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New conditions on fuzzy coupled coincidence fixed point theorem

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Abstract

Recently, Choudhury *et al.* proved a coupled coincidence point theorem in a partial order fuzzy metric space. In this paper, we give a new version of the result of Choudhury *et al.* by removing some restrictions. In our result, the mappings are not required to be compatible, continuous or commutable, and the t-norm is not required to be of Hadžić-type. Finally, two examples are presented to illustrate the main result of this paper.

MSC: 54E70; 47H25

Keywords: fuzzy metric space; contraction mapping; coincidence fixed point; partial order

1 Introduction

The concept of fuzzy metric spaces was defined in different ways [1–3]. Grabiec [4] presented a fuzzy version of Banach contraction principle in a fuzzy metric space of Kramosi and Michalek's sense. Fang [5] proved some fixed point theorems in fuzzy metric spaces, which improve, generalize, unify, and extend some main results of Edelstein [6], Istratescu [7], Sehgal and Bharucha-Reid [8].

In order to obtain a Hausdorff topology, George and Veeramani [9, 10] modified the concept of fuzzy metric space due to Kramosil and Michalek [11]. Many fixed point theorems in complete fuzzy metric spaces in the sense of George and Veeramani [9, 10] have been obtained. For example, Singh and Chauhan [12] proved some common fixed point theorems for four mappings in GV fuzzy metric spaces. Gregori and Sapena [13] proved that each fuzzy contractive mapping has a unique fixed point in a complete GV fuzzy metric space in which fuzzy contractive sequences are Cauchy.

The coupled fixed point theorem and its applications in metric spaces are firstly obtained by Bhaskar and Lakshmikantham [14]. Recently, some authors considered coupled fixed point theorems in fuzzy metric spaces; see [15–18].

In [15], the authors gave the following results.

Theorem 1.1 [15, Theorem 2.5] *Let $a * b > ab$ for all $a, b \in [0,1]$ and $(X, M, *)$ be a complete fuzzy metric space such that M has n-property. Let $F : X \times X \to X$ and $g : X \to X$ be two functions such that*

$$M\bigl(F(x,y), F(u,v), kt\bigr) \geq M(gx, gu, t) * M(gy, gv, t)$$

for all $x, y, u, v \in X$, where $0 < k < 1$, $F(X \times X) \subseteq g(X)$ and g is continuous and commutes with F. Then there exists a unique $x \in X$ such that $x = gx = F(x, x)$.

Let $\Phi = \{\phi : \mathbb{R}^+ \to \mathbb{R}^+\}$, where $\mathbb{R}^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfies the following conditions:

(ϕ-1) ϕ is non-decreasing;
(ϕ-2) ϕ is upper semicontinuous from the right;
(ϕ-3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$ where $\phi^{n+1}(t) = \phi(\phi^n(t))$, $n \in \mathbb{N}$.

In [16], Hu proved the following result.

Theorem 1.2 [16, Theorem 1] *Let $(X, M, *)$ be a complete fuzzy metric space, where $*$ is a continuous t-norm of H-type. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings and let there exist $\phi \in \Phi$ such that*

$$M\bigl(F(x, y), F(u, v), \phi(t)\bigr) \ge M(gx, gu, t) * M(gy, gv, t)$$

for all $x, y, u, v \in X$, $t > 0$. Suppose that $F(X \times X) \subseteq g(X)$, and g is continuous; F and g are compatible. Then there exists $x \in X$ such that $x = gx = F(x, x)$, that is, F and g have a unique common fixed point in X.

Choudhury *et al.* [17] gave the following coupled coincidence fixed point result in a partial order fuzzy metric space.

Theorem 1.3 [17, Theorem 3.1] *Let $(X, M, *)$ be a complete fuzzy metric space with a Hadžić type t-norm $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. Let $\preceq$ be a partial order defined on X. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has mixed g-monotone property and satisfies the following conditions*:

(i) *$F(X \times X) \subseteq g(X)$,*
(ii) *g is continuous and monotonic increasing,*
(iii) *(g, F) is a compatible pair,*
(iv) *$M(F(x, y), F(u, v), kt) \ge \gamma(M(g(x), g(u), t) * M(g(y), g(v), t))$ for all $x, y, u, v \in X$, $t > 0$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, where $k \in (0, 1)$, $\gamma : [0, 1] \to [0, 1]$ is a continuous function such that $\gamma(a) * \gamma(a) \ge a$ for each $0 \le a \le 1$.*

Also suppose that X has the following properties:

(a) *if we have a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$,*
(b) *if we have a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N} \cup \{0\}$.*

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$, and $M(g(x_0), F(x_0, y_0), t) * M(g(y_0), F(y_0, x_0), t) > 0$ for all $t > 0$, then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$, that is, g and F have a coupled coincidence point in X.

Wang *et al.* [18] proved the following coupled fixed point result in a fuzzy metric space.

Theorem 1.4 [18, Theorem 3.1] *Let $(X, M, *)$ be a fuzzy metric space under a continuous t-norm $*$ of H-type. Let $\phi : (0, \infty) \to (0, \infty)$ be a function satisfying $\lim_{n\to\infty} \phi^n(t) = 0$ for any $t > 0$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings with $F(X \times X) \subseteq g(X)$ and* *assume that for any $t > 0$,*

$$M\big(F(x,y),F(u,v),\phi(t)\big) \geq M(gx,gu,t) * M(gy,gv,t)$$

for all $x,y,u,v \in X$. Suppose that $F(X \times X)$ is complete and g and F are w-compatible, then g and F have a unique common fixed point $x^* \in X$, that is, $x^* = g(x^*) = F(x^*,x^*)$.

In this paper, by modifying the conditions on the result of Choudhury *et al.* [17], we give a new coupled coincidence fixed point theorem in partial order fuzzy metric spaces. In our result, we do not require that the t-norm is of Hadžić-type [19], the mappings are compatible [16], commutable, continuous or monotonic increasing. Our proof method is different from the one of Choudhury *et al.* Finally, some examples are presented to illustrate our result.

2 Preliminaries

Definition 2.1 [9] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a,b,c,d \in [0,1]$.

Typical examples of the continuous t-norm are $a *_1 b = ab$ and $a *_2 b = \min\{a,b\}$ for all $a,b \in [0,1]$.

A t-norm $*$ is said to be positive if $a * b > 0$ for all $a,b \in (0,1]$. Obviously, $*_1$ and $*_2$ are positive t-norms.

Definition 2.2 [9] The 3-tuple $(X,M,*)$ is called a fuzzy metric space if X is an arbitrary non-empty set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions for each $x,y,z \in X$ and $t,s > 0$:

(GV-1) $M(x,y,t) > 0$,
(GV-2) $M(x,y,t) = 1$ if and only if $x = y$,
(GV-3) $M(x,y,t) = M(y,x,t)$,
(KM-4) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ is continuous,
(KM-5) $M(x,y,t+s) \geq M(x,z,t) * M(y,z,s)$.

Lemma 2.1 [4] *Let $(X,M,*)$ be a fuzzy metric space. Then $M(x,y,*)$ is non-decreasing for all $x,y \in X$.*

Lemma 2.2 [20] *Let $(X,M,*)$ be a fuzzy metric space. Then M is a continuous function on $X^2 \times (0,\infty)$.*

Definition 2.3 [9] Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called an M-Cauchy sequence, if for each $\epsilon \in (0,1)$ and $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,t) > 1 - \epsilon$ for all $m,n \geq n_0$. The fuzzy metric space $(X,M,*)$ is called M-complete if every M-Cauchy sequence is convergent.

Let $(X, \preceq)$ be a partially ordered set and F be a mapping from X to itself. A sequence $\{x_n\}$ in X is said to be non-decreasing if for each $n \in \mathbb{N}$, $x_n \preceq x_{n+1}$. A mapping $g : X \to X$ is called monotonic increasing if for all $x, y \in X$ with $x \preceq y$, $g(x) \preceq g(y)$.

Definition 2.4 [21] Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be two mappings. The mapping F is said to have the mixed g-monotone property if for all $x_1, x_2 \in X$, $g(x_1) \preceq g(x_2)$ implies $F(x_1, y) \preceq F(x_2, y)$ for all $y \in X$, and for all $y_1, y_2 \in X$, $g(y_1) \preceq g(y_2)$ implies $F(x, y_1) \succeq F(x, y_2)$ for all $x \in X$.

Definition 2.5 [14] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if

$$F(x,y) = g(x), \qquad F(y,x) = g(y).$$

Here (gx, gy) is called a coupled point of coincidence.

3 Main results

Lemma 3.1 *Let $\gamma : [0,1] \to [0,1]$ be a left continuous function and $*$ be a continuous t-norm. Assume that $\gamma(a) * \gamma(a) > a$ for all $a \in (0,1)$. Then $\gamma(1) = 1$.*

Proof Let $\{a_n\} \subseteq (0,1)$ be a non-decreasing sequence with $\lim_{n\to\infty} a_n = 1$. By hypothesis we have

$$\gamma(a_n) * \gamma(a_n) > a_n, \quad n \in \mathbb{N}.$$

Since γ is left continuous and $*$ is continuous, we get

$$\gamma(1) * \gamma(1) \geq 1,$$

which implies that $\gamma(1) * \gamma(1) = 1$. Since $\gamma(1) \geq \gamma(1) * \gamma(1)$, one has $\gamma(1) = 1$. This completes the proof. □

Theorem 3.1 *Let $(X, M, *)$ be a fuzzy metric space with a continuous and positive t-norm. Let $\preceq$ be a partial order defined on X. Let $\phi : (0,\infty) \to (0,\infty)$ be a function satisfying $\phi(t) \leq t$ for all $t > 0$ and let $\gamma : [0,1] \to [0,1]$ be a left continuous and increasing function satisfying $\gamma(a) * \gamma(a) > a$ for all $a \in (0,1)$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has the mixed g-monotone property and assume that $g(X)$ is complete. Suppose that the following conditions hold*:

(i) $F(X \times X) \subseteq g(X)$,

(ii) *we have*

$$M\big(F(x,y), F(u,v), \phi(t)\big) \geq \gamma\Big(M\big(g(x), g(u), t\big) * M\big(g(y), g(v), t\big)\Big), \tag{3.1}$$

for all $x, y, u, v \in X$, $t > 0$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$,

(iii) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$,*

(iv) *if a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N} \cup \{0\}$.*

If there exist $x_0, y_0 \in X$ *such that* $g(x_0) \preceq F(x_0, y_0)$, $g(y_0) \succeq F(y_0, x_0)$ *and* $M(g(x_0), F(x_0, y_0), t) * M(g(y_0), F(y_0, x_0), t) > 0$ *for all* $t > 0$, *then there exist* $x^*, y^* \in X$ *such that* $g(x^*) = F(x^*, y^*)$ *and* $g(y^*) = F(y^*, x^*)$.

Proof Let $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq g(y_0)$. Define the sequences $\{x_n\}$ and $\{y_n\}$ in X by

$$g(x_{n+1}) = F(x_n, y_n) \quad \text{and} \quad g(y_{n+1}) = F(y_n, x_n), \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Along the lines of the proof of [17], we see that

$$g(x_n) \preceq g(x_{n+1}) \quad \text{and} \quad g(y_n) \succeq g(y_{n+1}), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.2}$$

By (3.1) and (3.2) we have

$$\begin{aligned} M\big(g(x_1), g(x_2), t\big) &\geq M\big(g(x_1), g(x_2), \phi(t)\big) \\ &= M\big(F(x_0, y_0), F(x_1, y_1), \phi(t)\big) \\ &\geq \gamma\big(M\big(g(x_0), g(x_1), t\big) * M\big(g(y_0), g(y_1), t\big)\big) \\ &> M\big(g(x_0), g(x_1), t\big) * M\big(g(y_0), g(y_1), t\big) > 0, \quad \forall t > 0, \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} M\big(g(y_1), g(y_2), t\big) &\geq M\big(g(y_1), g(y_2), \phi(t)\big) \\ &= M\big(F(y_0, x_0), F(y_1, x_1), \phi(t)\big) \\ &\geq \gamma\big(M\big(g(y_0), g(y_1), t\big) * M\big(g(x_0), g(x_1), t\big)\big) \\ &> M\big(g(y_0), g(y_1), t\big) * M\big(g(x_0), g(x_1), t\big) > 0, \quad \forall t > 0. \end{aligned} \tag{3.4}$$

Since $*$ is positive, we have

$$M\big(g(x_1), g(x_2), t\big) * M\big(g(y_1), g(y_2), t\big) > 0, \quad \forall t > 0.$$

Repeating the process (3.3) and (3.4), we get

$$M\big(g(x_2), g(x_3), t\big) > 0 \quad \text{and} \quad M\big(g(y_2), g(y_3), t\big) > 0, \quad \forall t > 0,$$

and further we have

$$M\big(g(x_2), g(x_3), t\big) * M\big(g(y_2), g(y_3), t\big) > 0, \quad \forall t > 0.$$

Continuing the above process, we get, for each $n \in \mathbb{N}$,

$$M\big(g(x_n), g(x_{n+1}), t\big) > 0, \quad \forall t > 0,$$

and

$$M\big(g(y_n), g(y_{n+1}), t\big) > 0, \quad \forall t > 0.$$

Since $*$ is positive, one has

$$M\big(g(x_n), g(x_{n+1}), t\big) * M\big(g(y_n), g(y_{n+1}), t\big) > 0, \quad \forall n \in \mathbb{N}, \forall t > 0.$$

Now we prove by induction that, for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $k \geq n$, one has

$$M\big(g(x_n), g(x_k), t\big) * M\big(g(y_n), g(y_k), t\big) > 0, \quad \forall t > 0. \tag{3.5}$$

Obviously (3.5) holds for $k = n$. Assume that (3.5) holds for some $k \in \mathbb{N}$ with $k > n$. Then we have

$$M\big(g(x_n), g(x_{k+1}), t\big) \geq M\big(g(x_n), g(x_k), t/2\big) * M\big(g(x_k), g(x_{k+1}), t/2\big).$$

Since $M(g(x_n), g(x_k), t/2) > 0$, $M(g(x_k), g(x_{k+1}), t/2) > 0$, and $*$ is positive, we have

$$M\big(g(x_n), g(x_{k+1}), t\big) > 0, \quad \forall t > 0.$$

Similarly, we have

$$M\big(g(y_n), g(y_{k+1}), t\big) > 0, \quad \forall t > 0.$$

Therefore, (3.5) holds for all $k \in \mathbb{N}$ with $k \geq n$.

Now we use the method of Wang [22] to show that both $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Fix $t > 0$. Let

$$a_n = \inf_{k \geq n} M\big(g(x_n), g(x_k), t\big) * M\big(g(y_n), g(y_k), t\big).$$

For $k \geq n + 1$, by (3.1) and (3.2) we have

$$\begin{aligned} M\big(g(x_{n+1}), g(x_k), t\big) &\geq M\big(g(x_{n+1}), g(x_k), \phi(t)\big) \\ &\geq \gamma\big(M\big(g(x_n), g(x_{k-1}), t\big) * M\big(g(y_n), g(y_{k-1}), t\big)\big). \end{aligned}$$

Similarly,

$$M\big(g(y_{n+1}), g(y_k), t\big) \geq \gamma\big(M\big(g(x_n), g(x_{k-1}), t\big) * M\big(g(y_n), g(y_{k-1}), t\big)\big).$$

So, by (3.5) and the hypothesis we have

$$\begin{aligned} &M\big(g(x_{n+1}), g(x_k), t\big) * M\big(g(y_{n+1}), g(y_k), t\big) \\ &\quad \geq *^2\big(\gamma\big(M\big(g(x_n), g(x_{k-1}), t\big) * M\big(g(y_n), g(y_{k-1}), t\big)\big)\big) \\ &\quad \geq M\big(g(x_n), g(x_{k-1}), t\big) * M\big(g(y_n), g(y_{k-1}), t\big) > 0, \end{aligned} \tag{3.6}$$

which implies that

$$a_{n+1} \geq a_n > 0.$$

Since $\{a_n\}$ is bounded, there exists $a \in (0,1]$ such that $\lim_{n\to\infty} a_n = a$. Assume that $a < 1$. Since γ is increasing, we have

$$\begin{aligned} &*^2\left(\gamma\left(M\big(g(x_n), g(x_{k-1}), t\big) * M\big(g(y_n), g(y_{k-1}), t\big)\right)\right) \\ &\quad \geq *^2\Big(\gamma\Big(\inf_{k\geq n+1}\big(M\big(g(x_n), g(x_{k-1}), t\big) * M\big(g(y_n), g(y_{k-1}), t\big)\big)\Big)\Big) \end{aligned}$$

and further

$$\begin{aligned} &\inf_{k\geq n+1} *^2\left(\gamma\left(\left(M\big(g(x_n), g(x_{k-1}), t\big) * M\big(g(y_n), g(y_{k-1}), t\big)\right)\right)\right) \\ &\quad \geq *^2\Big(\gamma\Big(\inf_{k\geq n+1}\big(M\big(g(x_n), g(x_{k-1}), t\big) * M\big(g(y_n), g(y_{k-1}), t\big)\big)\Big)\Big). \end{aligned} \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$\begin{aligned} &\inf_{k\geq n+1}\big(M\big(g(x_{n+1}), g(x_k), t\big) * M\big(g(y_{n+1}), g(y_k), t\big)\big) \\ &\quad \geq *^2\Big(\gamma\Big(\inf_{k\geq n+1}\big(M\big(g(x_n), g(x_{k-1}), t\big) * M\big(g(y_n), g(y_{k-1}), t\big)\big)\Big)\Big), \end{aligned}$$

i.e.,

$$a_{n+1} \geq \gamma(a_n) * \gamma(a_n), \quad \forall n \in \mathbb{N}.$$

Since γ is left continuous, by hypothesis we get

$$a \geq \gamma(a) * \gamma(a) > a.$$

This is a contradiction. So $a = 1$.

For any given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - a_n < \epsilon \quad \text{for all } n \geq n_0.$$

Thus for each $k \geq n \geq n_0$,

$$M\big(g(x_n), g(x_k), t\big) * M\big(g(y_n), g(y_k), t\big) > 1 - \epsilon,$$

which implies that

$$\min\big\{M\big(g(x_n), g(x_k), t\big), M\big(g(y_n), g(y_k), t\big)\big\} > 1 - \epsilon.$$

It follows that both $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences. Since $g(X)$ is complete, there exist $x^*, y^* \in X$ such that $g(x_n) \to g(x^*)$ and $g(y_n) \to g(y^*)$ as $n \to \infty$.

By hypothesis, we have

$$g(x_n) \preceq g\big(x^*\big) \quad \text{and} \quad g(y_n) \succeq g\big(y^*\big), \quad n \in \mathbb{N}. \tag{3.8}$$

Now, for all $t > 0$, by (3.1) and (3.8) we have

$$\begin{aligned} M\big(F(x^*,y^*),g(x^*),t\big) &\geq M\big(F(x^*,y^*),F(x_n,y_n),t/2\big) * M\big(F(x_n,y_n),g(x^*),t/2\big) \\ &\geq M\big(F(x^*,y^*),F(x_n,y_n),\phi(t/2)\big) * M\big(F(x_n,y_n),g(x^*),\phi(t/2)\big) \\ &\geq \gamma\big(M\big(g(x^*),g(x_n),t/2\big) * M\big(g(y^*),g(y_n),t/2\big)\big) \\ &\quad * M\big(F(x_n,y_n),g(x^*),\phi(t/2)\big). \end{aligned} \tag{3.9}$$

Since γ is left continuous and $*$ is continuous, letting $n \to \infty$ in (3.9), we get

$$\begin{aligned} M\big(F(x^*,y^*),g(x^*),t\big) &\geq \lim_{n\to\infty}\big[\gamma\big(M\big(g(x^*),g(x_n),t/2\big) * M\big(g(y^*),g(y_n),t/2\big)\big) \\ &\quad * M\big(F(x_n,y_n),g(x^*),\phi(t/2)\big)\big] \\ &= \gamma(1*1)*1 = 1, \quad \forall t > 0. \end{aligned}$$

It follows that $F(x^*,y^*) = g(x^*)$. Similarly, we can prove that $F(y^*,x^*) = g(y^*)$. This completes the proof. □

If $\phi(t) = t$ for all $t > 0$ in Theorem 3.1, we get the following corollary.

Corollary 3.1 *Let $(X,M,*)$ be a fuzzy metric space with a positive t-norm. Let $\preceq$ be a partial order defined on X. Let $\gamma : [0,1] \to [0,1]$ be a left continuous and increasing function satisfying $\gamma(a) * \gamma(a) > a$ for all $a \in (0,1)$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings such that F has mixed g-monotone property and assume that $g(X)$ is complete. Suppose that the following conditions hold*:

(i) *$F(X \times X) \subseteq g(X)$.*

(ii) *We have*

$$M\big(F(x,y),F(u,v),t\big) \geq \gamma\big(M\big(g(x),g(u),t\big) * M\big(g(y),g(v),t\big)\big),$$

for all $x,y,u,v \in X$, $t > 0$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$.

(iii) *If we have a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N} \cup \{0\}$.*

(iv) *If we have a non-increasing sequence $\{y_n\} \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N} \cup \{0\}$.*

If there exist $x_0, y_0 \in X$ such that $g(x_0) \preceq F(x_0,y_0)$, $g(y_0) \succeq F(y_0,x_0)$ and $M(g(x_0),F(x_0,y_0),t) * M(g(y_0),F(y_0,x_0),t) > 0$ for all $t > 0$, then there exist $x^*, y^* \in X$ such that $g(x^*) = F(x^*,y^*)$ and $g(y^*) = F(y^*,x^*)$.

Letting $g(x) = x$ for all $x \in X$ in Theorem 3.1 and Corollary 3.1, we get the following corollaries.

Corollary 3.2 *Let $(X,M,*)$ be a complete fuzzy metric space with a positive t-norm. Let $\preceq$ be a partial order defined on X. Let $\phi : (0,\infty) \to (0,\infty)$ be a function satisfying $\phi(t) \leq t$ for all $t > 0$ and let $\gamma : [0,1] \to [0,1]$ be a left continuous and increasing function satisfying $\gamma(a) * \gamma(a) > a$ for all $a \in (0,1)$. Let $F : X \times X \to X$ and assume F has mixed monotone property. Suppose that the following conditions hold*:

(i) *We have*

$$M\big(F(x,y),F(u,v),\phi(t)\big) \geq \gamma\big(M(x,u,t) * M(y,v,t)\big),$$

for all $x,y,u,v \in X$, $t > 0$ *with* $x \preceq u$ *and* $y \succeq v$.

(ii) *If we have a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* $n \in \mathbb{N} \cup \{0\}$.

(iii) *If we have a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \succeq y$ *for all* $n \in \mathbb{N} \cup \{0\}$.

If there exist $x_0, y_0 \in X$ *such that* $x_0 \preceq F(x_0,y_0)$, $y_0 \succeq F(y_0,x_0)$ *and* $M(x_0,F(x_0,y_0),t) * M(y_0,F(y_0,x_0),t) > 0$ *for all* $t > 0$, *then there exist* $x^*, y^* \in X$ *such that* $x^* = F(x^*,y^*)$ *and* $y^* = F(y^*,x^*)$.

Corollary 3.3 *Let* $(X,M,*)$ *be a complete fuzzy metric space with a positive t-norm. Let* $\preceq$ *be a partial order defined on* X. *Let* $\gamma : [0,1] \to [0,1]$ *be a left continuous and increasing function satisfying* $\gamma(a) * \gamma(a) > a$ *for all* $a \in (0,1)$. *Let* $F : X \times X \to X$ *and assume* F *has mixed monotone property. Suppose that the following conditions hold*:

(i) *We have*

$$M\big(F(x,y),F(u,v),t\big) \geq \gamma\big(M(x,u,t) * M(y,v,t)\big),$$

for all $x,y,u,v \in X$, $t > 0$ *with* $x \preceq u$ *and* $y \succeq v$.

(ii) *If we have a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \preceq x$ *for all* $n \in \mathbb{N} \cup \{0\}$.

(iii) *If we have a non-increasing sequence* $\{y_n\} \to y$, *then* $y_n \succeq y$ *for all* $n \in \mathbb{N} \cup \{0\}$.

If there exist $x_0, y_0 \in X$ *such that* $x_0 \preceq F(x_0,y_0)$, $y_0 \succeq F(y_0,x_0)$, *and* $M(x_0,F(x_0,y_0),t) * M(y_0,F(y_0,x_0),t) > 0$ *for all* $t > 0$, *then there exist* $x^*, y^* \in X$ *such that* $x^* = F(x^*,y^*)$ *and* $y^* = F(y^*,x^*)$.

First, we illustrate Theorem 3.1 by modifying [17, Example 3.4] as follows.

Example 3.1 Let $(X,\preceq)$ is the partially ordered set with $X = [0,1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preceq$. Define $M : X^2 \times (0,\infty)$ by

$$M(x,y,t) = e^{-|x-y|/t}, \quad \forall x,y \in X, \forall t > 0.$$

Let $a * b = ab$ for all $a,b \in [0,1]$. Then $(X,M,*)$ is a (complete) fuzzy metric space.

Let $\psi(t) = t$ for all $t > 0$ and $\gamma(s) = s^{\frac{1}{3}}$ for all $s \in [0,1]$. It is easy to see that $\gamma(s) * \gamma(s) > s$ for all $s \in (0,1)$.

Define the mappings $g : X \to X$ by

$$g(x) = x^2, \quad \forall x \in X,$$

and $F : X \times X \to X$ by

$$F(x,y) = \frac{x^2 - y^2}{3} + \frac{2}{3}, \quad \forall x,y \in X.$$

Then $F(X \times X) \subseteq g(X)$, F satisfies the mixed g-monotone property; see [17, Example 3.4]. Obviously $g(X)$ is complete.

Let $x_0 = 0$ and $y_0 = 1$, then $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$; see [17, Example 3.4]. Moreover, $M(g(x_0), F(x_0, y_0), t) * M(g(y_0), F(y_0, x_0), t) > 0$ for all $t > 0$.

Next we show that for all $t > 0$ and all $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$, *i.e.*, $x \leq u$ and $y \geq v$, one has

$$M\big(F(x,y), F(u,v), t\big) \geq \big(M\big(g(x), g(u), t\big)M\big(g(y), g(v), t\big)\big)^{\frac{1}{3}}. \tag{3.10}$$

We prove the above inequality by a contradiction. Assume

$$M\big(F(x,y), F(u,v), t\big) < \big(M\big(g(x), g(u), t\big)M\big(g(y), g(v), t\big)\big)^{\frac{1}{3}}.$$

Then

$$e^{-|(x^2-y^2)/3-(u^2-v^2)/3|/t} < e^{-(|x^2-u^2|+|y^2-v^2|)/3t},$$

i.e.,

$$\big|\big(x^2 - u^2\big) - \big(y^2 - v^2\big)\big| > \big|x^2 - u^2\big| + \big|y^2 - v^2\big|.$$

This is a contradiction. Thus, (3.10) holds. Therefore, all the conditions of Theorem 3.1 are satisfied. Then by Theorem 3.1 we conclude that there exist x^*, y^* such that $g(x^*) = F(x^*, y^*)$ and $g(y^*) = F(y^*, x^*)$. It is easy to see that $(x^*, y^*) = (\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}})$, as desired.

Example 3.2 Let $(X, \preceq)$ is the partially ordered set with $X = [0,1) \cup \{2\}$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preceq$. Define a mapping $M : X^2 \times (0, \infty)$ by $M(x, x, t) = e^{-|x-y|}$ for all $x, y \in X$ and $t > 0$. Let $a * b = ab$ for all $a, b \in [0, 1]$. Then $(X, M, *)$ is a fuzzy metric space but not complete.

Define the mappings $g : X \to X$ and $F : X \times X \to X$ by

$$g(x) = \begin{cases} \frac{1}{2}(1-x), & \text{if } 0 \leq x < 1, \\ 0, & \text{if } x = 2, \end{cases}$$

and $F(x,y) = \frac{y-x}{16} + \frac{1}{8}$ for all $x, y \in X$. Then $F(X \times X) \subseteq g(X)$, F satisfies the mixed g-monotone property, and $g(X)$ is complete. Take $(x_0, y_0) = (\frac{23}{28}, \frac{1}{4})$. By a simple calculation we see that $g(x_0) \leq F(x_0, y_0)$ and $g(y_0) \geq F(y_0, x_0)$. Moreover, $M(g(x_0), F(x_0, y_0), t) * M(g(y_0), F(y_0, x_0), t) > 0$ for all $t > 0$.

Let $\phi(t) = t$ for all $t > 0$. Let γ be a function from $[0, 1]$ to $[0, 1]$ defined by

$$\gamma(s) = \begin{cases} \sqrt[3]{s}, & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \sqrt[4]{s}, & \text{if } \frac{1}{2} < s \leq 1. \end{cases}$$

Obviously, γ is left continuous and increasing, and $\gamma(s) * \gamma(s) > s$ for all $s \in (0, 1)$.

Let $t > 0$ and $x, y, u, v \in X$ with $g(x) \leq g(u)$ and $g(y) \geq g(v)$, *i.e.*, $u \leq x$ and $y \leq v$, since

$$\begin{aligned} M\big(F(x,y), F(u,v), \phi(t)\big) &= e^{-|F(x,y)-F(u,v)|} = e^{-|\frac{x-y}{16} - \frac{u-v}{16}|} \\ &\geq \max\big\{e^{-\frac{|x-y|+|y-v|}{6}}, e^{-\frac{|x-y|+|y-v|}{8}}\big\} \\ &\geq \gamma\big(M\big(g(x), g(u), t\big) * M\big(g(y), g(v), t\big)\big). \end{aligned}$$

Hence (3.1) is satisfied. Therefore, all the conditions of Theorem 3.1 are satisfied. Then by Theorem 3.1 F and g have a coincidence point. It is easy to check that $(x^*, y^*) = (\frac{3}{4}, \frac{3}{4})$.

The above two examples cannot be applied to [17, Theorem 3.1], since $*$ is not of Hadžić-type, or g is not monotonic increasing or continuous, or $M(x,y,t) \nrightarrow 1$ as $t \to \infty$ for all $x, y \in X$.

4 Conclusion

In this paper, we prove a new coupled coincidence fixed point result in a partial order fuzzy metric space in which some restrictions required in [17, Theorem 3.1] are removed, such that the conditions required in our result are fewer than the ones required in [17, Theorem 3.1]. The purpose of this paper is to give some new conditions on the coupled coincidence fixed point theorem. Our result is not an improvement of [17, Theorem 3.1], since we add some other restrictions such as requiring that the function γ is increasing and $M(g(x_0), F(x_0, y_0), t) * M(g(y_0), F(y_0, x_0), t) > 0$ for all $t > 0$. As pointed out in the conclusion part of [17], it still is an interesting open problem to find simpler or fewer conditions on the coupled coincidence fixed point theorem in a fuzzy metric space.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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