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On generalized equilibrium problems and strictly pseudocontractive mappings in Hilbert spaces

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Abstract

In this article, a mean iterative algorithm is investigated for finding a common element in the solution set of generalized equilibrium problems and in the fixed point set of strictly pseudocontractive mappings. Strong convergence of the mean iterative algorithm is obtained in the framework of Hilbert spaces.

Keywords: variational inequality; strictly pseudocontractive mapping; nonexpansive mapping; inverse-strongly monotone mapping; equilibrium problem

1 Introduction-preliminaries

Many important problems have reformulations which require finding solutions of classical variational inequalities, for instance, image recovery, inverse problems, transportation problems, fixed point problems and optimization problems; see [1–11] and the references therein. Equilibrium problems, which include the classical variational inequalities as special cases, have been recently extensively investigated; see [12–26] and the references therein. In this paper, we study an equilibrium problem via fixed point methods. Global convergence of the fixed point algorithm is obtained. Throughout this paper, we always assume that H is a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H , and let P_C be a metric projection from H onto C .

Let $S : C \rightarrow C$ be a mapping. In this paper, we use $F(S)$ to denote the fixed point set of S . Recall that the mapping S is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C.$$

S is said to be k -strictly pseudocontractive if there exists a constant $k \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k \|(x - Sx) - (y - Sy)\|^2 \quad \forall x, y \in C.$$

The class of strictly pseudocontractive mappings was introduced by Browder and Petryshyn [27] in 1967. It is easy to see that the class of strictly pseudocontractive mappings includes the class of nonexpansive mappings as a special case. If $k = 1$, then it is called a pseudocontractive mapping.

Recall that a set-valued mapping $T : H \rightarrow 2^H$ is said to be monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle > 0$. A monotone mapping $T : H \rightarrow 2^H$ is maximal if the graph $\text{Graph}(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g \rangle \geq 0$ for all $(y, g) \in \text{Graph}(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H , and let $N_C v$ be a normal cone to C at $v \in C$, i.e.,

$$N_C v = \{w \in H : \langle v - u, w \rangle \geq 0 \ \forall u \in C\},$$

and define a mapping R on C by

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Rv$ if and only if $\langle Av, u - v \rangle \geq 0$ for all $u \in C$; see [28] and the references therein.

Let $A : C \rightarrow H$ be a mapping. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0 \quad \forall x, y \in C.$$

A is said to be inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2 \quad \forall x, y \in C.$$

For such a case, A is also called an α -inverse-strongly monotone mapping. It is easy to see that if A is an inverse-strongly monotone mapping, then the mapping $I - A$ is a strictly pseudocontractive mapping.

Let $T : C \rightarrow H$ be an inverse-strongly monotone mapping, and let F be a bifunction of $C \times C$ into \mathbb{R} , where \mathbb{R} denotes the set of real numbers. In this paper, we consider the following generalized equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) + \langle Tx, y - x \rangle \geq 0 \quad \forall y \in C. \tag{1.1}$$

In this paper, we use $\text{EP}(F, T)$ to denote the solution set of problem (1.1).

Next, we give two special cases of problem (1.1).

- (a) If $T \equiv 0$, then the generalized equilibrium problem (1.1) is reduced to the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } F(x, y) \geq 0 \quad \forall y \in C. \tag{1.2}$$

In this paper, we use $\text{EP}(F)$ to denote the solution set of problem (1.2). We remark here that problem (1.2) was first introduced by Fan [29].

- (b) If $F \equiv 0$, then problem (1.1) is reduced to the classical variational inequality: Find $x \in C$ such that

$$\langle Tx, y - x \rangle \geq 0 \quad \forall y \in C. \tag{1.3}$$

In this paper, we use $VI(C, T)$ to denote the solution set of variational inequality (1.3). It is well known that $x \in C$ is a solution to (1.3) if and only if x is a fixed point of the mapping $P_C(I - rT)$, where $r > 0$ is a constant and I is the identity mapping.

To study the generalized equilibrium problem (1.1), we may assume that F satisfies the following conditions:

- (A1) $F(x, x) = 0$ for all $x \in C$;
- (A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
- (A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x, y);$$

- (A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semicontinuous.

Recently, many authors investigated problems (1.1), (1.2) and (1.3) based on iterative methods. In 2003, Takahashi and Toyoda [30] investigated fixed points of nonexpansive mappings and solutions of variational inequality (1.3). They obtained the following results. Let A be an α -inverse-strongly monotone mapping of C into H , and let S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C, A) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by

$$x_0 \in C, \quad x_{n+1} = \alpha_n x_n + (1 - \alpha_n) SP_C(x_n - \lambda_n A x_n) \quad \forall n \geq 0,$$

where $\lambda_n \in [a, b]$ for some $a, b \in (0, 2\alpha)$ and $\alpha_n \in [c, d]$ for some $c, d \in (0, 1)$. Then $\{x_n\}$ converges weakly to $z \in F(S) \cap VI(C, A)$, where $z = \lim_{n \rightarrow \infty} P_{F(S) \cap VI(C, A)} x_n$.

Recently, Tada and Takahashi [14] investigated fixed points of nonexpansive mappings and solutions of equilibrium problem (1.2). They obtained the following result. Let F be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4), and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(F) \neq \emptyset$. Let $\{x_n\}$ and $\{u_n\}$ be sequences generated by $x_1 = x \in H$, and let

$$\begin{cases} u_n \in C \text{ such that } F(u_n, u) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0 & \forall u \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S u_n, \end{cases}$$

where $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n \rightarrow \infty} r_n > 0$. Then $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(F)$, where $w = \lim_{n \rightarrow \infty} P_{F(S) \cap EP(F)} x_n$.

In this paper, motivated by the above results, we investigate fixed points of strictly pseudocontractive mappings and solutions of equilibrium problem (1.1). Weak convergence theorems for common solutions are established in Hilbert spaces. Applications of the main results are also provided. In order to prove our main results, we also need the following lemmas.

Lemma 1.1 [27, 31] *Let C be a nonempty closed convex subset of a real Hilbert space H , and let $S : C \rightarrow C$ be a k -strict pseudocontraction with a fixed point. Define $S : C \rightarrow C$ by $S_a x = ax + (1 - a)Sx$ for each $x \in C$. If $a \in [k, 1)$, then S_a is nonexpansive with $F(S_a) = F(S)$.*

Lemma 1.2 [1] *Let C be a nonempty closed convex subset of H , and let $F : C \times C \rightarrow \mathbb{R}$ be a bifunction satisfying (A1)-(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \forall y \in C.$$

Further, define

$$T_r x = \left\{ z \in C : F(z, y) + \frac{1}{r} \langle y - z, z - x \rangle \geq 0 \quad \forall y \in C \right\}$$

for all $r > 0$ and $x \in H$. Then the following hold:

- (a) T_r is single-valued;
- (b) T_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y \rangle;$$

- (c) $F(T_r) = \text{EP}(F)$;
- (d) $\text{EP}(F)$ is closed and convex.

Lemma 1.3 [32] *Let H be a Hilbert space and $0 < p \leq t_n \leq q < 1$ for all $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in H such that*

$$\limsup_{n \rightarrow \infty} \|x_n\| \leq r, \quad \limsup_{n \rightarrow \infty} \|y_n\| \leq r$$

and

$$\lim_{n \rightarrow \infty} \|t_n x_n + (1 - t_n) y_n\| = r$$

hold for some $r \geq 0$. Then $\lim_{n \rightarrow \infty} \|x_n - y_n\| = 0$.

Lemma 1.4 [33] *Let C be a nonempty closed convex subset of a Hilbert space H , and let $S : C \rightarrow C$ be a k -strict pseudocontraction. Then*

- (a) S is $\frac{1+k}{1-k}$ -Lipschitz;
- (b) $I - S$ is demi-closed, i.e., if $\{x_n\}$ is a sequence in C with $x_n \rightarrow x$ and $x_n - Sx_n \rightarrow 0$, then $x \in F(S)$.

2 Main results

Now, we are in a position to show the main results of the article.

Theorem 2.1 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow H$ be a λ -inverse-strongly monotone mapping, and let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4). Let $S : C \rightarrow C$ be a k -strict pseudocontraction. Assume that $\mathcal{F} := \text{EP}(F, T) \cap F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$. Let $\{r_n\}$ be a sequence in $(0, 2\lambda)$, and let $\{e_n\}$ be a bounded sequence in C . Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in C, \\ F(u_n, u) + \langle Tx_n, u - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0 \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n (\delta_n u_n + (1 - \delta_n) S u_n) + \gamma_n e_n \quad \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ satisfy the following restrictions: $0 < a \leq \alpha_n \leq a' < 1$, $0 \leq k \leq \delta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\lambda$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the sequence $\{x_n\}$ converges weakly to some point $\bar{x} \in \mathcal{F}$, where $\bar{x} = \lim_{n \rightarrow \infty} P_{\mathcal{F}} x_n$.

Proof First, we show that the sequences $\{x_n\}$ and $\{u_n\}$ are bounded. Putting $S_n = \delta_n I + (1 - \delta_n)S$, we see from Lemma 1.1 that S_n is nonexpansive and $F(S_n) = F(S)$. Note that

$$\begin{aligned} \|(I - r_n T)x - (I - r_n T)y\|^2 &= \|(x - y) - r_n(Tx - Ty)\|^2 \\ &= \|x - y\|^2 - 2r_n \langle x - y, Tx - Ty \rangle + r_n^2 \|Tx - Ty\|^2 \\ &\leq \|x - y\|^2 - r_n(2\lambda - r_n) \|Tx - Ty\|^2 \\ &\leq \|x - y\|^2 \quad \forall x, y \in C. \end{aligned}$$

This proves that the mapping $I - r_n T$ is also nonexpansive. Fixing $p \in \mathcal{F}$, we find from Lemma 1.2 that $p = Sp = T_{r_n}(I - r_n T)p$. Since

$$\|u_n - p\| \leq \|T_{r_n}(I - r_n T)x_n - T_{r_n}(I - r_n T)p\| \leq \|x_n - p\|,$$

we find that

$$\begin{aligned} \|x_{n+1} - p\| &\leq \alpha_n \|x_n - p\| + \beta_n \|S_n u_n - p\| + \gamma_n \|e_n - p\| \\ &\leq \alpha_n \|x_n - p\| + \beta_n \|u_n - p\| + \gamma_n \|e_n - p\| \\ &\leq (1 - \gamma_n) \|x_n - p\| + \gamma_n \|e_n - p\|. \end{aligned} \tag{2.1}$$

This implies that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. This shows that $\{x_n\}$ is bounded, so is $\{u_n\}$. Since $\|\cdot\|^2$ is convex, we find that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + \beta_n \|S_n u_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + \beta_n \|u_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + \beta_n \|(I - r_n T)x_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\ &\leq (1 - \gamma_n) \|x_n - p\|^2 - r_n(2\lambda - r_n) \beta_n \|Tx_n - Tp\|^2 + \gamma_n \|e_n - p\|^2. \end{aligned}$$

It follows that

$$r_n(2\lambda - r_n) \beta_n \|Tx_n - Tp\|^2 \leq (1 - \gamma_n) \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n \|e_n - p\|^2.$$

This yields that

$$\lim_{n \rightarrow \infty} \|Tx_n - Tp\| = 0. \tag{2.2}$$

Using Lemma 1.2, we see that

$$\begin{aligned} \|u_n - p\|^2 &\leq \langle (I - r_n T)x_n - (I - r_n T)p, u_n - p \rangle \\ &= \frac{1}{2} (\|(I - r_n T)x_n - (I - r_n T)p\|^2 + \|u_n - p\|^2 \\ &\quad - \|(I - r_n T)x_n - (I - r_n T)p - (u_n - p)\|^2) \\ &\leq \frac{1}{2} (\|x_n - p\|^2 + \|u_n - p\|^2 - \|x_n - u_n - r_n(Tx_n - Tp)\|^2) \end{aligned}$$

$$= \frac{1}{2} (\|x_n - p\|^2 + \|u_n - p\|^2 - (\|x_n - u_n\|^2 - 2r_n \langle x_n - u_n, Tx_n - Tp \rangle + r_n^2 \|Tx_n - Tp\|^2)).$$

This implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2 + 2r_n \|x_n - u_n\| \|Tx_n - Tp\|.$$

Since $\|\cdot\|^2$ is convex, we find that

$$\begin{aligned} \|x_{n+1} - p\|^2 &\leq \alpha_n \|x_n - p\|^2 + \beta_n \|S_n u_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\ &\leq \alpha_n \|x_n - p\|^2 + \beta_n \|u_n - p\|^2 + \gamma_n \|e_n - p\|^2 \\ &\leq \|x_n - p\|^2 - \beta_n \|x_n - u_n\|^2 + 2r_n \beta_n \|x_n - u_n\| \|Tx_n - Tp\| + \gamma_n \|e_n - p\|^2. \end{aligned}$$

It follows that

$$\beta_n \|x_n - u_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2r_n \|x_n - u_n\| \|Tx_n - Tp\| + \gamma_n \|e_n - p\|^2.$$

Using the restrictions imposed on the sequences, we obtain from (2.2) that

$$\lim_{n \rightarrow \infty} \|x_n - u_n\| = 0. \tag{2.3}$$

Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to \bar{x} . Using (2.3), we also find that $\{u_{n_i}\}$ converges weakly to \bar{x} . Note that

$$F(u_n, u) + \langle Tx_n, u - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0 \quad \forall u \in C.$$

From (A2), we see that

$$\langle Tx_n, u - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq F(u, u_n) \quad \forall u \in C.$$

Replacing n by n_i , we arrive at

$$\langle Tx_{n_i}, u - u_{n_i} \rangle + \frac{1}{r_{n_i}} \langle u - u_{n_i}, u_{n_i} - x_{n_i} \rangle \geq F(u, u_{n_i}) \quad \forall u \in C. \tag{2.4}$$

For t with $0 < t \leq 1$ and $u \in C$, let $u_t = tu + (1 - t)\bar{x}$. Since $u \in C$ and $\bar{x} \in C$, we have $u_t \in C$. It follows from (2.4) that

$$\begin{aligned} \langle u_t - u_{n_i}, Tu_t \rangle &\geq \langle u_t - u_{n_i}, Tu_t \rangle - \langle Tx_{n_i}, u_t - u_{n_i} \rangle \\ &\quad - \left\langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + F(u_t, u_{n_i}) \\ &= \langle u_t - u_{n_i}, Tu_t - Tu_{n_i} \rangle + \langle u_t - u_{n_i}, Tu_{n_i} - Tx_{n_i} \rangle \\ &\quad - \left\langle u_t - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle + F(u_t, u_{n_i}). \end{aligned} \tag{2.5}$$

Using (2.3), we have $Tu_{n_i} - Tx_{n_i} \rightarrow 0$ as $i \rightarrow \infty$. Using the monotonicity of T , we see that $\langle u_t - u_{n_i}, Tu_t - Tu_{n_i} \rangle \geq 0$. It follows from (A4) that

$$\langle u_t - \bar{x}, Tu_t \rangle \geq F(u_t, \bar{x}). \tag{2.6}$$

Using (A1) and (A4), we see from (2.6) that

$$\begin{aligned} 0 &= F(u_t, u_t) \leq tF(u_t, u) + (1-t)F(u_t, \bar{x}) \\ &\leq tF(u_t, u) + (1-t)\langle u_t - \bar{x}, Tu_t \rangle \\ &= tF(u_t, u) + (1-t)t\langle u - \bar{x}, Tu_t \rangle. \end{aligned}$$

It follows that $F(u_t, u) + (1-t)\langle u - \bar{x}, Tu_t \rangle \geq 0$. Letting $t \rightarrow 0$ in the above inequality, we arrive at $F(\bar{x}, u) + \langle u - \bar{x}, T\bar{x} \rangle \geq 0$. Hence, $\bar{x} \in EP(F, T)$.

Next, we are in a position to show that $\bar{x} \in F(S)$. Note that $\lim_{n \rightarrow \infty} \|x_n - p\|$ exists. We may assume that $\lim_{n \rightarrow \infty} \|x_n - p\| = d > 0$. Note that

$$\lim_{n \rightarrow \infty} \|x_{n+1} - p\| = \lim_{n \rightarrow \infty} \|\beta_n(S_n u_n - p + \gamma_n(e_n - x_n)) + (1 - \beta_n)(x_n - p + \gamma_n(e_n - x_n))\| = d.$$

Since

$$\begin{aligned} \|S_n u_n - p + \gamma_n(e_n - x_n)\| &\leq \|S_n u_n - p\| + \gamma_n \|e_n - x_n\| \\ &\leq \|u_n - p\| + \gamma_n \|e_n - x_n\| \\ &\leq \|x_n - p\| + \gamma_n \|e_n - x_n\|, \end{aligned}$$

we find that $\lim_{n \rightarrow \infty} \|S_n x_n - p + \gamma_n(e_n - x_n)\| \leq d$. Since $\|x_n - p + \gamma_n(e_n - x_n)\| \leq \|x_n - p\| + \gamma_n \|e_n - x_n\|$, we find that $\lim_{n \rightarrow \infty} \|x_n - p + \gamma_n(e_n - x_n)\| \leq d$. Using Lemma 1.3, we obtain that $\lim_{n \rightarrow \infty} \|S_n u_n - x_n\| = 0$. In view of

$$Su_n - x_n = \frac{S_n u_n - x_n}{1 - \delta_n} + \frac{\delta_n(x_n - u_n)}{1 - \delta_n},$$

it follows that $\lim_{n \rightarrow \infty} \|Su_n - x_n\| = 0$. Note that $\|Sx_n - x_n\| \leq \|Sx_n - Su_n\| + \|Su_n - x_n\|$. Using Lemma 1.4, we find that $\lim_{n \rightarrow \infty} \|Sx_n - x_n\| = 0$. It follows from Lemma 1.4 that $\bar{x} \in F(S)$. This proves that $\bar{x} \in \mathcal{F}$. Assume that there exists another subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly to x' . We can find that $x' \in \mathcal{F}$. If $\bar{x} \neq x'$, we get from the Opial condition [34] that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|x_n - \bar{x}\| &= \liminf_{i \rightarrow \infty} \|x_{n_i} - \bar{x}\| < \liminf_{i \rightarrow \infty} \|x_{n_i} - x'\| \\ &= \lim_{n \rightarrow \infty} \|x_n - x'\| = \liminf_{j \rightarrow \infty} \|x_{n_j} - x'\| \\ &< \liminf_{j \rightarrow \infty} \|x_{n_j} - \bar{x}\| = \lim_{n \rightarrow \infty} \|x_n - \bar{x}\|. \end{aligned}$$

This derives a contradiction. Hence, we have $\bar{x} = x'$. This implies that $x_n \rightarrow \bar{x} \in \mathcal{F}$. The proof is completed. \square

From Theorem 2.1, the following result is not hard to derive.

Corollary 2.2 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4). Let $S : C \rightarrow C$ be a k -strict pseudocontraction. Assume that $\mathcal{F} := \text{EP}(F) \cap F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be sequences in $(0, 1)$. Let $\{r_n\}$ be a positive number sequence, and let $\{e_n\}$ be a bounded sequence in C . Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in C, \\ F(u_n, u) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0 \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n (\delta_n u_n + (1 - \delta_n) S u_n) + \gamma_n e_n \quad \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ satisfy the following restrictions: $0 < a \leq \alpha_n \leq a' < 1$, $0 \leq k \leq \delta_n \leq b < 1$, $0 < c \leq r_n \leq d < +\infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the sequence $\{x_n\}$ converges weakly to some point $\bar{x} \in \mathcal{F}$, where $\bar{x} = \lim_{n \rightarrow \infty} P_{\mathcal{F}} x_n$.

Corollary 2.3 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $T : C \rightarrow H$ be a λ -inverse-strongly monotone mapping, and let F be a bifunction from $C \times C$ to \mathbb{R} which satisfies (A1)-(A4). Let $S : C \rightarrow C$ be a nonexpansive mapping. Assume that $\mathcal{F} := \text{EP}(F, T) \cap F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$. Let $\{r_n\}$ be a positive real number sequence, and let $\{e_n\}$ be a bounded sequence in C . Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in C, \\ F(u_n, u) + \langle T x_n, u - u_n \rangle + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0 \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n S u_n + \gamma_n e_n \quad \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{r_n\}$ satisfy the following restrictions: $0 < a \leq \alpha_n \leq a' < 1$, $0 < c \leq r_n \leq d < 2\lambda$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the sequence $\{x_n\}$ converges weakly to some point $\bar{x} \in \mathcal{F}$, where $\bar{x} = \lim_{n \rightarrow \infty} P_{\mathcal{F}} x_n$.

3 Applications

The computation of common fixed points is important in the study of many real world problems, including inverse problems; for instance, it is not hard to show that the split feasibility problem and the convex feasibility problem in signal processing and image reconstruction can both be formulated as a problem of finding fixed points of certain operators, respectively; for more details, see [35, 36] and the references therein.

First, we consider the following common fixed point problem.

Theorem 3.1 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $S_m : C \rightarrow C$ be a k_m -strict pseudocontraction for each $1 \leq m \leq N$, where N is some positive integer. Assume that $\mathcal{F} := \bigcap_{m=1}^{\infty} F(S)$ is not empty. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$. Let $\{e_n\}$ be a bounded sequence in C . Let $\{x_n\}$ be a sequence generated in the following manner:*

$$x_1 \in C, \quad x_{n+1} = \alpha_n x_n + \beta_n \left(\delta_n x_n + (1 - \delta_n) \sum_{i=1}^N \mu_i S_i x_n \right) + \gamma_n e_n \quad \forall n \geq 1.$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\delta_n\}$ and $\{r_n\}$ satisfy the following restrictions: $0 < a \leq \alpha_n \leq a' < 1$, $0 \leq k \leq \delta_n \leq b < 1$, $0 < c \leq r_n \leq d < 2\lambda$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the sequence $\{x_n\}$ converges weakly to some point $\bar{x} \in \mathcal{F}$, where $\bar{x} = \lim_{n \rightarrow \infty} P_{\mathcal{F}}x_n$.

Proof Using the definition of strict pseudocontractions, we see that a mapping T is said to be a k -strict pseudocontraction iff

$$\langle x - y, (I - T)x - (I - T)y \rangle \geq \frac{1 - k}{2} \|(I - T)x - (I - T)y\|^2.$$

Define a mapping $S : C \rightarrow C$ by $S = \sum_{m=1}^N \mu_m S_m$. Next, we prove that $F(S) = \bigcap_{m=1}^N F(S_m)$ and S is a k -strict pseudocontraction, where $k = \max\{k_m : 1 \leq m \leq N\}$. Note that

$$\begin{aligned} \langle x - y, (I - S)x - (I - S)y \rangle &= \sum_{m=1}^N \mu_m \langle x - y, (I - S_m)x - (I - S_m)y \rangle \\ &\geq \sum_{m=1}^N \mu_m \frac{1 - k_m}{2} \|(I - S_m)x - (I - S_m)y\|^2 \\ &\geq \sum_{m=1}^N \mu_m \frac{1 - k}{2} \|(I - S_m)x - (I - S_m)y\|^2 \\ &\geq \frac{1 - k}{2} \|(I - S)x - (I - S)y\|^2. \end{aligned}$$

This proves that S is a k -strict pseudocontraction, where $k = \max\{k_m : 1 \leq m \leq N\}$. Next, we show that $F(S) = \bigcap_{m=1}^N F(S_m)$. It is clear to see that $F(S) \supseteq \bigcap_{m=1}^N F(S_m)$. It suffices to prove that $\bigcap_{m=1}^N F(S_m) \supseteq F(S)$. Let $x \in F(S)$ and write $T_m = I - S_m$. Let $y \in \bigcap_{m=1}^N F(S_m)$. For any $i, j \in \{1, 2, \dots, N\}$ and $i \neq j$, we have

$$\begin{aligned} \|x - y\|^2 &= \left\| \sum_{m=1}^N \mu_m (y - S_m x) \right\|^2 \\ &\leq \sum_{m=1}^N \mu_m \|y - S_m x\|^2 - \mu_i \mu_j \|S_i x - S_j x\|^2 \\ &\leq \sum_{m=1}^N \mu_m (\|y - x\|^2 + k_m \|T_m x\|^2) - \mu_i \mu_j \|S_i x - S_j x\|^2 \\ &\leq \|y - x\|^2 + k \sum_{i=1}^N \mu_m \|T_m x\|^2 - \mu_i \mu_j \|S_i x - S_j x\|^2. \end{aligned}$$

This shows that

$$\mu_i \mu_j \|S_i x - S_j x\|^2 \leq k \sum_{m=1}^N \mu_m \|T_m x\|^2.$$

Since $\sum_{i=1}^N \mu_m T_m x = 0$, we find that $\|S_i x - S_j x\| = 0$. This proves that $S_i x = S_j x$. Since x is a fixed point of S , we obtain $\bigcap_{m=1}^N F(S_m) \supseteq F(S)$. This proves that $F(S) = \bigcap_{m=1}^N F(S_m)$. Putting $T = 0$, $F = 0$ and $r_n = 1$, we find from Theorem 2.1 the desired conclusion immediately. \square

Next, we study an optimization problem: Find a minimizer of a convex and lower semi-continuous functional $h(x)$ defined on a closed convex subset C of a Hilbert space H .

We denote by Δ the set of solutions of the optimization problem. Let R be a bifunction from $C \times C$ to R defined by $R(x, y) = h(y) - h(x)$. We consider the following equilibrium problem:

$$\text{Find } x \in C \text{ such that } R(x, y) \geq 0 \quad \forall y \in C.$$

It is obvious that $EP(R) = \Delta$. In addition, we also find that $R(x, y)$ satisfies the conditions (A1)-(A4).

Theorem 3.2 *Let C be a nonempty closed convex subset of a real Hilbert space H . Let $h(x)$ be a convex and lower semicontinuous functional defined on C with a nonempty minimizer set. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be sequences in $(0, 1)$. Let $\{r_n\}$ be a positive real number sequence, and let $\{e_n\}$ be a bounded sequence in C . Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} x_1 \in C, \\ h(u) - h(u_n) + \frac{1}{r_n} \langle u - u_n, u_n - x_n \rangle \geq 0 \quad \forall u \in C, \\ x_{n+1} = \alpha_n x_n + \beta_n u_n + \gamma_n e_n \quad \forall n \geq 1. \end{cases}$$

Assume that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{r_n\}$ satisfy the following restrictions: $0 < a \leq \alpha_n \leq a' < 1, 0 \leq k \leq \delta_n \leq b < 1, 0 < c \leq r_n \leq d < +\infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Then the sequence $\{x_n\}$ converges weakly to some point $\bar{x} \in \Delta$, where $\bar{x} = \lim_{n \rightarrow \infty} P_{\Delta} x_n$.

Remark 3.3 A special form of the optimization problem is to take $h(x) = \|x\|$, which is known as the minimum norm point problem. We also remark here that if we take $F = 0$ and $S = I$, then we easily obtain convergence theorems of solutions of variational inequality (1.3).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by CH. All authors participated in the research and performed steps of the proof in this research. All authors read and approved the final manuscript.

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