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Common fixed point theorems for weakly compatible mappings satisfying contractive conditions of integral type

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Abstract

Two common fixed theorems for weakly compatible mappings satisfying general contractive conditions of integral type in metric spaces are proved and an illustrative example is provided. The results obtained in this paper substantially extend and improve several previous results, particularly Theorem 2.1 of Branciari (Int. J. Math. Math. Sci. 29(9):531-536, 2002), Theorem 2 of Rhoades (Int. J. Math. Math. Sci. 2003(63):4007-4013, 2003) and Theorem 2 of Vijayaraju *et al.* (Int. J. Math. Math. Sci. 2005(15):2359-2364, 2005). A nontrivial example with uncountably many points is also provided to support the results presented herein.

MSC: 54H25

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1 Introduction and preliminaries

In 2002, Branciari [1] introduced the notion of contractive mappings of integral type in metric spaces and proved the following fixed point theorem for the contractive mapping of integral type, which is a nice generalization of the Banach contraction principle.

Theorem 1.1 ([1]) *Let (X, d) be a complete metric space, $c \in (0, 1)$, and let $f : X \rightarrow X$ be a mapping such that*

$$\int_0^{d(fx, fy)} \varphi(s) ds \leq c \int_0^{d(x, y)} \varphi(s) ds, \quad \forall x, y \in X,$$

where $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, +\infty)$ and such that for all $\varepsilon > 0$,

$$\int_0^\varepsilon \varphi(s) ds > 0.$$

Then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n \rightarrow \infty} f^n x = a$.

Afterward, the researchers [2–17] and others extended the result to more general contractive conditions of integral type. In particular, Rhoades [13] proved the following extension of Theorem 1.1.

Theorem 1.2 ([13]) *Let (X, d) be a complete metric space, $k \in [0, 1)$, $f : X \rightarrow X$ be a mapping such that*

$$\int_0^{d(fx, fy)} \varphi(s) ds \leq k \int_0^{m(x, y)} \varphi(s) ds, \quad \forall x, y \in X,$$

where

$$m(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2} \right\},$$

$\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a Lebesgue integrable mapping which is summable on each compact subset of $[0, +\infty)$ and such that for all $\varepsilon > 0$,

$$\int_0^\varepsilon \varphi(s) ds > 0.$$

Then f has a unique fixed point $a \in X$ and, for each $x \in X$, $\lim_{n \rightarrow \infty} f^n x = a$.

Vijayaraju *et al.* [17] extended further Theorems 1.1 and 1.2 from a single mapping to a pair of mappings. Using a rational expression for a contractive condition of integral type, Vetro [16] extended also Theorem 1.1 and proved the following common fixed point theorem for weakly compatible mappings.

Theorem 1.3 ([16]) *Let (X, d) be a metric space and let A, B, S and T be self-mappings of X with $S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$ such that*

$$\int_0^{d(Sx, Ty)} \varphi(s) ds \leq \alpha \int_0^{m(x, y)} \varphi(s) ds + \beta \int_0^{M(x, y)} \varphi(s) ds, \quad \forall x, y \in X,$$

where

$$m(x, y) = d(By, Ty) \frac{1 + d(Ax, Sx)}{1 + d(Ax, By)},$$

$$M(x, y) = \max \{ d(Ax, By), d(Ax, Sx), d(By, Ty) \},$$

$\alpha > 0, \beta > 0, \alpha + \beta < 1$ and $\varphi : [0, +\infty) \rightarrow [0, +\infty)$ is a Lebesgue integrable mapping on each compact subset of $[0, +\infty)$ and such that for all $\varepsilon > 0$,

$$\int_0^\varepsilon \varphi(s) ds > 0.$$

Suppose that one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subset of X and the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible. Then A, B, S and T have a unique common fixed point in X .

Motivated and inspired by the results in [1–17], in this paper we introduce more general contractive mappings of integral type, which include the contractive mappings of integral type in [1, 4, 13, 16, 17] as special cases, and we establish the existence and uniqueness

of common fixed points for these contractive mappings of integral type with weak compatibility. Our results extend, improve and unify the corresponding results in [1, 4, 13, 16, 17]. A nontrivial example with uncountably many points is also provided to support the results presented herein.

Throughout this paper, we assume that $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, where \mathbb{N} denotes the set of all positive integers and

- $\Phi = \{\varphi : \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ satisfies that } \varphi \text{ is Lebesgue integrable, summable on each compact subset of } \mathbb{R}^+ \text{ and } \int_0^\varepsilon \varphi(t) dt > 0 \text{ for each } \varepsilon > 0\}$,
- $\Psi = \{\psi : \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ is upper semi-continuous on } \mathbb{R}^+ \setminus \{0\}, \psi(0) = 0 \text{ and } \psi(t) < t \text{ for each } t > 0\}$,
- $\Psi_1 = \{\psi : \psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \text{ is nondecreasing on } \mathbb{R}^+, \psi(t) < t \text{ and } \sum_{n=1}^\infty \psi^n(t) < +\infty \text{ for each } t > 0\}$.

Recall that a pair of self-mappings f and g in a metric space (X, d) are said to be weakly compatible if for all $t \in X$ the equality $ft = gt$ implies $fgt = gft$.

Lemma 1.4 ([9]) *Let $\varphi \in \Phi$ and $\{r_n\}_{n \in \mathbb{N}}$ be a nonnegative sequence. Then*

$$\lim_{n \rightarrow \infty} \int_0^{r_n} \varphi(t) dt = 0$$

if and only if $\lim_{n \rightarrow \infty} r_n = 0$.

2 Common fixed point theorems

Now we show two common fixed point theorems for four contractive mappings of integral type in metric spaces.

Theorem 2.1 *Let A, B, S and T be self-mappings of a metric space (X, d) such that*

- (C1) $S(X) \subseteq B(X)$ and $T(X) \subseteq A(X)$;
- (C2) the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible;
- (C3) one of $A(X), B(X), S(X)$ and $T(X)$ is a complete subset of X and

$$\int_0^{d(Sx, Ty)} \varphi(t) dt \leq \psi \left(\max \left\{ \int_0^{m_i(x, y)} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right), \quad \forall x, y \in X, \quad (2.1)$$

where (ψ, φ) is in $\Psi \times \Phi$ and

$$\begin{aligned} m_1(x, y) &= d(By, Ty) \frac{1 + d(Ax, Sx)}{1 + d(Ax, By)}, \\ m_2(x, y) &= d(Ax, Sx) \frac{1 + d(By, Ty)}{1 + d(Ax, By)}, \\ m_3(x, y) &= \frac{d(Sx, By)d(Ty, Ax)}{1 + d(Ax, By)}, \\ m_4(x, y) &= \max \left\{ d(Ax, By), d(Ax, Sx), d(By, Ty), \right. \\ &\quad \left. \frac{1}{2} [d(Sx, By) + d(Ty, Ax)] \right\}. \end{aligned} \quad (2.2)$$

Then A, B, S and T have a unique common fixed point in X .

Proof Let $x_0 \in X$. It follows from (C1) that there exist two sequences $\{y_n\}_{n \in \mathbb{N}}$ and $\{x_n\}_{n \in \mathbb{N}_0}$ in X satisfying

$$y_{2n+1} = Sx_{2n} = Bx_{2n+1} \quad \text{and} \quad y_{2n+2} = Tx_{2n+1} = Ax_{2n+2}, \quad \forall n \in \mathbb{N}_0. \tag{2.3}$$

Put $d_n = d(y_n, y_{n+1})$ for each $n \in \mathbb{N}$.

Firstly we show that A, B, S and T have at most a common fixed point in X . Suppose that u and v are two different common fixed points of A, B, S and T in X . It follows from (2.1), (2.2) and $(\psi, \varphi) \in \Psi \times \Phi$ that

$$\begin{aligned} m_1(u, v) &= d(Bv, Tv) \frac{1 + d(Au, Su)}{1 + d(Au, Bv)} = 0, \\ m_2(u, v) &= d(Au, Su) \frac{1 + d(Bv, Tv)}{1 + d(Au, Bv)} = 0, \\ m_3(u, v) &= \frac{d(Su, Bv)d(Tv, Au)}{1 + d(Au, Bv)} = \frac{d^2(u, v)}{1 + d(u, v)}, \\ m_4(u, v) &= \max \left\{ d(Au, Bv), d(Au, Su), d(Bv, Tv), \frac{1}{2} [d(Su, Bv) + d(Tv, Au)] \right\} = d(u, v) \end{aligned}$$

and

$$\begin{aligned} \int_0^{d(u,v)} \varphi(t) dt &= \int_0^{d(Su, Tv)} \varphi(t) dt \\ &\leq \psi \left(\max \left\{ \int_0^{m_i(u,v)} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \\ &= \psi \left(\max \left\{ 0, 0, \int_0^{\frac{d^2(u,v)}{1+d(u,v)}} \varphi(t) dt, \int_0^{d(u,v)} \varphi(t) dt \right\} \right) \\ &= \psi \left(\int_0^{d(u,v)} \varphi(t) dt \right) < \int_0^{d(u,v)} \varphi(t) dt, \end{aligned}$$

which is a contradiction. Hence A, B, S and T have at most a common fixed point in X .

Secondly we show that A, B, S and T have a common fixed point $Aa \in X$ if there exist $a, b \in X$ satisfying

$$Aa = Sa = Bb = Tb. \tag{2.4}$$

Assume that (2.4) holds for some $a, b \in X$. Put $c = Aa$. Note that (C2) implies that

$$Sc = SAa = ASa = Ac \quad \text{and} \quad Bc = BTb = TBb = Tc. \tag{2.5}$$

Suppose that $c \neq Tc$. In view of (2.1), (2.2), (2.4), (2.5) and $(\psi, \varphi) \in \Psi \times \Phi$, we infer that

$$\begin{aligned} m_1(a, c) &= d(Bc, Tc) \frac{1 + d(Aa, Sa)}{1 + d(Aa, Bc)} = 0, \\ m_2(a, c) &= d(Aa, Sa) \frac{1 + d(Bc, Tc)}{1 + d(Aa, Bc)} = 0, \end{aligned}$$

$$\begin{aligned}
 m_3(a, c) &= \frac{d(Sa, Bc)d(Tc, Aa)}{1 + d(Aa, Bc)} = \frac{d(c, Bc)d(Tc, c)}{1 + d(c, Bc)} = \frac{d^2(c, Tc)}{1 + d(c, Tc)}, \\
 m_4(a, c) &= \max \left\{ d(Aa, Bc), d(Aa, Sa), d(Bc, Tc), \frac{1}{2} [d(Sa, Bc) + d(Tc, Aa)] \right\} \\
 &= \max \left\{ d(c, Bc), 0, 0, \frac{1}{2} [d(c, Bc) + d(Tc, c)] \right\} = d(c, Tc)
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^{d(c, Tc)} \varphi(t) dt &= \int_0^{d(Sa, Tc)} \varphi(t) dt \\
 &\leq \psi \left(\max \left\{ \int_0^{m_i(a, c)} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \\
 &= \psi \left(\max \left\{ 0, 0, \int_0^{\frac{d^2(c, Tc)}{1 + d(c, Tc)}} \varphi(t) dt, \int_0^{d(c, Tc)} \varphi(t) dt \right\} \right) \\
 &= \psi \left(\int_0^{d(c, Tc)} \varphi(t) dt \right) < \int_0^{d(c, Tc)} \varphi(t) dt,
 \end{aligned}$$

which is impossible. Consequently, $c = Tc = Bc$. Similarly we conclude that $c = Ac = Sc$. That is, c is a common fixed point of A, B, S and T .

Thirdly we show that (2.4) holds for some $a, b \in X$. In order to prove (2.4), we have to consider three possible cases as follows.

Case 1. There exists $n_0 \in \mathbb{N}$ satisfying $d_{2n_0} = 0$. We claim that $d_{2n_0+1} = 0$. Otherwise $d_{2n_0+1} > 0$. Using (2.1)-(2.3) and $(\psi, \varphi) \in \Psi \times \Phi$, we deduce that

$$\begin{aligned}
 m_1(x_{2n_0}, x_{2n_0+1}) &= d(Bx_{2n_0+1}, Tx_{2n_0+1}) \frac{1 + d(Ax_{2n_0}, Sx_{2n_0})}{1 + d(Ax_{2n_0}, Bx_{2n_0+1})} \\
 &= d(y_{2n_0+1}, y_{2n_0+2}) \frac{1 + d(y_{2n_0}, y_{2n_0+1})}{1 + d(y_{2n_0}, y_{2n_0+1})} = d_{2n_0+1}, \\
 m_2(x_{2n_0}, x_{2n_0+1}) &= d(Ax_{2n_0}, Sx_{2n_0}) \frac{1 + d(Bx_{2n_0+1}, Tx_{2n_0+1})}{1 + d(Ax_{2n_0}, Bx_{2n_0+1})} \\
 &= d(y_{2n_0}, y_{2n_0+1}) \frac{1 + d(y_{2n_0+1}, y_{2n_0+2})}{1 + d(y_{2n_0}, y_{2n_0+1})} = 0, \\
 m_3(x_{2n_0}, x_{2n_0+1}) &= \frac{d(Sx_{2n_0}, Bx_{2n_0+1})d(Tx_{2n_0+1}, Ax_{2n_0})}{1 + d(Ax_{2n_0}, Bx_{2n_0+1})} \\
 &= \frac{d(y_{2n_0+1}, y_{2n_0+1})d(y_{2n_0+2}, y_{2n_0})}{1 + d(y_{2n_0}, y_{2n_0+1})} = 0, \\
 m_4(x_{2n_0}, x_{2n_0+1}) &= \max \left\{ d(Ax_{2n_0}, Bx_{2n_0+1}), d(Ax_{2n_0}, Sx_{2n_0}), d(Bx_{2n_0+1}, Tx_{2n_0+1}), \right. \\
 &\quad \left. \frac{1}{2} [d(Sx_{2n_0}, Bx_{2n_0+1}) + d(Tx_{2n_0+1}, Ax_{2n_0})] \right\} \\
 &= \max \left\{ d(y_{2n_0}, y_{2n_0+1}), d(y_{2n_0}, y_{2n_0+1}), d(y_{2n_0+1}, y_{2n_0+2}), \right. \\
 &\quad \left. \frac{1}{2} [d(y_{2n_0+1}, y_{2n_0+1}) + d(y_{2n_0+2}, y_{2n_0})] \right\} = \max\{d_{2n_0}, d_{2n_0+1}\} = d_{2n_0+1}
 \end{aligned}$$

and

$$\begin{aligned} \int_0^{d_{2n_0+1}} \varphi(t) dt &= \int_0^{d(y_{2n_0+1}, y_{2n_0+2})} \varphi(t) dt = \int_0^{d(Sx_{2n_0}, Tx_{2n_0+1})} \varphi(t) dt \\ &\leq \psi \left(\max \left\{ \int_0^{m_i(x_{2n_0}, x_{2n_0+1})} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \\ &= \psi \left(\max \left\{ \int_0^{d_{2n_0+1}} \varphi(t) dt, 0, 0, \int_0^{d_{2n_0+1}} \varphi(t) dt \right\} \right) \\ &< \int_0^{d_{2n_0+1}} \varphi(t) dt, \end{aligned}$$

which is a contradiction. Hence $d_{2n_0+1} = 0$. It follows that

$$Ax_{2n_0} = y_{2n_0} = y_{2n_0+1} = Sx_{2n_0} \quad \text{and} \quad Bx_{2n_0+1} = y_{2n_0+1} = y_{2n_0+2} = Tx_{2n_0+1}.$$

Put $a = x_{2n_0}$ and $b = x_{2n_0+1}$. It is easy to see that (2.4) holds and y_{2n_0} is a common fixed point of A, B, S and T .

Case 2. There exists $n_0 \in \mathbb{N}$ satisfying $d_{2n_0-1} = 0$. As in the proof of Case 1, we infer similarly that (2.4) holds for $a = x_{2n_0}$ and $b = x_{2n_0-1}$, and y_{2n_0-1} is a common fixed point of A, B, S and T .

Case 3. $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Now we claim that $d_{2n} \leq d_{2n-1}$ for all $n \in \mathbb{N}$. Suppose that $d_{2n} > d_{2n-1}$ for some $n \in \mathbb{N}$. By virtue of (2.1), (2.2) and $(\psi, \varphi) \in \Psi \times \Phi$, we arrive at

$$\begin{aligned} m_1(x_{2n}, x_{2n-1}) &= d(Bx_{2n-1}, Tx_{2n-1}) \frac{1 + d(Ax_{2n}, Sx_{2n})}{1 + d(Ax_{2n}, Bx_{2n-1})} \\ &= d(y_{2n-1}, y_{2n}) \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n-1})} = d_{2n-1} \frac{1 + d_{2n}}{1 + d_{2n-1}} < d_{2n}, \\ m_2(x_{2n}, x_{2n-1}) &= d(Ax_{2n}, Sx_{2n}) \frac{1 + d(Bx_{2n-1}, Tx_{2n-1})}{1 + d(Ax_{2n}, Bx_{2n-1})} \\ &= d(y_{2n}, y_{2n+1}) \frac{1 + d(y_{2n-1}, y_{2n})}{1 + d(y_{2n}, y_{2n-1})} = d_{2n}, \\ m_3(x_{2n}, x_{2n-1}) &= \frac{d(Sx_{2n}, Bx_{2n-1})d(Tx_{2n-1}, Ax_{2n})}{1 + d(Ax_{2n}, Bx_{2n-1})} \\ &= \frac{d(y_{2n+1}, y_{2n-1})d(y_{2n}, y_{2n})}{1 + d(y_{2n}, y_{2n-1})} = 0, \\ m_4(x_{2n}, x_{2n-1}) &= \max \left\{ d(Ax_{2n}, Bx_{2n-1}), d(Ax_{2n}, Sx_{2n}), d(Bx_{2n-1}, Tx_{2n-1}), \right. \\ &\quad \left. \frac{1}{2} [d(Sx_{2n}, Bx_{2n-1}) + d(Tx_{2n-1}, Ax_{2n})] \right\} \\ &= \max \left\{ d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \right. \\ &\quad \left. \frac{1}{2} [d(y_{2n+1}, y_{2n-1}) + d(y_{2n}, y_{2n})] \right\} \\ &= \max \{d_{2n-1}, d_{2n}\} = d_{2n} \end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
 \int_0^{d_{2n}} \varphi(t) dt &= \int_0^{d(y_{2n+1}, y_{2n})} \varphi(t) dt = \int_0^{d(Sx_{2n}, Tx_{2n-1})} \varphi(t) dt \\
 &\leq \psi \left(\max \left\{ \int_0^{m_i(x_{2n}, x_{2n-1})} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \\
 &= \psi \left(\max \left\{ \int_0^{d_{2n-1} \frac{1+d_{2n}}{1+d_{2n-1}}} \varphi(t) dt, \int_0^{d_{2n}} \varphi(t) dt, 0, \int_0^{d_{2n}} \varphi(t) dt \right\} \right) \\
 &= \psi \left(\int_0^{d_{2n}} \varphi(t) dt \right) < \int_0^{d_{2n}} \varphi(t) dt,
 \end{aligned} \tag{2.7}$$

which is absurd. Hence $d_{2n} \leq d_{2n-1}$ for each $n \in \mathbb{N}$. As in the proofs of (2.6) and (2.7), we infer similarly that $d_{2n+1} \leq d_{2n}$ for all $n \in \mathbb{N}$. Consequently, $\{d_n\}_{n \in \mathbb{N}}$ is a nonincreasing positive sequence, which means that there exists a constant $r \geq 0$ with

$$\lim_{n \rightarrow \infty} d_n = r. \tag{2.8}$$

Suppose that $r > 0$. Making use of (2.1), (2.2), (2.6), (2.8) and $(\psi, \varphi) \in \Psi \times \Phi$ and Lemma 1.4, we get that

$$\begin{aligned}
 \int_0^r \varphi(t) dt &= \limsup_{n \rightarrow \infty} \int_0^{d_{2n}} \varphi(t) dt = \limsup_{n \rightarrow \infty} \int_0^{d(y_{2n+1}, y_{2n})} \varphi(t) dt \\
 &= \limsup_{n \rightarrow \infty} \int_0^{d(Sx_{2n}, Tx_{2n-1})} \varphi(t) dt \\
 &\leq \limsup_{n \rightarrow \infty} \psi \left(\max \left\{ \int_0^{m_i(x_{2n}, x_{2n-1})} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \\
 &\leq \psi \left(\limsup_{n \rightarrow \infty} \max \left\{ \int_0^{d_{2n-1} \frac{1+d_{2n}}{1+d_{2n-1}}} \varphi(t) dt, \int_0^{d_{2n}} \varphi(t) dt, 0, \right. \right. \\
 &\quad \left. \left. \int_0^{\max\{d_{2n-1}, d_{2n}\}} \varphi(t) dt \right\} \right) \\
 &= \psi \left(\max \left\{ \int_0^r \varphi(t) dt, \int_0^r \varphi(t) dt, 0, \int_0^r \varphi(t) dt \right\} \right) \\
 &= \psi \left(\int_0^r \varphi(t) dt \right) < \int_0^r \varphi(t) dt,
 \end{aligned}$$

which is a contradiction. Hence $r = 0$. That is,

$$\lim_{n \rightarrow \infty} d_n = 0. \tag{2.9}$$

In order to prove that $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence, by (2.9) we need only to prove that $\{y_{2n}\}_{n \in \mathbb{N}}$ is a Cauchy sequence. Suppose that $\{y_{2n}\}_{n \in \mathbb{N}}$ is not a Cauchy sequence. It follows that there exists $\varepsilon > 0$ such that for each even integer $2k$ there are even integers $2m(k)$, $2n(k)$ with $2m(k) > 2n(k) > 2k$ and

$$d(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon. \tag{2.10}$$

For every even integer $2k$, let $2m(k)$ be the least even integer exceeding $2n(k)$ satisfying (2.10). It follows that

$$d(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon, \quad \forall k \in \mathbb{N}. \tag{2.11}$$

Note that

$$\begin{aligned} d(y_{2n(k)}, y_{2m(k)}) &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d_{2m(k)-2} + d_{2m(k)-1}, \quad \forall k \in \mathbb{N}; \\ |d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)})| &\leq d_{2n(k)}, \quad \forall k \in \mathbb{N}; \\ |d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| &\leq d_{2m(k)-1}, \quad \forall k \in \mathbb{N}; \\ |d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)+1}, y_{2m(k)})| &\leq d_{2m(k)-1}, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{2.12}$$

In terms of (2.9)-(2.12), we know that

$$\begin{aligned} \varepsilon &= \lim_{k \rightarrow \infty} d(y_{2n(k)}, y_{2m(k)}) = \lim_{k \rightarrow \infty} d(y_{2n(k)+1}, y_{2m(k)}) \\ &= \lim_{k \rightarrow \infty} d(y_{2n(k)}, y_{2m(k)-1}) = \lim_{k \rightarrow \infty} d(y_{2n(k)+1}, y_{2m(k)-1}). \end{aligned} \tag{2.13}$$

In light of (2.1), (2.2), (2.9), (2.13), $(\psi, \varphi) \in \Psi \times \Phi$ and Lemma 1.4, we deduce that

$$\begin{aligned} m_1(x_{2n(k)}, x_{2m(k)-1}) &= d(Bx_{2m(k)-1}, Tx_{2m(k)-1}) \frac{1 + d(Ax_{2n(k)}, Sx_{2n(k)})}{1 + d(Ax_{2n(k)}, Bx_{2m(k)-1})} \\ &= d(y_{2m(k)-1}, y_{2m(k)}) \frac{1 + d(y_{2n(k)}, y_{2n(k)+1})}{1 + d(y_{2n(k)}, y_{2m(k)-1})} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

$$\begin{aligned} m_2(x_{2n(k)}, x_{2m(k)-1}) &= d(Ax_{2n(k)}, Sx_{2n(k)}) \frac{1 + d(Bx_{2m(k)-1}, Tx_{2m(k)-1})}{1 + d(Ax_{2n(k)}, Bx_{2m(k)-1})} \\ &= d(y_{2n(k)}, y_{2n(k)+1}) \frac{1 + d(y_{2m(k)-1}, y_{2m(k)})}{1 + d(y_{2n(k)}, y_{2m(k)-1})} \rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

$$\begin{aligned} m_3(x_{2n(k)}, x_{2m(k)-1}) &= \frac{d(Sx_{2n(k)}, Bx_{2m(k)-1})d(Tx_{2m(k)-1}, Ax_{2n(k)})}{1 + d(Ax_{2n(k)}, Bx_{2m(k)-1})} \\ &= \frac{d(y_{2n(k)+1}, y_{2m(k)-1})d(y_{2m(k)}, y_{2n(k)})}{1 + d(y_{2n(k)}, y_{2m(k)-1})} \rightarrow \frac{\varepsilon^2}{1 + \varepsilon} \quad \text{as } k \rightarrow \infty, \end{aligned}$$

$$\begin{aligned} m_4(x_{2n(k)}, x_{2m(k)-1}) &= \max \left\{ d(Ax_{2n(k)}, Bx_{2m(k)-1}), d(Ax_{2n(k)}, Sx_{2n(k)}), d(Bx_{2m(k)-1}, Tx_{2m(k)-1}), \right. \\ &\quad \left. \frac{1}{2} [d(Sx_{2n(k)}, Bx_{2m(k)-1}) + d(Tx_{2m(k)-1}, Ax_{2n(k)})] \right\} \\ &= \max \left\{ d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)}, y_{2n(k)+1}), d(y_{2m(k)-1}, y_{2m(k)}), \right. \\ &\quad \left. \frac{1}{2} [d(y_{2n(k)+1}, y_{2m(k)-1}) + d(y_{2m(k)}, y_{2n(k)})] \right\} \rightarrow \varepsilon \quad \text{as } k \rightarrow \infty \end{aligned}$$

and

$$\begin{aligned} \int_0^\varepsilon \varphi(t) dt &= \limsup_{k \rightarrow \infty} \int_0^{d(y_{2n(k)+1}, y_{2m(k)})} \varphi(t) dt = \limsup_{k \rightarrow \infty} \int_0^{d(Sx_{2n(k)}, Tx_{2m(k)-1})} \varphi(t) dt \\ &\leq \limsup_{k \rightarrow \infty} \psi \left(\max \left\{ \int_0^{m_i(x_{2n(k)}, x_{2m(k)-1})} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \\ &\leq \psi \left(\limsup_{k \rightarrow \infty} \max \left\{ \int_0^{m_i(x_{2n(k)}, x_{2m(k)-1})} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \\ &= \psi \left(\max \left\{ 0, 0, \int_0^{\frac{\varepsilon^2}{1+\varepsilon}} \varphi(t) dt, \int_0^\varepsilon \varphi(t) dt \right\} \right) \\ &= \psi \left(\int_0^\varepsilon \varphi(t) dt \right) < \int_0^\varepsilon \varphi(t) dt, \end{aligned}$$

which is a contradiction. Therefore $\{y_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence.

Assume that $A(X)$ is complete. Notice that $\{y_{2n}\}_{n \in \mathbb{N}} \subseteq A(X)$, which implies that $\{y_{2n}\}_{n \in \mathbb{N}}$ converges to a point $c \in A(X)$. Obviously $\lim_{n \rightarrow \infty} y_n = c$. Put $a \in A^{-1}c$. It follows that $Aa = c$. Suppose that $Sa \neq c$. In view of (2.1)-(2.3), $(\psi, \varphi) \in \Psi \times \Phi$, Lemma 1.4 and $\lim_{n \rightarrow \infty} y_n = c$, we infer that

$$\begin{aligned} m_1(a, x_{2n-1}) &= d(Bx_{2n-1}, Tx_{2n-1}) \frac{1 + d(Aa, Sa)}{1 + d(Aa, Bx_{2n-1})} \\ &= d(y_{2n-1}, y_{2n}) \frac{1 + d(c, Sa)}{1 + d(c, y_{2n-1})} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ m_2(a, x_{2n-1}) &= d(Aa, Sa) \frac{1 + d(Bx_{2n-1}, Tx_{2n-1})}{1 + d(Aa, Bx_{2n-1})} \\ &= d(c, Sa) \frac{1 + d(y_{2n-1}, y_{2n})}{1 + d(c, y_{2n-1})} \rightarrow d(c, Sa) \quad \text{as } n \rightarrow \infty, \\ m_3(a, x_{2n-1}) &= \frac{d(Sa, Bx_{2n-1})d(Tx_{2n-1}, Aa)}{1 + d(Aa, Bx_{2n-1})} \\ &= \frac{d(Sa, y_{2n-1})d(y_{2n}, c)}{1 + d(c, y_{2n-1})} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \\ m_4(a, x_{2n-1}) &= \max \left\{ d(Aa, Bx_{2n-1}), d(Aa, Sa), d(Bx_{2n-1}, Tx_{2n-1}), \right. \\ &\quad \left. \frac{1}{2} [d(Sa, Bx_{2n-1}) + d(Tx_{2n-1}, Aa)] \right\} \\ &= \max \left\{ d(c, y_{2n-1}), d(c, Sa), d(y_{2n-1}, y_{2n}), \right. \\ &\quad \left. \frac{1}{2} [d(Sa, y_{2n-1}) + d(y_{2n}, c)] \right\} \rightarrow d(c, Sa) \quad \text{as } n \rightarrow \infty \end{aligned}$$

and

$$\begin{aligned} \int_0^{d(Sa, c)} \varphi(t) dt &= \limsup_{n \rightarrow \infty} \int_0^{d(Sa, y_{2n})} \varphi(t) dt = \limsup_{n \rightarrow \infty} \int_0^{d(Sa, Tx_{2n-1})} \varphi(t) dt \\ &\leq \limsup_{n \rightarrow \infty} \psi \left(\max \left\{ \int_0^{m_i(a, x_{2n-1})} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \end{aligned}$$

$$\begin{aligned} &\leq \psi \left(\limsup_{n \rightarrow \infty} \max \left\{ \int_0^{m_i(a, x_{2n-1})} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \\ &= \psi \left(\max \left\{ 0, \int_0^{d(c, Sa)} \varphi(t) dt, 0, \int_0^{d(c, Sa)} \varphi(t) dt \right\} \right) \\ &= \psi \left(\int_0^{d(c, Sa)} \varphi(t) dt \right) < \int_0^{d(c, Sa)} \varphi(t) dt, \end{aligned}$$

which is a contradiction. Therefore, $Sa = c$, which together with (C1) means that $c \in B(X)$. Put $b \in B^{-1}c$, that is, $Bb = c$. Suppose that $c \neq Tb$. By means of (2.1), (2.2) and $(\psi, \varphi) \in \Psi \times \Phi$, we get that

$$\begin{aligned} m_1(a, b) &= d(Bb, Tb) \frac{1 + d(Aa, Sa)}{1 + d(Aa, Bb)} = d(c, Tb), \\ m_2(a, b) &= d(Aa, Sa) \frac{1 + d(Bb, Tb)}{1 + d(Aa, Bb)} = 0, \\ m_3(a, b) &= \frac{d(Sa, Bb)d(Tb, Aa)}{1 + d(Aa, Bb)} = 0, \\ m_4(a, b) &= \max \left\{ d(Aa, Bb), d(Aa, Sa), d(Bb, Tb), \frac{1}{2} [d(Sa, Bb) + d(Tb, Aa)] \right\} \\ &= \max \left\{ 0, 0, d(c, Tb), \frac{1}{2} [0 + d(Tb, c)] \right\} \\ &= d(c, Tb) \end{aligned}$$

and

$$\begin{aligned} \int_0^{d(c, Tb)} \varphi(t) dt &= \int_0^{d(Sa, Tb)} \varphi(t) dt \leq \psi \left(\max \left\{ \int_0^{m_i(a, b)} \varphi(t) dt : 1 \leq i \leq 4 \right\} \right) \\ &= \psi \left(\max \left\{ \int_0^{d(c, Tb)} \varphi(t) dt, 0, 0, \int_0^{d(c, Tb)} \varphi(t) dt \right\} \right) \\ &= \psi \left(\int_0^{d(c, Tb)} \varphi(t) dt \right) < \int_0^{d(c, Tb)} \varphi(t) dt, \end{aligned}$$

which is impossible. That is, $c = Tb$. Hence (2.4) holds.

Assume that $T(X)$ is complete. Notice that $\{y_{2n}\}_{n \in \mathbb{N}} \subseteq T(X)$, which implies that $\{y_{2n}\}_{n \in \mathbb{N}}$ converges to a point $c \in T(X)$. Obviously $\lim_{n \rightarrow \infty} y_n = c$. Put $b_1 \in T^{-1}c$. It follows that $Tb_1 = c$. Observe that $T(X) \subseteq A(X)$, which implies that there exists $a \in X$ with $Aa = Tb_1 = c$. As in the proof of completeness of $A(X)$, we infer that (2.4) holds. Similarly we conclude that (2.4) holds if one of $B(X)$ and $S(X)$ is complete. This completes the proof. \square

As in the proof of Theorem 2.1 we have the following result and omit its proof.

Theorem 2.2 *Let A, B, S and T be self-mappings of a metric space (X, d) satisfying (C1)-(C3) and*

$$\int_0^{d(Sx, Ty)} \varphi(t) dt \leq \psi \left(\int_0^{m_4(x, y)} \varphi(t) dt \right), \quad \forall x, y \in X, \tag{2.14}$$

where (ψ, φ) is in $\Psi \times \Phi$ and m_4 is defined by (2.2). Then A, B, S and T have a unique common fixed point in X .

Remark 2.3 Theorems 2.1 and 2.2 extend, improve and unify Theorem 2.1 in [1, 4], Theorem 2 in [13, 16, 17] and Corollary 3 in [17]. The following example reveals that Theorem 2.2 is both an indeed generalization of Theorem 2.1 in [1], Theorem 2 in [13, 17] and Corollary 3 in [17], and different from Theorems 3.1-3.3 in [9].

Example 2.4 Let $X = \mathbb{R}$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $S, T : X \rightarrow X$ be defined by

$$Sx = \begin{cases} 2, & \forall x \in X - \{1\}, \\ \frac{3}{2}, & x = 1, \end{cases} \quad Tx = 2, \quad \forall x \in X.$$

Now we claim that Theorem 2 and Corollary 3 in [17] cannot be used to prove the existence of common fixed points of the mappings S and T in X , and Theorem 2 in [13], Theorem 2.1 in [1] and Theorems 3.1-3.3 in [9] are useless in proving the existence of fixed points of the mapping S in X .

Suppose that there exists $(\varphi, \psi) \in \Phi \times \Psi_1$ satisfying the condition of Theorem 2 in [17], that is,

$$\int_0^{d(Sx, Ty)} \varphi(t) dt \leq \psi \left(\int_0^{M(x, y)} \varphi(t) dt \right), \quad \forall x, y \in X, \tag{2.15}$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, Sx), d(y, Ty), \frac{1}{2} [d(y, Sx) + d(x, Ty)] \right\}. \tag{2.16}$$

Put $x_0 = 1$ and $y_0 = \frac{3}{2}$. It follows from (2.15), (2.16) and $(\varphi, \psi) \in \Phi \times \Psi_1$ that

$$\begin{aligned} M(x_0, y_0) &= \max \left\{ d(x_0, y_0), d(x_0, Sx_0), d(y_0, Ty_0), \frac{1}{2} [d(y_0, Sx_0) + d(x_0, Ty_0)] \right\} \\ &= \max \left\{ d\left(1, \frac{3}{2}\right), d\left(1, \frac{3}{2}\right), d\left(\frac{3}{2}, 2\right), \frac{1}{2} \left[d\left(\frac{3}{2}, \frac{3}{2}\right) + d(1, 2) \right] \right\} \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \int_0^{\frac{1}{2}} \varphi(t) dt &= \int_0^{d(Sx_0, Ty_0)} \varphi(t) dt \leq \psi \left(\int_0^{M(x_0, y_0)} \varphi(t) dt \right) \\ &= \psi \left(\int_0^{\frac{1}{2}} \varphi(t) dt \right) < \int_0^{\frac{1}{2}} \varphi(t) dt, \end{aligned}$$

which is impossible.

Suppose that there exist $\varphi \in \Phi$ and $k \in [0, 1)$ satisfying the condition of Corollary 3 in [17], that is,

$$\int_0^{d(Sx, Ty)} \varphi(t) dt \leq k \int_0^{n(x, y)} \varphi(t) dt \quad \text{for each distinct } x, y \in X, \tag{2.17}$$

where

$$n(x, y) = \max \left\{ d(x, y), \frac{d(y, Ty)[1 + d(x, Sx)]}{1 + d(x, y)} \right\}. \tag{2.18}$$

Take $x_0 = 1$ and $y_0 = \frac{3}{2}$. It follows from (2.17), (2.18), $k \in [0, 1)$ and $\varphi \in \Phi$ that

$$\begin{aligned} n(x_0, y_0) &= \max \left\{ d(x_0, y_0), \frac{d(y_0, Ty_0)[1 + d(x_0, Sx_0)]}{1 + d(x_0, y_0)} \right\} \\ &= \max \left\{ d\left(1, \frac{3}{2}\right), \frac{d\left(\frac{3}{2}, 2\right)[1 + d\left(1, \frac{3}{2}\right)]}{1 + d\left(1, \frac{3}{2}\right)} \right\} \\ &= \frac{1}{2} \end{aligned}$$

and

$$\begin{aligned} \int_0^{\frac{1}{2}} \varphi(t) dt &= \int_0^{d(Sx_0, Ty_0)} \varphi(t) dt \leq k \int_0^{n(x_0, y_0)} \varphi(t) dt \\ &= k \int_0^{\frac{1}{2}} \varphi(t) dt < \int_0^{\frac{1}{2}} \varphi(t) dt, \end{aligned}$$

which is a contradiction.

Suppose that there exist $\varphi \in \Phi$ and $k \in [0, 1)$ satisfying the condition of Theorem 2 in [13], that is,

$$\int_0^{d(Sx, Sy)} \varphi(t) dt \leq k \int_0^{m(x, y)} \varphi(t) dt, \quad \forall x, y \in X, \tag{2.19}$$

where

$$m(x, y) = \max \left\{ d(x, y), d(x, Sx), d(y, Sy), \frac{1}{2}[d(y, Sx) + d(x, Sy)] \right\}. \tag{2.20}$$

Put $x_0 = 1$ and $y_0 = \frac{3}{2}$. It follows from (2.19), (2.20), $k \in [0, 1)$ and $\varphi \in \Phi$ that

$$\begin{aligned} m(x_0, y_0) &= \max \left\{ d(x_0, y_0), d(x_0, Sx_0), d(y_0, Sy_0), \frac{1}{2}[d(y_0, Sx_0) + d(x_0, Sy_0)] \right\} \\ &= \max \left\{ d\left(1, \frac{3}{2}\right), d(1, S1), d\left(\frac{3}{2}, S\frac{3}{2}\right), \frac{1}{2}\left[d\left(\frac{3}{2}, S1\right) + d\left(1, S\frac{3}{2}\right) \right] \right\} \\ &= \max \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \\ &= \frac{1}{2} \end{aligned}$$

and

$$\int_0^{\frac{1}{2}} \varphi(t) dt = \int_0^{d(Sx_0, Sy_0)} \varphi(t) dt \leq k \int_0^{m(x_0, y_0)} \varphi(t) dt = k \int_0^{\frac{1}{2}} \varphi(t) dt < \int_0^{\frac{1}{2}} \varphi(t) dt,$$

which is absurd. Observe that Theorem 2 in [13] generalizes Theorem 2.1 in [1], hence Theorem 2.1 in [1] cannot be used to prove the existence of fixed points of S in X .

Suppose that there exists $\varphi \in \Phi$ satisfying the condition of Theorem 3.1 in [9], that is,

$$\int_0^{d(Sx, Sy)} \varphi(t) dt \leq \alpha(d(x, y)) \int_0^{d(x, y)} \varphi(t) dt, \quad \forall x, y \in X, \tag{2.21}$$

where

$$\alpha : \mathbb{R}^+ \rightarrow [0, 1) \quad \text{and} \quad \limsup_{s \rightarrow t} \alpha(s) < 1, \quad \forall t > 0. \tag{2.22}$$

Put $x_0 = 1$ and $y_0 = \frac{3}{2}$. It follows from (2.21), (2.22) and $\varphi \in \Phi$ that

$$\begin{aligned} \int_0^{\frac{1}{2}} \varphi(t) dt &= \int_0^{d(Sx_0, Sy_0)} \varphi(t) dt \leq \alpha(d(x_0, y_0)) \int_0^{d(x_0, y_0)} \varphi(t) dt \\ &= \alpha\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \varphi(t) dt < \int_0^{\frac{1}{2}} \varphi(t) dt, \end{aligned}$$

which is impossible.

Suppose that there exists $\varphi \in \Phi$ satisfying the condition of Theorem 3.2 in [9], that is,

$$\begin{aligned} \int_0^{d(Sx, Sy)} \varphi(t) dt \\ \leq \alpha(d(x, y)) \int_0^{d(x, Sx)} \varphi(t) dt + \beta(d(x, y)) \int_0^{d(y, Sy)} \varphi(t) dt, \quad \forall x, y \in X, \end{aligned} \tag{2.23}$$

where

$$\begin{aligned} \alpha, \beta : \mathbb{R}^+ \rightarrow [0, 1) \text{ satisfy that } \alpha(t) + \beta(t) < 1, \quad \forall t \in \mathbb{R}^+, \\ \limsup_{s \rightarrow 0^+} \beta(s) < 1 \quad \text{and} \quad \limsup_{s \rightarrow t^+} \frac{\alpha(s)}{1 - \beta(s)} < 1, \quad \forall t > 0. \end{aligned} \tag{2.24}$$

Put $x_0 = 1$ and $y_0 = \frac{3}{2}$. It follows from (2.23), (2.24) and $\varphi \in \Phi$ that

$$\begin{aligned} \int_0^{\frac{1}{2}} \varphi(t) dt &= \int_0^{d(Sx_0, Sy_0)} \varphi(t) dt \\ &\leq \alpha(d(x_0, y_0)) \int_0^{d(x_0, Sx_0)} \varphi(t) dt + \beta(d(x_0, y_0)) \int_0^{d(y_0, Sy_0)} \varphi(t) dt \\ &= \left[\alpha\left(\frac{1}{2}\right) + \beta\left(\frac{1}{2}\right) \right] \int_0^{\frac{1}{2}} \varphi(t) dt < \int_0^{\frac{1}{2}} \varphi(t) dt, \end{aligned}$$

which is a contradiction.

Suppose that there exists $\varphi \in \Phi$ satisfying the condition of Theorem 3.3 in [9], that is,

$$\int_0^{d(Sx,Sy)} \varphi(t) dt \leq \gamma(d(x,y)) \left(\int_0^{d(x,Sx)} \varphi(t) dt + \int_0^{d(y,Sy)} \varphi(t) dt \right), \quad \forall x,y \in X, \quad (2.25)$$

where

$$\gamma : \mathbb{R}^+ \rightarrow \left[0, \frac{1}{2} \right) \quad \text{and} \quad \limsup_{s \rightarrow t^+} \frac{\gamma(s)}{1 - \gamma(s)} < 1, \quad \forall t > 0. \quad (2.26)$$

Put $x_0 = 1$ and $y_0 = \frac{3}{2}$. It follows from (2.25), (2.26) and $\varphi \in \Phi$ that

$$\begin{aligned} \int_0^{\frac{1}{2}} \varphi(t) dt &= \int_0^{d(Sx_0,Sy_0)} \varphi(t) dt \\ &\leq \gamma(d(x_0,y_0)) \left(\int_0^{d(x_0,Sx_0)} \varphi(t) dt + \int_0^{d(y_0,Sy_0)} \varphi(t) dt \right) \\ &= 2\gamma\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} \varphi(t) dt < \int_0^{\frac{1}{2}} \varphi(t) dt, \end{aligned}$$

which is impossible.

Next we prove, by using Theorem 2.2, that the mappings A, B, S and T have a unique common fixed point in X , where $A, B : X \rightarrow X$ are defined by

$$Ax = \frac{1}{2}x^2 \quad \text{and} \quad Bx = \frac{1}{4}x^3, \quad \forall x \in X.$$

Define two functions $\psi, \varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ by

$$\varphi(t) = 6t^5 + 3t^2 + t, \quad \forall t \in [0, +\infty) \quad \text{and} \quad \psi(t) = \begin{cases} t^2, & \forall t \in [0, \frac{1}{2}], \\ \frac{t}{4+t}, & \forall t \in (\frac{1}{2}, +\infty). \end{cases}$$

It is easy to see that (C1), (C2) and (C3) hold. Let $x, y \in X$. In order to verify (2.14), we have to consider two possible cases as follows.

Case 1. $x \in X \setminus \{1\}$. It is clear that

$$\int_0^{d(Sx,Ty)} \varphi(t) dt = 0 \leq \psi \left(\int_0^{m_4(x,y)} \varphi(t) dt \right).$$

Case 2. $x = 1$. Note that ψ is nondecreasing on $(\frac{1}{2}, +\infty)$,

$$\begin{aligned} m_4(1,y) &= \max \left\{ d(A1,By), d(A1,S1), d(By,Ty), \frac{1}{2} [d(S1,By) + d(Ty,A1)] \right\} \\ &= \max \left\{ \left| \frac{1}{2} - \frac{y^3}{4} \right|, \left| \frac{1}{2} - \frac{3}{2} \right|, \left| \frac{y^3}{4} - 2 \right|, \frac{1}{2} \left(\left| \frac{3}{2} - \frac{y^3}{4} \right| + \left| 2 - \frac{1}{2} \right| \right) \right\} \\ &\geq 1 \end{aligned}$$

and

$$\begin{aligned} \int_0^{d(S1, Ty)} \varphi(t) dt &= \int_0^{\frac{1}{2}} (6t^5 + 3t^2 + t) dt = \frac{17}{64} \\ &< \frac{5}{13} = \frac{\frac{5}{2}}{4 + \frac{5}{2}} = \psi\left(\frac{5}{2}\right) \\ &= \psi\left(\int_0^1 (6t^5 + 3t^2 + t) dt\right) \\ &\leq \psi\left(\int_0^{m_4(1,y)} \varphi(t) dt\right). \end{aligned}$$

Hence (2.14) holds. That is, the conditions of Theorem 2.2 are satisfied. Consequently, Theorem 2.2 implies that A, B, S and T have a unique common fixed point $2 \in X$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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