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Iterative algorithms based on hybrid method and Cesàro mean of asymptotically nonexpansive mappings for equilibrium problems

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Abstract
Using Cesàro means of a mapping, we modify the progress of Mann's iteration in hybrid method for asymptotically nonexpansive mappings in Hilbert spaces. Under suitable conditions, we prove that the iterative sequence converges strongly to a fixed point of an asymptotically nonexpansive mapping. We also introduce a new hybrid iterative scheme for finding a common element of the set of common fixed points of asymptotically nonexpansive mappings and the set of solutions of an equilibrium problem in Hilbert spaces.

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1 Introduction

Let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. A mapping $T:C\to C$ is said to be asymptotically nonexpansive if for each $n\geq 1$, there exists a nonnegative real number k_n satisfying $\lim_{n\to\infty}k_n=1$ such that

$$\|T^nx-T^ny\|\leq k_n\|x-y\|,\quad \forall x,y\in C;$$

when $k_n\equiv 1$, T is called nonexpansive.

The concept of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [1] in 1972. We denote by $F(T):=\{x\in C:Tx=x\}$ the set of fixed points of T. It is well known that if $T:H\to H$ is asymptotically nonexpansive, then $F(T)$ is nonempty convex.

In 1953, Mann [2] introduced the iteration as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1}=\alpha_nx_n+(1-\alpha_n)Tx_n. \tag{1.1}$$

In an infinite-dimensional Hilbert space, Mann iteration could conclude only weak convergence [3]. Attempts to modify the Mann iteration method (1.1) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [4] proposed the following modification of Mann iteration method for a nonexpansive mapping T in a Hilbert space:

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n = 0,1,2\ldots, \end{cases} \tag{1.2}$$

where P_K denotes the metric projection from H onto a closed convex subset K of H. The above method is also called CQ method or hybrid method.

In 2006, Kim and Xu [5] adapted the iteration (1.2) in a Hilbert space. More precisely, they introduced the following iteration process for asymptotically nonexpansive mappings:

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n = 0,1,2\ldots, \end{cases} \tag{1.3}$$

where

$$\theta_n = (1-\alpha_n)\big(k_n^2 - 1\big)(\operatorname{diam} C)^2 \to 0 \quad \text{as } n \to \infty.$$

They proved that $\{x_n\}$ converges in norm to $P_{F(T)}x_0$ under some conditions. Several authors (see [6, 7]) have studied the convergence of hybrid method.

Baillon [8] first proved that the following Cesàro mean iterative sequence weakly converges to a fixed point of a nonexpansive mapping in Hilbert spaces:

$$T_n x = \frac{1}{n+1} \sum_{i=0}^{n} T^i x.$$

Shimizu and Takahashi [9] proved a strong convergence theorem of the above iteration for an asymptotically nonexpansive mapping in Hilbert spaces.

Let C be a nonempty closed convex subset of a real Hilbert space H, let $f : C \times C \to \mathbb{R}$ be a functional, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem is to find $\hat{x} \in C$ such that

$$f(x,y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

The set of solutions of (1.4) is denoted by EP(f). Given a mapping $T : C \to X^*$, let $f(x,y) = \langle Tx, y-x \rangle$, $\forall x, y \in C$, then $z \in \mathrm{EP}(f)$ if and only if $\langle Tz, y-z \rangle \geq 0$, $\forall y \in C$, *i.e.*, z is the solution of the variational inequality.

There are several other problems, for example, the complementarity problem, fixed point problem and optimization problem, which can also be written in the form of an equilibrium problem. So, equilibrium problems provide us with a systematic framework to study a wide class of problems arising in financial economics, optimization and operation research *etc.*, which motivates the extensive concern. See, for example, [10–14]. In recent years, equilibrium problems have been deeply and thoroughly researched. See, for example, [15–20]. Some methods have been proposed to solve the equilibrium problem in a Hilbert space; see, for instance, [21–23]. In 2011, Jitpeera, Katchang, and Kumam [24] found a common element of the set of solutions for mixed equilibrium problem, the set of solutions of the variational inequality for a β-inverse strongly monotone mapping, and the set of fixed points of a family of finitely nonexpansive mappings in a real Hilbert space by using the viscosity and Cesàro mean approximation method.

Motivated by the above-mentioned results, in this paper we introduce the following iteration process for asymptotically nonexpansive mappings T with C a closed convex bounded subset of a real Hilbert space:

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n = 0,1,2\ldots, \end{cases} \tag{1.5}$$

where

$$\theta_n = (1-\alpha_n)\bigl(L_n^2 - 1\bigr)(\operatorname{diam} C)^2 \to 0 \quad \text{as } n \to \infty.$$

We shall prove that the above iterative sequence $\{x_n\}$ converges strongly to a fixed point of T under some proper conditions. In addition, we also introduce a new hybrid iterative scheme for finding a common element of the set of common fixed points of asymptotically nonexpansive mappings and the set of solutions of an equilibrium problem in Hilbert spaces.

We will use the notation $\rightharpoonup$ for weak convergence and $\to$ for strong convergence.

2 Preliminaries

Let H be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle \tag{2.1}$$

and

$$\bigl\|\lambda x + (1-\lambda)y\bigr\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. It is also known that H satisfies

(1) Opial's condition, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

(2) The Kadec-Klee property, that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ together implies $\|x_n - x\| \to 0$.

Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that

$$\|x - P_C(x)\| \le \|x - y\|, \quad \forall y \in C.$$

Such a mapping P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Furthermore, for $x \in H$ and $z \in C$,

$$z = P_C(x) \quad \text{if and only if} \quad \langle x - z, z - y\rangle \ge 0, \quad \forall y \in C.$$

We also need the following lemmas.

Lemma 2.1 (See [25]) *Let T be an asymptotically nonexpansive mapping defined on a bounded closed convex subset C of a Hilbert space H. Assume that $\{x_n\}$ is a sequence in C with the properties*: (i) $x_n \rightharpoonup z$; (ii) $Tx_n - x_n \to 0$. *Then $z \in F(T)$.*

Lemma 2.2 (See [9]) *Let C be a nonempty bounded subset of a Hilbert space. Let T be an asymptotically nonexpansive mapping from C into itself such that $F(T)$ is nonempty. Then, for any $\varepsilon > 0$, there exists a positive integer l_ε such that for any integer $l \ge l_\varepsilon$, there is a positive integer n_l satisfying*

$$\left\| \frac{1}{n+1}\sum_{j=0}^{n} T^j x - T^l\left(\frac{1}{n+1}\sum_{j=0}^{n} T^j x\right)\right\| < \varepsilon, \quad \forall x \in C, \forall n \ge n_l.$$

The equilibrium problem is to find $\hat{x} \in C$ such that

$$f(\hat{x}, y) \ge 0, \quad \forall y \in C.$$

The set of solutions of the above inequality is denoted by $\mathrm{EP}(f)$. For solving the equilibrium problem, let us assume that a bifunction f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, *i.e.*, $f(x,y) + f(y,x) \le 0$, for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} f\bigl(tz + (1-t)x, y\bigr) \le f(x,y);$$

(A4) for each $x \in C$, $f(x,\cdot)$ is convex and lower semi-continuous.

The following lemma appears implicitly in [21].

Lemma 2.3 ([21]) *Let C be a nonempty closed convex subset of H, let f be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying* (A1)-(A4), *and let $r > 0$ and $x \in H$. Then there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \ge 0, \quad \forall y \in C.$$

The following lemma was also given in [26].

Lemma 2.4 ([26]) *Assume that $f : C \times C \to \mathbb{R}$ satisfies* (A1)-(A4). *For $r > 0$ and $x \in H$, define a mapping $T_r : X \to C$ as follows*:

$$T_r(x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C \right\}$$

for all $z \in H$. Then the following hold:

(1) *T_r is single-valued*;

(2) *T_r is firmly nonexpansive, i.e.*,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H;$$

(3) $F(T_r) = \mathrm{EP}(f)$;

(4) $\mathrm{EP}(f)$ *is closed and convex*.

Lemma 2.5 ([26]) *Let C be a nonempty closed convex subset of H, let $f : C \times C \to \mathbb{R}$ be a functional, satisfying* (A1)-(A4), *and let $r > 0$. Then, for $x \in H$ and $q \in F(T_r)$*,

$$\|q - T_r x\|^2 + \|T_r x - x\|^2 \leq \|q - x\|^2.$$

3 Strong convergence theorem of modified Mann iteration based on hybrid method

Inspired by Kim and Xu's results (see [5]), Mann-type iteration (1.3) is modified to obtain the strong convergence theorem as follows.

Theorem 3.1 *Let C be a nonempty bounded closed convex subset of a real Hilbert space H and let $T : C \to C$ be an asymptotically nonexpansive mapping with k_n, denote $L_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j$. Assume that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0,1)$ such that $\alpha_n \leq a$ for all n and for some $0 < a < 1$. Define a sequence $\{x_n\}_{n=0}^{\infty}$ in C by the following algorithm*:

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1 - \alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n = 0, 1, 2 \ldots, \end{cases} \tag{3.1}$$

where

$$\theta_n = (1 - \alpha_n)\bigl(L_n^2 - 1\bigr)(\operatorname{diam} C)^2 \to 0 \quad \text{as } n \to \infty.$$

Then $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$.

Proof First note that T has a fixed point in C (see [1]); that is, $F(T)$ is nonempty. We divide the proof of this theorem into four steps as below.

Step 1. We show that C_n and Q_n are closed and convex for each $n \in \mathbb{N} \cup \{0\}$.

From the definition of C_n and Q_n, it is obvious that C_n is closed and Q_n is closed and convex for each $n \in \mathbb{N} \cup \{0\}$. We prove that C_n is convex. Since $\|y_n - z\|^2 \le \|x_n - z\|^2 + \theta_n$ is equivalent to

$$2\langle x_n - y_n, z\rangle \le \|x_n\|^2 - \|y_n\|^2 + \theta_n,$$

it follows that C_n is convex.

Step 2. We show that $F(T) \subset C_n$, $\forall n \in \mathbb{N} \cup \{0\}$.

Let $p \in F(T)$ and $n \in \mathbb{N} \cup \{0\}$. Then from

$$\begin{aligned} \|y_n - p\|^2 &= \left\| \alpha_n x_n + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n - p \right\|^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} \|T^j x_n - p\|\right)^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)\left(\frac{1}{n+1}\sum_{j=0}^{n} k_j \|x_n - p\|\right)^2 \\ &\le \alpha_n \|x_n - p\|^2 + (1-\alpha_n)L_n^2 \|x_n - p\|^2 \\ &= \|x_n - p\|^2 + (1-\alpha_n)\big(L_n^2 - 1\big)\|x_n - p\|^2 \\ &\le \|x_n - p\|^2 + \theta_n, \end{aligned}$$

we have $p \in C_n$. Next, we show that $F(T) \subset Q_n$, $\forall n \in \mathbb{N} \cup \{0\}$. We prove this by induction. For $n = 0$, we have $F(T) \subset C = Q_0$. Suppose that $F(T) \subset Q_n$, then $\emptyset \ne F(T) \subset C_n \cap Q_n$ and there exists a unique element $x_{n+1} \in C_n \cap Q_n$ such that $x_{n+1} = P_{C_n \cap Q_n}(x_0)$. Then

$$\langle x_{n+1} - z, x_0 - x_{n+1}\rangle \ge 0, \quad \forall z \in C_n \cap Q_n.$$

In particular,

$$\langle x_{n+1} - p, x_0 - x_{n+1}\rangle \ge 0, \quad \forall p \in F(T).$$

It follows that $F(T) \subset Q_{n+1}$ and hence $F(T) \subset Q_n$ for each n. Thus we obtain $F(T) \subset C_n \cap Q_n$, $\forall n \in \mathbb{N} \cup \{0\}$. This means that $\{x_n\}$ is well defined.

Step 3. We show that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

It follows from the definition of Q_n that $x_n = P_{Q_n}(x_0)$. Therefore

$$\|x_n - x_0\| \le \|z - x_0\| \quad \text{for all } z \in Q_n \text{ and all } n \in \mathbb{N} \cup \{0\}.$$

Let $z \in F(T) \subset Q_n$ for all $n \in \mathbb{N} \cup \{0\}$. Then

$$\|x_n - x_0\| \le \|z - x_0\| \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

On the other hand, from $x_{n+1} = P_{C_n \cap Q_n}(x_0) \in Q_n$, we have

$$\|x_n - x_0\| \le \|x_{n+1} - x_0\| \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Therefore, the sequence $\{\|x_n - x_0\|\}$ is nondecreasing. Since C is bounded, we obtain that $\lim_{n\to\infty}\|x_n - x_0\|$ exists. This implies that $\{x_n\}$ is bounded. Noticing again that $x_{n+1} = P_{C_n\cap Q_n}(x_0) \in Q_n$ and $x_n = P_{Q_n}(x_0)$, we have $\langle x_{n+1} - x_n, x_n - x_0\rangle \geq 0$. It follows from (2.1) that

$$\begin{aligned}\|x_{n+1} - x_n\|^2 &= \big\|(x_{n+1} - x_0) - (x_n - x_0)\big\|^2 \\ &= \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2\end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Since $x_{n+1} = P_{C_n\cap Q_n} \in C_n$, we have $\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n$. Noticing that $\lim_{n\to\infty} k_n = 1$, then $L_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j \to 1$ as $n \to \infty$. Hence

$$\|y_n - x_{n+1}\| \leq \|x_n - x_{n+1}\| + \sqrt{\theta_n} \to 0.$$

Let $A_n = \frac{1}{n+1}\sum_{j=0}^{n} T^j$. Also since $\alpha_n \leq a < 1$ for all n, then

$$\begin{aligned}\|x_n - A_n x_n\| &= \frac{1}{1-\alpha_n}\|y_n - x_n\| \\ &\leq \frac{1}{1-a}\big(\|y_n - x_{n+1}\| + \|x_n - x_{n+1}\|\big) \to 0.\end{aligned}$$

From Lemma 2.2, we have

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\big\|A_n x_n - T^l(A_n x_n)\big\| \leq \limsup_{l\to\infty}\limsup_{n\to\infty}\sup_{x\in C}\big\|A_n x - T^l(A_n x)\big\| = 0.$$

Therefore,

$$\limsup_{l\to\infty}\limsup_{n\to\infty}\big\|A_n x_n - T^l(A_n x_n)\big\| = 0.$$

Put $k_\infty = \sup\{k_n : n \geq 1\} < \infty$, then

$$\begin{aligned}\big\|x_n - T^l x_n\big\| &\leq \|x_n - A_n x_n\| + \big\|A_n x_n - T^l(A_n x_n)\big\| + \big\|T^l(A_n x_n) - T^l x_n\big\| \\ &\leq \|x_n - A_n x_n\| + \big\|A_n x_n - T^l(A_n x_n)\big\| + k_l\|x_n - A_n x_n\| \\ &\leq (1 + k_\infty)\|x_n - A_n x_n\| + \big\|A_n x_n - T^l(A_n x_n)\big\| \to 0\end{aligned}$$

as $n, l \to \infty$. Thus, we have

$$\begin{aligned}\|Tx_n - x_n\| \leq{} & \big\|Tx_n - T^{l+1}x_n\big\| + \big\|T^{l+1}x_n - T^{l+1}x_{n+1}\big\| \\ & + \big\|T^{l+1}x_{n+1} - x_{n+1}\big\| + \|x_{n+1} - x_n\| \\ \leq{} & k_\infty\big\|x_n - T^l x_n\big\| + \big\|x_{n+1} + T^l x_{n+1}\big\| + (k_\infty + 1)\|x_{n+1} - x_n\| \to 0.\end{aligned}$$

Step 4. We show that $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$.

Put $w = P_{F(T)}(x_0)$. Since $\{x_n\}$ is bounded, let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w'$. By Lemma 2.1, we get $w' \in F(T)$. Since $x_n = P_{Q_n}(x_0)$ and $w \in F(T) \subset Q_n$, we have $\|x_n - x_0\| \le \|w - x_0\|$. It follows from $w = P_{F(T)}(x_0)$ and the weak lower semicontinuity of the norm that

$$\|w - x_0\| \le \|w' - x_0\| \le \liminf_{k\to\infty} \|x_{n_k} - x_0\| \le \limsup_{k\to\infty} \|x_{n_k} - x_0\| \le \|w - x_0\|.$$

Thus, we obtain that $\lim_{k\to\infty} \|x_{n_k} - x_0\| = \|w' - x_0\| = \|w - x_0\|$. Using the Kadec-Klee property of H, we get $\lim_{k\to\infty} x_{n_k} = w' = w$. Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $P_{F(T)}(x_0)$. □

Remark 3.2 It is not difficult to see from the proof above that the boundedness of C can be discarded if T is a nonexpansive mapping.

4 Strong convergence theorem for equilibrium problems

In this section, we prove a strong convergence theorem for finding a common element of the set of zero points of an asymptotically nonexpansive mapping T and the set of solutions of an equilibrium problem in a Hilbert space.

Theorem 4.1 *Let H be a real Hilbert space, and let C be a nonempty bounded closed convex subset of H, let $f : C \times C \to \mathbb{R}$ be a functional, satisfying* (A1)-(A4). *Let $T : C \to C$ be an asymptotically nonexpansive mapping with k_n, denote $L_n = \frac{1}{n+1}\sum_{j=0}^{n} k_j$ such that $F(T) \cap \mathrm{EP}(f) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} x_0 \in C \quad \text{is arbitrary}, \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \\ u_n \in C, \quad \text{such that } f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \quad \forall y \in C, \\ C_n = \{z \in C : \|u_n - z\|^2 \le \|x_n - z\|^2 + \theta_n\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \quad n = 0, 1, 2, \ldots, \end{cases} \tag{4.1}$$

where

$$\theta_n = (1-\alpha_n)\big(L_n^2 - 1\big)(\operatorname{diam} C)^2 \to 0, \quad n \to \infty.$$

Suppose that $\{\alpha_n\} \subset [0,1]$ and there exists $a \in (0,1)$ such that $\alpha_n \le a$, $\forall n \in \mathbb{N}$, and $\{r_n\} \subset (0,\infty)$ such that $\liminf_{n\to\infty} r_n > 0$. Then $\{x_n\}$ converges strongly to $P_{F(T)\cap \mathrm{EP}(f)}(x_0)$.

Proof We divide the proof of this theorem into four steps as below.

Step 1. Similar to the proof of Step 1 in Theorem 3.1, it is easy to see that C_n and Q_n are closed convex sets for each $n \in \mathbb{N} \cup \{0\}$.

Step 2. We show that $F(T) \cap \mathrm{EP}(f) \subset C_n \cap Q_n$, $\forall n \in \mathbb{N} \cup \{0\}$.

Let $p \in F(T) \cap \mathrm{EP}(f)$. Putting $u_n = T_{r_n} y_n$, $\forall n \in \mathbb{N} \cup \{0\}$, by (2) of Lemma 2.4, we have T_{r_n} is relatively nonexpansive. Noticing that relatively nonexpansive mappings are nonexpansive in Hilbert spaces, then for any $n \in \mathbb{N} \cup \{0\}$,

$$\|u_n - p\|^2 = \|T_{r_n} y_n - p\|^2 = \|T_{r_n} y_n - T_{r_n} p\|^2 \le \|y_n - p\|^2.$$

From the proof of Step 2 in Theorem 3.1, we have $\|u_n - p\|^2 \le \|x_n - p\|^2 + \theta_n$. Thus $p \in C_n$, hence $F(T) \cap \mathrm{EP}(f) \subset C_n$. Similar to the proof of Step 2 in Theorem 3.1, it is easy to see that $F(T) \cap \mathrm{EP}(f) \subset Q_n$. Therefore we have $F(T) \cap \mathrm{EP}(f) \subset C_n \cap Q_n$, $\forall n \in \mathbb{N} \cup \{0\}$. This means that $\{x_n\}$ is well defined.

Step 3. We show that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Similar to the proof of Step 3 in Theorem 3.1, we may obtain that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.2}$$

Since $x_{n+1} = P_{C_n \cap Q_n} \in C_n$, then $\|u_n - x_{n+1}\|^2 \le \|x_n - x_{n+1}\|^2 + \theta_n$. Noticing that $\theta_n \to 0$, we have

$$\|u_n - x_{n+1}\| \le \|x_n - x_{n+1}\| + \sqrt{\theta_n} \to 0.$$

Hence

$$\lim_{n\to\infty} \|x_n - u_n\| \le \lim_{n\to\infty} \|x_n - x_{n+1}\| + \lim_{n\to\infty} \|x_{n+1} - u_n\| \to 0. \tag{4.3}$$

For $p \in F(T) \cap \mathrm{EP}(f) \subset C_n$, we have

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \theta_n. \tag{4.4}$$

Since $u_n = T_{r_n} y_n$, by (4.4) and Lemma 2.5, we get that

$$\begin{aligned} \|u_n - y_n\|^2 &= \|T_{r_n} y_n - y_n\|^2 \\ &\le \|y_n - p\|^2 - \|T_{r_n} y_n - p\|^2 \\ &\le \|x_n - p\|^2 + \theta_n - \|u_n - p\|^2 \to 0. \end{aligned} \tag{4.5}$$

Let $A_n = \frac{1}{n+1} \sum_{j=0}^{n} T^j$. Since $\alpha_n \le a < 1$, $\forall n \in \mathbb{N}$, then by (4.3) and (4.5), we have

$$\begin{aligned} \|x_n - A_n x_n\| &= \frac{1}{1 - \alpha_n} \|y_n - x_n\| \\ &\le \frac{1}{1 - a} \bigl(\|y_n - u_n\| + \|u_n - x_n\|\bigr) \to 0. \end{aligned} \tag{4.6}$$

From Lemma 2.2, it follows that

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \bigl\|A_n x_n - T^l(A_n x_n)\bigr\| \le \limsup_{l\to\infty} \limsup_{n\to\infty} \sup_{x\in C} \bigl\|A_n x - T^l(A_n x)\bigr\| = 0.$$

Therefore

$$\limsup_{l\to\infty} \limsup_{n\to\infty} \bigl\|A_n x_n - T^l(A_n x_n)\bigr\| = 0. \tag{4.7}$$

Put $k_\infty = \sup\{k_n : n \geq 1\} < \infty$, then

$$\begin{aligned}\|y_n - T^l x_n\| &= \|\alpha_n x_n + (1-\alpha_n)A_n x_n - T^l x_n\| \\ &\leq \alpha_n \|x_n - A_n x_n\| + \|A_n x_n - T^l(A_n x_n)\| + \|T^l(A_n x_n) - T^l x_n\| \\ &\leq \alpha_n \|x_n - A_n x_n\| + \|A_n x_n - T^l(A_n x_n)\| + k_l \|x_n - A_n x_n\| \\ &\leq (1+k_\infty)\|x_n - A_n x_n\| + \|A_n x_n - T^l(A_n x_n)\|.\end{aligned}$$

Let $n, l \to \infty$, we get that $\|y_n - T^l x_n\| \to 0$. Thus, by (4.3) and (4.5), we have

$$\|x_n - T^l x_n\| \leq \|x_n - u_n\| + \|u_n - y_n\| + \|y_n - T^l x_n\| \to 0. \tag{4.8}$$

Therefore, by (4.8) and (4.2), we obtain that

$$\begin{aligned}\|Tx_n - x_n\| &\leq \|Tx_n - T^{l+1}x_n\| + \|T^{l+1}x_n - T^{l+1}x_{n+1}\| + \|T^{l+1}x_{n+1} - x_{n+1}\| \\ &\quad + \|x_{n+1} - x_n\| \\ &\leq k_\infty \|x_n - T^l x_n\| + (k_\infty + 1)\|x_{n+1} - x_n\| + \|T^{l+1}x_{n+1} - x_{n+1}\| \\ &\to 0.\end{aligned}$$

Step 4. We show that $\{x_n\}$ converges strongly to $P_{F(T)\cap \mathrm{EP}(f)}(x_0)$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup w'$. By Lemma 2.1, we have $w' \in F(T)$. Next we show $w' \in \mathrm{EP}(f)$.

From (4.3) and (4.5), we get that $u_{n_k} \rightharpoonup w'$, $y_{n_k} \rightharpoonup w'$. Since $u_n = T_{r_n} y_n$, then

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \geq 0, \quad \forall y \in C.$$

Replacing n by n_k, we have from Condition (A2) that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, u_{n_k} - y_{n_k}\rangle \geq -f(u_{n_k}, y) \geq f(y, u_{n_k}), \quad \forall y \in C.$$

Let $k \to \infty$, since $\liminf_{n\to\infty} r_n > 0$, by (4.5) and Condition (A4), we get that

$$f(y, w') \leq 0, \quad \forall y \in C.$$

For $t \in (0,1)$, $y \in C$, let $y_t = ty + (1-t)w'$, then $y_t \in C$, thus $f(y_t, w') \leq 0$. By Condition (A1), we get that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, w') \leq tf(y_t, y).$$

Dividing by t, we have

$$f(y_t, y) \geq 0, \quad \forall y \in C.$$

Let $t \to 0$. From Condition (A3), we obtain that $f(w', y) \geq 0$, $\forall y \in C$. Therefore, $w' \in \mathrm{EP}(f)$.

Denote $w = P_{F(T)\cap \mathrm{EP}(f)}(x_0)$. Since $x_{n+1} = P_{C_n\cap Q_n}(x_0)$, $w \in F(T) \cap \mathrm{EP}(f) \subset C_n \cap Q_n$, then $\|x_{n+1} - x_0\| \le \|w - x_0\|$. Since the norm is weakly lower semicontinuous, we have

$$\|w - x_0\| \le \|w' - x_0\| \le \liminf_{k\to\infty} \|x_{n_k} - x_0\| \le \limsup_{k\to\infty} \|x_{n_k} - x_0\| \le \|w - x_0\|.$$

Hence $\lim_{k\to\infty} \|x_{n_k} - x_0\| = \|w' - x_0\| = \|w - x_0\|$. Using the Kadec-Klee property of H, we get $\lim_{k\to\infty} x_{n_k} = w' = w$. Since $\{x_{n_k}\}$ is an arbitrary subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $P_{F(T)\cap \mathrm{EP}(f)}(x_0)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper is proposed by JZ and YC. JZ prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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References
1. Goebel, K, Kirk, WA: A fixed point theorem for asymptotically nonexpansive mappings. Proc. Am. Math. Soc. **35**, 171-174 (1972)
2. Mann, WR: Mean value methods in iteration. Proc. Am. Math. Soc. **4**, 506-510 (1953)
3. Genel, A, Lindenstrass, L: An example concerning fixed points. Isr. J. Math. **22**, 81-86 (1975)
4. Nakajo, K, Takahashi, W: Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. J. Math. Anal. Appl. **279**, 372-379 (2003)
5. Kim, TH, Xu, HK: Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups. Nonlinear Anal. **64**, 1140-1152 (2006)
6. Yanes, CM, Xu, HK: Strong convergence of the CQ method for fixed point iteration processes. Nonlinear Anal. **64**, 2400-2411 (2006)
7. Nilsrakoo, W, Seajung, S: Weak and strong convergence theorems for countable Lipschitzian mappings and its applications. Nonlinear Anal. **69**, 2695-2708 (2008)
8. Baillon, JB: Un théorème de type ergodique pour les contractions nonlinéairs dans un espaces de Hilbert. C. R. Acad. Sci. Paris Sér. A-B **280**, 1511-1541 (1975)
9. Shimizu, T, Takahashi, W: Strong convergence theorem for asymptotically nonexpansive mappings. Nonlinear Anal. **26**, 265-272 (1996)
10. Li, J: The generalized projection operator on reflexive Banach spaces and its applications. J. Math. Anal. Appl. **306**, 55-71 (2005)
11. Li, J, Lin, SL, Zhang, CJ: On the existence of Nash equilibriums for infinite matrix games. Nonlinear Anal. **10**, 42-53 (2009)
12. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**, 1127-1138 (1991)
13. Qin, X, Su, Y: Strong convergence theorems for relatively nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 1958-1965 (2007)
14. Wlodarczyk, K, Klim, D: Equilibria and fixed points of set-valued maps with nonconvex and noncompact domains and ranges. Nonlinear Anal. **65**, 918-932 (2006)
15. Chadli, O, Chbani, Z, Riahi, H: Equilibrium problems and noncoercive variational inequalities. Optimization **49**, 1-12 (1999)
16. Ansari, QH, Yao, JC: An existence results for the generalized vector equilibrium problem. Appl. Math. Lett. **12**, 53-56 (1999)
17. Lan, H: Common fixed-point iterative processes with errors for generalized asymptotically quasi-nonexpansive mappings. Comput. Math. Appl. **52**, 1403-1412 (2006)
18. Zhao, J, He, S: A hybrid iteration scheme for equilibrium problems and common fixed point problems of generalized quasi-ϕ-asymptotically nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2012**, 33 (2012)
19. Chang, S, Wang, L, Tang, YK: Strong convergence theorems of nonlinear operator equations for countable family of multivalued total quasi-ϕ-asymptotically nonexpansive mappings with applications. Fixed Point Theory Appl. **2012**, 69 (2012)

20. Zegeye, H, Shahzad, N: Strong convergence theorems for variational inequality problems and quasi-ϕ-asymptotically nonexpansive mappings. J. Glob. Optim. **54**, 101-116 (2012)
21. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Program. **63**, 123-145 (1994)
22. Combettes, PL, Hirstoaga, SA: Equilibrium programming in Hilbert spaces. J. Nonlinear Convex Anal. **6**, 117-136 (2005)
23. Takahashi, S, Takahashi, W: Viscosity approximation methods for equilibrium problems and fixed point in Hilbert space. J. Math. Anal. Appl. **331**, 506-515 (2007)
24. Jitpeera, T, Katchang, P, Kumam, P: A viscosity of Cesàro mean approximation methods for a mixed equilibrium, variational inequalities, and fixed point problems. Fixed Point Theory Appl. **2011**, 945051 (2011)
25. Browder, FE: Nonlinear operators and nonlinear equations of evolution in Banach spaces. Proc. Symp. Pure Math. **18**, 78-81 (1976)
26. Takahashi, W, Zembayashi, K: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. Nonlinear Anal. **70**, 45-57 (2009)

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