem for this paper.

Theorem 3.1 *Let (X, d) be a complete bounded metric space with uniform normal structure, and let $\Im = \{F(t) : t \in G\}$ be an asymptotically regular uniformly Lipschitzian semigroup of self-mappings on $\prod_{i=1}^{r} X$ with property $(*)$ and satisfying*

$$k \cdot \widetilde{k} < \frac{1}{N(X)},$$

where $k = \liminf_{t\to\infty} k(t)$ and $\widetilde{k} = \limsup_{t\to\infty} k(t)$.

Then there exist some $x^1, x^2, x^3, \ldots, x^r \in X$ such that $F(t)(x^1, x^2, x^3, \ldots, x^r) = x^1, \ldots, F(t)(x^2, x^3, \ldots, x^r, x^1) = x^2$ and $F(t)(x^r, x^1, x^2, \ldots, x^{r-1}) = x^r$ for all $t \in G$.

Proof First, we choose a constant c such that $N(X) < \widehat{c} < 1$ and $k\widetilde{k} < \frac{1}{\sqrt{c}}$. We can select a sequence $\{t_n\} \in w(\infty)$, from Lemma 2.1 we find that $\{t_{n+1} - t_n\} \in w(\infty)$ and $\lim_{n\to\infty} k(t_n) = k$.

Now fix $x_0^1, x_0^2, x_0^3, \ldots, x_0^r \in X$. Then, by Lemma 3.1, we can inductively construct sequences $\{x_l^1\}_{l=1}^{\infty}, \{x_l^2\}_{l=1}^{\infty}, \ldots, \{x_l^r\}_{l=1}^{\infty} \subset X$ such that $x_{l+1}^1 \in \bigcap_{n=1}^{\infty} \operatorname{ad}\{F(t_i)(x_l^1, x_l^2, x_l^3, \ldots, x_l^r) : i \ge n\}$, $x_{l+1}^2 \in \bigcap_{n=1}^{\infty} \operatorname{ad}\{F(t_i)(x_l^2, x_l^3, \ldots, x_l^r, x_l^1) : i \ge n\}, \ldots, x_{l+1}^r \in \bigcap_{n=1}^{\infty} \operatorname{ad}\{F(t_i)(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}) : i \ge n\}$ for each integer $l \ge 0$,

(III)

$$\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr), x_{l+1}^1\bigr) \le c \cdot A^1\bigl(\bigl\{F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr\}\bigr),$$

$$\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr), x_{l+1}^2\bigr) \le c \cdot A^2\bigl(\bigl\{F(t_n)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr)\bigr\}\bigr),$$

$$\vdots$$

$$\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr), x_{l+1}^r\bigr) \le c.A^r\bigl(\bigl\{F(t_n)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr)\bigr\}\bigr),$$

where

$$\begin{aligned} &A^1\bigl(\bigl\{F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr\}\bigr) \\ &\quad = \limsup_{n\to\infty} \bigl\{d\bigl(F(t_i)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr), F(t_j)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) : i, j \ge n\bigr\}, \end{aligned}$$

$$\begin{aligned}
&A^2\big(\big\{F(t_n)\big(x_l^2,x_l^3,\ldots,x_l^r,x_l^1\big)\big\}\big)\\
&\quad=\limsup_{n\to\infty}\big\{d\big(F(t_i)\big(x_l^2,x_l^3,\ldots,x_l^r,x_l^1\big),F(t_j)\big(x_l^2,x_l^3,\ldots,x_l^r,x_l^1\big)\big):i,j\geq n\big\},\\
&\vdots\\
&A^r\big(\big\{F(t_n)\big(x_l^r,x_l^1,x_l^2,\ldots,x_l^{r-1}\big)\big\}\big)\\
&\quad=\limsup_{n\to\infty}\big\{d\big(F(t_i)\big(x_l^r,x_l^1,x_l^2,\ldots,x_l^{r-1}\big),F(t_j)\big(x_l^r,x_l^1,x_l^2,\ldots,x_l^{r-1}\big)\big):i,j\geq n\big\};
\end{aligned}$$

(IV)

$$\begin{aligned}
&d\big(x_{l+1}^1,w\big)\leq\limsup_{n\to\infty}d\big(F(t_n)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big),w\big),\\
&d\big(x_{l+1}^2,w\big)\leq\limsup_{n\to\infty}d\big(F(t_n)\big(x_l^2,x_l^3,\ldots,x_l^r,x_l^1\big),w\big),\\
&\vdots\\
&d\big(x_{l+1}^r,w\big)\leq\limsup_{n\to\infty}d\big(F(t_n)\big(x_l^r,x_l^1,x_l^2,\ldots,x_l^{r-1}\big),w\big)\quad\forall w\in X.
\end{aligned}$$

Let

$$\begin{aligned}
&D_l=\limsup_{n\to\infty}\big[d\big(x_{l+1}^1,F(t_n)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big)\big)+\cdots+d\big(x_{l+1}^r,F(t_n)\big(x_l^r,x_l^1,x_l^2,\ldots,x_l^{r-1}\big)\big)\big],\\
&\quad\text{and}\quad h=c\cdot k\widetilde{k}<1.
\end{aligned}$$

Observe that for each $i>j\geq 1$, using (IV) we have

$$\begin{aligned}
&d\big(F(t_i)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big),F(t_j)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big)\big)\\
&\quad=d\big(F(t_j)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big),F(t_j)F(t_i-t_j)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big)\big)\\
&\quad=d\big(F(t_j)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big),F(t_j)\big(F(t_i-t_j)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big),\\
&\qquad F(t_i-t_j)\big(x_l^2,x_l^3,\ldots,x_l^r,x_l^1\big),\ldots,F(t_i-t_j)\big(x_l^r,x_l^1,x_l^2,\ldots,x_l^{r-1}\big)\big)\big)\\
&\quad\leq\frac{k(t_j)}{r}\big[d\big(x_l^1,F(t_i-t_j)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big)\big)+\cdots\\
&\qquad+d\big(x_l^r,F(t_i-t_j)\big(x_l^r,x_l^1,x_l^2,\ldots,x_l^{r-1}\big)\big)\big]\\
&\quad\leq\frac{k(t_j)}{r}\limsup_{n\to\infty}\big[d\big(F(t_n)\big(x_{l-1}^1,x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r\big),F(t_i-t_j)\big(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\big)\big)\\
&\qquad+d\big(F(t_n)\big(x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r,x_{l-1}^1\big),F(t_i-t_j)\big(x_l^2,x_l^3,\ldots,x_l^r,x_l^1\big)\big)+\cdots\\
&\qquad+d\big(F(t_n)\big(x_{l-1}^r,x_{l-1}^1,x_{l-1}^2,\ldots,x_{l-1}^{r-1}\big),F(t_i-t_j)\big(x_l^r,x_l^1,x_l^2,\ldots,x_l^{r-1}\big)\big)\big].
\end{aligned}\tag{9}$$

By the asymptotic regularity of $\Im=\{F(t):t\in G\}$ on $\prod_{i=1}^r X$, we see that

$$\begin{aligned}
&\limsup_{n\to\infty}\big[d\big(F(t_n)\big(x_{l-1}^1,x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r\big),F(t_n+t_i-t_j)\big(x_{l-1}^1,x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r\big)\big)\big]=0,\\
&\limsup_{n\to\infty}\big[d\big(F(t_n)\big(x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r,x_{l-1}^1\big),F(t_n+t_i-t_j)\big(x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r,x_{l-1}^1\big)\big)\big]=0,
\end{aligned}$$

$$\vdots$$

$$\limsup_{n\to\infty}\bigl[d\bigl(F(t_n)\bigl(x_{l-1}^r,x_{l-1}^1,x_{l-1}^2,\ldots,x_{l-1}^{r-1}\bigr),F(t_n+t_i-t_j)\bigl(x_{l-1}^r,x_{l-1}^1,x_{l-1}^2,\ldots,x_{l-1}^{r-1}\bigr)\bigr)\bigr]=0,$$

which implies

$$\begin{aligned}
&\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_{l-1}^1,x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r\bigr),F(t_i-t_j)\bigl(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\bigr)\bigr)\\
&\quad\le \limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_{l-1}^1,x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r\bigr),F(t_n+t_i-t_j)\bigl(x_{l-1}^1,x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r\bigr)\bigr)\\
&\qquad+\limsup_{n\to\infty} d\bigl(F(t_n+t_i-t_j)\bigl(x_{l-1}^1,x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r\bigr),F(t_i-t_j)\bigl(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\bigr)\bigr)\\
&\quad\le \limsup_{n\to\infty} d\bigl(F(t_i-t_j)\bigl(F(t_n)\bigl(x_{l-1}^1,x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r\bigr),F(t_n)\bigl(x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r,x_{l-1}^1\bigr),\ldots,\\
&\qquad F(t_n)\bigl(x_{l-1}^r,x_{l-1}^1,x_{l-1}^2,\ldots,x_{l-1}^{r-1}\bigr)\bigr),F(t_i-t_j)\bigl(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\bigr)\bigr)\\
&\quad\le \frac{k(t_i-t_j)}{r}\bigl[d\bigl(F(t_n)\bigl(x_{l-1}^1,x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r\bigr),x_l^1\bigr)\\
&\qquad+d\bigl(F(t_n)\bigl(x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r,x_{l-1}^1\bigr),x_l^2\bigr)+\cdots+d\bigl(F(t_n)\bigl(x_{l-1}^r,x_{l-1}^1,x_{l-1}^2,\ldots,x_{l-1}^{r-1}\bigr),x_l^r\bigr)\bigr]\\
&\le \frac{k(t_i-t_j)}{r}D_{l-1}.
\end{aligned}\tag{10}$$

Similarly, one can show that

$$\begin{aligned}
&\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_{l-1}^2,x_{l-1}^3,\ldots,x_{l-1}^r,x_{l-1}^1\bigr),F(t_i-t_j)\bigl(x_l^2,x_l^3,\ldots,x_l^r,x_l^1\bigr)\bigr)\\
&\quad\le \frac{k(t_i-t_j)}{r}D_{l-1},
\end{aligned}\tag{11}$$

$$\vdots$$

$$\begin{aligned}
&\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_{l-1}^r,x_{l-1}^1,x_{l-1}^2,\ldots,x_{l-1}^{r-1}\bigr),F(t_i-t_j)\bigl(x_l^r,x_l^1,x_l^2,\ldots,x_l^{r-1}\bigr)\bigr)\\
&\quad\le \frac{k(t_i-t_j)}{r}D_{l-1}.
\end{aligned}\tag{12}$$

Then it follows from (9), (10), (11) and (12) that for each $i>j\ge 1$,

$$d\bigl(F(t_i)\bigl(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\bigr),F(t_j)\bigl(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\bigr)\bigr)\le \frac{k(t_j)}{r}\cdot k(t_i-t_j)\cdot D_{l-1},$$

which implies that for each $n\ge 1$,

$$\begin{aligned}
&\sup\bigl\{d\bigl(F(t_i)\bigl(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\bigr),F(t_j)\bigl(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\bigr)\bigr):i,j\ge n\bigr\}\\
&\quad=\sup\bigl\{d\bigl(F(t_i)\bigl(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\bigr),F(t_j)\bigl(x_l^1,x_l^2,x_l^3,\ldots,x_l^r\bigr)\bigr):i>j\ge n\bigr\}\\
&\quad\le \sup\left\{\frac{k(t_j)}{r}\cdot k(t_i-t_j)\cdot D_{l-1}:i>j\ge n\right\}\\
&\quad\le \frac{D_{l-1}}{r}\cdot\sup\bigl\{k(t_j):j\ge n\bigr\}\cdot\sup\bigl\{k(t_i-t_j):i>j\ge n\bigr\}\\
&\quad\le \frac{D_{l-1}}{r}\cdot\sup\bigl\{k(t_j):j\ge n\bigr\}\cdot\sup\bigl\{k(t):G\ni t\ge t_{n+1}-t_n\bigr\}.
\end{aligned}\tag{13}$$

Hence, by using (III) and (9), we have

$$\begin{aligned}
D_l &= \limsup_{n\to\infty}\bigl[d\bigl(x_{l+1}^1, F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) + d\bigl(x_{l+1}^2, F(t_n)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr)\bigr) + \cdots \\
&\quad + d\bigl(x_{l+1}^r, F(t_n)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr)\bigr)\bigr] \\
&\le c\cdot\bigl[A^1\bigl(\bigl\{F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr\}\bigr) + A^2\bigl(\bigl\{F(t_n)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr)\bigr\}\bigr) + \cdots \\
&\quad + A^r\bigl(\bigl\{F(t_n)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr)\bigr\}\bigr)\bigr] \\
&\le c\cdot\limsup_{n\to\infty}\bigl\{d\bigl(F(t_i)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr), F(t_j)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&\quad + d\bigl(F(t_i)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr), F(t_j)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr)\bigr) + \cdots \\
&\quad + d\bigl(F(t_i)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr), F(t_j)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr)\bigr) : i,j \ge n\bigr\} \\
&\le c\cdot D_{l-1}\cdot\limsup_{n\to\infty} k(t_n)\cdot\limsup_{n\to\infty}\bigl\{k(t) : G \ni t \ge t_{n+1} - t_n\bigr\} \\
&\le c\cdot k\widetilde{k}\cdot D_{l-1} \le hD_{l-1} \le h^2 D_{l-2} \le \cdots \\
&= h^l D_0.
\end{aligned} \tag{14}$$

Hence, by the asymptotic regularity of $\Im$ on $\prod_{i=1}^r X$, we have, for each integer $n \ge 1$,

$$\begin{aligned}
d\bigl(x_{l+1}^1, x_l^1\bigr) &\le d\bigl(x_{l+1}^1, F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) + d\bigl(x_l^1, F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&\le d\bigl(x_{l+1}^1, F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&\quad + \limsup_{m\to\infty} d\bigl(F(t_m)\bigl(x_{l-1}^1, x_{l-1}^2, x_{l-1}^3, \ldots, x_{l-1}^r\bigr), F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&\le d\bigl(x_{l+1}^1, F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&\quad + \limsup_{m\to\infty} d\bigl(F(t_m)\bigl(x_{l-1}^1, x_{l-1}^2, x_{l-1}^3, \ldots, x_{l-1}^r\bigr), \\
&\quad F(t_m + t_n)\bigl(x_{l-1}^1, x_{l-1}^2, x_{l-1}^3, \ldots, x_{l-1}^r\bigr)\bigr) \\
&\quad + \limsup_{m\to\infty} d\bigl(F(t_m + t_n)\bigl(x_{l-1}^1, x_{l-1}^2, x_{l-1}^3, \ldots, x_{l-1}^r\bigr), F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&\le d\bigl(x_{l+1}^1, F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&\quad + \frac{k(t_n)}{r}\cdot\limsup_{m\to\infty}\bigl[d\bigl(x_l^1, F(t_m)\bigl(x_{l-1}^1, x_{l-1}^2, x_{l-1}^3, \ldots, x_{l-1}^r\bigr)\bigr) \\
&\quad + d\bigl(x_l^2, F(t_m)\bigl(x_{l-1}^2, x_{l-1}^3, \ldots, x_{l-1}^r, x_{l-1}^1\bigr)\bigr) + \cdots \\
&\quad + d\bigl(x_l^r, F(t_m)\bigl(x_{l-1}^r, x_{l-1}^1, x_{l-1}^2, \ldots, x_{l-1}^{r-1}\bigr)\bigr)\bigr] \\
&\le d\bigl(x_{l+1}^1, F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) + \frac{k(t_n)}{r}\cdot D_{l-1},
\end{aligned} \tag{15}$$

which implies

$$\begin{aligned}
d\bigl(x_{l+1}^1, x_l^1\bigr) \le \limsup_{n\to\infty}\bigl[&d\bigl(x_{l+1}^1, F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&+ d\bigl(x_{l+1}^2, F(t_n)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr)\bigr) + \cdots \\
&+ d\bigl(x_{l+1}^r, F(t_n)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr)\bigr)\bigr]
\end{aligned}$$

$$\begin{aligned}
&+ \frac{1}{r} D_{l-1} \cdot \limsup_{n\to\infty} k(t_n) \\
&\le D_l + \frac{1}{r} \cdot k \cdot D_{l-1}. \qquad (16)
\end{aligned}$$

It follows from (14) that

$$d\bigl(x_{l+1}^1, x_l^1\bigr) \le D_l + \frac{1}{r} \cdot k \cdot D_{l-1} \le \left(h^l + \frac{1}{r} k h^{l-1}\right) D_0 \le h^{l-1} \cdot 2D_0 \max\left\{h, \frac{k}{r}\right\}.$$

Similarly, one can deduce that

$$d\bigl(x_{l+1}^2, x_l^2\bigr) \le h^{l-1} \cdot 2D_0 \max\left\{h, \frac{k}{r}\right\}, \qquad (17)$$

$$\vdots$$

$$d\bigl(x_{l+1}^r, x_l^r\bigr) \le h^{l-1} \cdot 2D_0 \max\left\{h, \frac{k}{r}\right\}. \qquad (18)$$

Thus, we have $\sum_{l=0}^{\infty} d(x_{l+1}^1, x_l^1) \le 2D_0 \max\{h, \frac{k}{r}\} \sum_{l=0}^{\infty} h^{l-1} < \infty, \ldots, \sum_{l=0}^{\infty} d(x_{l+1}^r, x_l^r) < \infty$. Consequently, $\{x_l^1\}, \ldots, \{x_l^r\}$ are Cauchy and hence convergent as X is complete. Let $x^1 = \lim_{l\to\infty} x_l^1, \ldots, x^r = \lim_{l\to\infty} x_l^r$, then, for each $s \in G$, by the continuity of $F(s)$ we have

$$\begin{aligned}
&\lim_{l\to\infty} d\bigl(F(s)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr), F(s)\bigl(x^1, x^2, x^3, \ldots, x^r\bigr)\bigr) = 0, \\
&\lim_{l\to\infty} d\bigl(F(s)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr), F(s)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr)\bigr) = 0, \\
&\vdots \\
&\lim_{l\to\infty} d\bigl(F(s)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr), F(s)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr)\bigr) = 0.
\end{aligned}$$

On the other hand, from (15) we have actually proven the following inequalities:

$$\begin{aligned}
&\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr), x_l^1\bigr) \le \frac{k(t_n)}{r} \cdot D_{l-1} \le \frac{1}{r} k(t_n) h^{l-1} D_0, \\
&\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_l^2, x_l^3, \ldots, x_l^r, x_l^1\bigr), x_l^2\bigr) \le \frac{1}{r} k(t_n) h^{l-1} D_0, \\
&\vdots \\
&\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_l^r, x_l^1, x_l^2, \ldots, x_l^{r-1}\bigr), x_l^r\bigr) \le \frac{1}{r} k(t_n) h^{l-1} D_0.
\end{aligned}$$

Since $\lim_{n\to\infty} k(t_n) = k$, it follows that

$$\begin{aligned}
&\limsup_{n\to\infty} d\bigl(x^1, F(t_n)\bigl(x^1, x^2, x^3, \ldots, x^r\bigr)\bigr) \\
&\quad = d\bigl(x^1, x_l^1\bigr) + \limsup_{n\to\infty} d\bigl(x_l^1, F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&\qquad + \limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr), F(t_n)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr)
\end{aligned}$$

$$\begin{aligned}
&\le d\bigl(x^1, x_l^1\bigr) + \frac{1}{r}\limsup_{n\to\infty} k(t_n)h^{l-1}D_0 \\
&\le d\bigl(x^1, x_l^1\bigr) + \frac{1}{r}kh^{l-1}D_0 \to 0, \quad l\to\infty.
\end{aligned}$$

Similarly, one can obtain that

$$\begin{aligned}
&\limsup_{n\to\infty} d\bigl(x^2, F(t_n)\bigl(x^2, x^3, \ldots, x^r, x^1\bigr)\bigr) \le d\bigl(x^2, x_l^2\bigr) + \frac{1}{r}kh^{l-1}D_0 \to 0, \quad l\to\infty, \\
&\vdots \\
&\limsup_{n\to\infty} d\bigl(x^r, F(t_n)\bigl(x^r, x^1, x^2, \ldots, x^{r-1}\bigr)\bigr) \le d\bigl(x^r, x_l^r\bigr) + \frac{1}{r}kh^{l-1}D_0 \to 0, \quad l\to\infty,
\end{aligned}$$

i.e., $\lim_{n\to\infty} d(x^1, F(t_n)(x^1, x^2, x^3, \ldots, x^r)) = 0, \ldots, \lim_{n\to\infty} d(x^r, F(t_n)(x^r, x^1, x^2, \ldots, x^{r-1})) = 0$. Hence, for each $s \in G$, by the continuity of $F(s)$, we deduce

$$\begin{aligned}
&d\bigl(x^1, F(s)\bigl(x^1, x^2, x^3, \ldots, x^r\bigr)\bigr) \\
&\quad = \lim_{l\to\infty} d\bigl(x_l^1, F(s)\bigl(x_l^1, x_l^2, x_l^3, \ldots, x_l^r\bigr)\bigr) \\
&\quad \le \lim_{l\to\infty}\limsup_{n\to\infty} d\bigl(x_l^1, F(t_n + s)\bigl(x_{l-1}^1, x_{l-1}^2, x_{l-1}^3, \ldots, x_{l-1}^r\bigr)\bigr) \\
&\quad \le \lim_{l\to\infty}\limsup_{n\to\infty} d\bigl(x_l^1, F(t_n)\bigl(x_{l-1}^1, x_{l-1}^2, x_{l-1}^3, \ldots, x_{l-1}^r\bigr)\bigr) \\
&\qquad + \lim_{l\to\infty}\limsup_{n\to\infty} d\bigl(F(t_n)\bigl(x_{l-1}^1, x_{l-1}^2, \ldots, x_{l-1}^r\bigr), F(t_n + s)\bigl(x_{l-1}^1, x_{l-1}^2, \ldots, x_{l-1}^r\bigr)\bigr) \\
&\quad \le \lim_{l\to\infty} D_{l-1} \le \lim_{l\to\infty} h^{l-1}D_0 = 0.
\end{aligned}$$

Similarly, we get that

$$\begin{aligned}
&d\bigl(x^2, F(s)\bigl(x^2, x^3, \ldots, x^r, x^1\bigr)\bigr) = 0, \\
&d\bigl(x^3, F(s)\bigl(x^3, x^4, \ldots, x^r, x^1, x^2\bigr)\bigr) = 0, \\
&\vdots \\
&d\bigl(x^r, F(s)\bigl(x^r, x^1, x^2, \ldots, x^{r-1}\bigr)\bigr) = 0.
\end{aligned}$$

Then we have $d(x^1, F(s)(x^1, x^2, x^3, \ldots, x^r)) = 0, \ldots, d(x^2, F(s)(x^2, x^3, \ldots, x^r, x^1)) = 0$, *i.e.*, $F(s)(x^1, x^2, x^3, \ldots, x^r) = x^1$, $F(s)(x^2, x^3, \ldots, x^r, x^1) = x^2, \ldots, F(s)(x^r, x^1, x^2, \ldots, x^{r-1}) = x^r$ for each $s \in G$. □

The following corollary is related to the simplest uniformly Lipschitzian semigroup defined in Definition 3.2.

Corollary 3.1 *Let (X, d) be a complete bounded metric space with uniform normal structure, and let $\Im = \{F^n : n \in N\}$ be the simplest asymptotically regular uniformly Lipschitzian semigroup of self-mappings on $\prod_{i=1}^r X$ with property (P) and satisfying*

$$k < \frac{1}{\sqrt{N(X)}}.$$

Then there exist some $x^1, x^2, x^3, \dots, x^r \in X$ such that $F(x^1, x^2, x^3, \dots, x^r) = x^1, \dots, F(x^2, x^3, \dots, x^r, x^1) = x^2$ and $F(x^r, x^1, x^2, \dots, x^{r-1}) = x^r$ for all $t \in G$.

From Remark 2.1 and Theorem 3.1, we immediately obtain the following corollary.

Corollary 3.2 *Let (X, d) be a complete bounded metric space with property (P) and uniform normal structure, and let $\Im = \{F(t) : t \in G\}$ be an asymptotically regular semigroup on $\prod_{i=1}^{r} X$ satisfying*

$$k \cdot \widetilde{k} < \frac{1}{N(X)}.$$

Then there exist some $x^1, x^2, x^3, \dots, x^r \in X$ such that $F(s)(x^1, x^2, x^3, \dots, x^r) = x^1$, $F(s)(x^2, x^3, \dots, x^r, x^1) = x^2, \dots, F(s)(x^r, x^1, x^2, \dots, x^{r-1}) = x^r$ for all $t \in G$.

For $r = 1, 2, 3$ in Theorem 3.1, we get the following two corollaries which are due to Soliman [27].

Corollary 3.3 [27] *Let (X, d) be a complete bounded metric space with property $(*)$ and uniform normal structure, and let $\Im = \{F(t) : t \in G\}$ be an asymptotically regular semigroup on $X \times X \times X$ satisfying*

$$k \cdot \widetilde{k} < \frac{1}{N(X)}.$$

Then there exist some $x, y, z \in X$ such that $F(s)(x, y, z) = x$, $F(s)(y, z, x) = y$ and $F(s)(z, x, y) = z$ for all $t \in G$.

Corollary 3.4 [27] *Let (X, d) be a complete bounded metric space with property (P) and uniform normal structure, and let $\Im = \{F(t) : t \in G\}$ be an asymptotically regular semigroup on $X \times X \times X$ satisfying*

$$k \cdot \widetilde{k} < \frac{1}{\tilde{N}(X)}.$$

Then there exist some $x, y, z \in X$ such that $F(s)(x, y, z) = x$, $F(s)(y, z, x) = y$ and $F(s)(z, x, y) = z$ for all $t \in G$.

Remark 3.2 It is well known that the Lipschitzian mapping is uniformly continuous. It is natural to ask if there is a contractive mapping definition which does not force it to be continuous. It was answered affirmatively by Kannan. It is clear that Lipschitzian mappings are always continuous and Kannan type mappings are not necessarily continuous. It will be interesting to establish Theorem 3.1 for representative $\psi = \{F(t) : t \in G\}$ on $\prod_{i=1}^{r} X$ satisfying the following condition:

$$\begin{aligned}
&d\big(F(t)\big(x^1, x^2, \dots, x^r\big), F(t)\big(y^1, y^2, \dots, y^r\big)\big) \\
&\quad \le \frac{\beta}{r}\big[d\big(x^1, F(t)\big(x^1, x^2, \dots, x^r\big)\big) + d\big(y^1, F(t)\big(y^1, y^2, \dots, y^r\big)\big) \\
&\qquad + d\big(x^2, F(t)\big(x^2, x^3, \dots, x^r, x^1\big)\big) + d\big(y^2, F(t)\big(y^2, y^3, \dots, y^r, y^1\big)\big)
\end{aligned}$$

$$\vdots$$

$$+ d\big(x^r, F(t)\big(x^r, x^1, \ldots, x^{r-1}\big)\big) + d\big(y^r, F(t)\big(y^r, y^1, \ldots, y^{r-1}\big)\big)\big]$$

for all $x^1, x^2, \ldots, x^r, y^1, y^2, \ldots, y^r \in X$ and $0 < \beta$.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
The author is grateful to anonymous referees for their fruitful comments.

Received: 24 January 2014 Accepted: 4 July 2014 Published: 18 Aug 2014

References

1. Boyd, DW, Wong, JS: On nonlinear contractions. Proc. Am. Math. Soc. **20**, 458-464 (1969)
2. Casini, E, Maluta, E: Fixed points of uniformly Lipschitzian mappings in metric spaces with uniform normal structure. Nonlinear Anal. **9**, 103-108 (1985)
3. Ćirić, LB: A generalization of Banach's contraction principle. Proc. Am. Math. Soc. **45**, 267-273 (1974)
4. Dorić, D, Kadelburg, Z, Radenović, S: Coupled fixed point results for mappings without mixed monotone property. Appl. Math. Lett. (2012). doi:10.1016/j.aml.2012.02.022
5. Dunford, N, Schwartz, JT: Linear Operators. Interscience, New York (1958)
6. Guo, D, Lakshmikantham, V: Coupled fixed points of nonlinear operators with applications. Nonlinear Anal. **11**(5), 623-632 (1987)
7. Bhaskar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**, 1379-1393 (2006)
8. Agarwal, RP, Sintunavarat, W, Kumam, P: Coupled coincidence point and common coupled fixed point theorems lacking the mixed monotone property. Fixed Point Theory Appl. **2013**, Article ID 22 (2013)
9. Cho, YJ, Gupta, A, Karapınar, E, Kumam, P, Sintunavarat, W: Tripled proximity point theorem in metric spaces. Math. Inequal. Appl. **16**(4), 1197-1216 (2013)
10. Imdad, M, Soliman, AH, Choudhury, BS, Das, P: On *n*-tupled coincidence point results in metric spaces. J. Oper. **2013**, Article ID 532867 (2013)
11. Karapınar, E, Kumam, P, Sintunavarat, W: Coupled fixed point theorems in cone metric spaces with a *c*-distance and applications. Fixed Point Theory Appl. **2012**, Article ID 194 (2012)
12. Kutbi, MA, Roldan, A, Sintunavarat, W, Martinez-Moreno, J, Roldan, C: F-closed sets and coupled fixed point theorems without the mixed monotone property. Fixed Point Theory Appl. **2013**, Article ID 330 (2013)
13. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**, 4341-4349 (2009)
14. Sabetghadam, F, Masiha, HP, Sanatpour, AH: Some coupled fixed point theorems in cone metric spaces. Fixed Point Theory Appl. **2009**, Article ID 125426 (2009)
15. Sintunavarat, W, Cho, YJ, Kumam, P: Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces. Fixed Point Theory Appl. **2011**, Article ID 81 (2011)
16. Wairojjana, N, Sintunavarat, W, Kumam, P: Common tripled fixed point theorems for *W*-compatible mappings along with the CLR_g property in abstract metric spaces. J. Inequal. Appl. **2014**, Article ID 133 (2014)
17. Samet, B, Vetro, C: Coupled fixed point, *F*-invariant set fixed point of *N*-order. Ann. Funct. Anal. **1**, 46-56 (2010)
18. Imdad, M, Sharma, A, Rao, KPR: Generalized *n*-tupled fixed point theorems for nonlinear contractions. Afr. Math. (2013). doi:10.1007/s13370-013-0217-8
19. Sharma, A: Results on *n*-tupled fixed points in complete asymptotically regular metric spaces. J. Egypt. Math. Soc. (2013, in press)
20. Brodskii, MS, Milman, DP: On the center of a convex set. Dokl. Akad. Nauk SSSR **59**, 837-840 (1948) (in Russian)
21. Kirk, WA: A fixed point theorem for mappings which do not increase distance. Am. Math. Mon. **72**, 1004-1006 (1965)
22. Kijima, Y, Takahashi, W: A fixed point theorem for nonexpansive mappings in metric spaces. Kodai Math. Semin. Rep. **21**, 326-330 (1969)
23. Khamsi, MA: On metric spaces with uniform normal structure. Proc. Am. Math. Soc. **106**, 723-726 (1989)
24. Maluta, E: Uniformly normal structure and related coefficients. Pac. J. Math. **111**, 357-369 (1984)
25. Lim, T-C, Xu, H-K: Uniformly Lipschitzian mappings in metric spaces with uniform normal structure. Nonlinear Anal., Theory Methods Appl. **25**(11), 1231-1235 (1995)
26. Yao, J-C, Zeng, L-C: Fixed point theorem for asymptotically regular semigroups in metric spaces with uniform normal structure. J. Nonlinear Convex Anal. **8**(1), 153-163 (2007)
27. Soliman, AH: A tripled fixed point theorem for semigroups of Lipschitzian mappings on metric spaces with uniform normal structure. Fixed Point Theory Appl. **2013**, Article ID 346 (2013)
28. Soliman, AH: A coupled fixed point theorem for nonexpansive one parameter semigroup. J. Adv. Math. Stud. **7**(2), 2-14 (2014)

10.1186/1687-1812-2014-168
Cite this article as: Soliman: **Results on *n*-tupled fixed points in metric spaces with uniform normal structure.** *Fixed Point Theory and Applications* **2014, 2014:168**