Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Some results on zero points of m-accretive operators in reflexive Banach spaces

Chang Qun Wu[1], Songtao Lv[2*] and Yunpeng Zhang[3]

*Correspondence: sqlvst@yeah.net
[2]School of Mathematics and Information Science, Shangqiu Normal University, Shangqiu, Henan, China
Full list of author information is available at the end of the article

Abstract

A modified proximal point algorithm is proposed for treating common zero points of a finite family of m-accretive operators. A strong convergence theorem is established in a reflexive, strictly convex Banach space with the uniformly Gâteaux differentiable norm.

Keywords: accretive operator; nonexpansive mapping; resolvent; fixed point; zero point

1 Introduction and preliminaries

Let E be a Banach space and let E^* be the dual of E. Let $\langle\cdot,\cdot\rangle$ denote the pairing between E and E^*. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E.$$

A Banach space E is said to strictly convex if and only if $\|x\| = \|y\| = \|(1-\lambda)x + \lambda y\|$ for $x, y \in E$ and $0 < \lambda < 1$ implies that $x = y$. Let $U_E = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists for each $x, y \in U_E$. In this case, E is said to be smooth. The norm of E is said to be uniformly Gâteaux differentiable if for each $y \in U_E$, the limit is attained uniformly for all $x \in U_E$. The norm of E is said to be Fréchet differentiable if for each $x \in U_E$, the limit is attained uniformly for all $y \in U_E$. The norm of E is said to be uniformly Fréchet differentiable if the limit is attained uniformly for all $x, y \in U_E$. It is well known that (uniform) Fréchet differentiability of the norm of E implies (uniform) Gâteaux differentiability of the norm of E.

Let $\rho_E : [0,\infty) \to [0,\infty)$ be the modulus of smoothness of E by

$$\rho_E(t) = \sup\left\{\frac{\|x+y\| - \|x-y\|}{2} - 1 : x \in U_E, \|y\| \le t\right\}.$$

A Banach space E is said to be uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$. It is well known that if the norm of E is uniformly Gâteaux differentiable, then the duality mapping J is single valued and uniformly norm to weak* continuous on each bounded subset of E.

Recall that a closed convex subset C of a Banach space E is said to have a normal structure if for each bounded closed convex subset K of C which contains at least two points, there exists an element x of K which is not a diametral point of K, *i.e.*, $\sup\{\|x-y\| : y \in K\} < d(K)$, where $d(K)$ is the diameter of K.

©2014 Wu et al.; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Let D be a nonempty subset of a set C. Let $Proj_D : C \to D$. Q is said to be

(1) sunny if for each $x \in C$ and $t \in (0,1)$, we have $Proj_D(tx + (1-t)Proj_D x) = Proj_D x$;

(2) a contraction if $Proj_D^2 = Proj_D$;

(3) a sunny nonexpansive retraction if $Proj_D$ is sunny, nonexpansive, and a contraction.

D is said to be a nonexpansive retract of C if there exists a nonexpansive retraction from C onto D. The following result, which was established in [1–3], describes a characterization of sunny nonexpansive retractions on a smooth Banach space.

Let E be a smooth Banach space and let C be a nonempty subset of E. Let $Proj_C : E \to C$ be a retraction and J_φ be the duality mapping on E. Then the following are equivalent:

(1) $Proj_C$ is sunny and nonexpansive;

(2) $\langle x - Proj_C x, J_\varphi(y - Proj_C x)\rangle \leq 0$, $\forall x \in E$, $y \in C$;

(3) $\|Proj_C x - Proj_C y\|^2 \leq \langle x - y, J_\varphi(Proj_C x - Proj_C y)\rangle$, $\forall x, y \in E$.

It is well known that if E is a Hilbert space, then a sunny nonexpansive retraction $Proj_C$ is coincident with the metric projection from E onto C. Let C be a nonempty closed convex subset of a smooth Banach space E, let $x \in E$, and let $x_0 \in C$. Then we have from the above that $x_0 = Proj_C x$ if and only if $\langle x - x_0, J_\varphi(y - x_0)\rangle \leq 0$ for all $y \in C$, where $Proj_C$ is a sunny nonexpansive retraction from E onto C. For more additional information on nonexpansive retracts, see [4] and the references therein.

Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the set of fixed points of T. Recall that T is said to be an α-contractive mapping iff there exists a constant $\alpha \in [0,1)$ such that $\|Tx - Ty\| \leq \alpha\|x - y\|$, $\forall x, y \in C$. The Picard iterative process is an efficient method to study fixed points of α-contractive mappings. It is well known that α-contractive mappings have a unique fixed point. T is said to be nonexpansive iff $\|Tx - Ty\| \leq \|x - y\|$, $\forall x, y \in C$. It is well known that nonexpansive mappings have fixed points if the set C is closed and convex, and the space E is uniformly convex. The Krasnoselski-Mann iterative process is an efficient method for studying fixed points of nonexpansive mappings. The Krasnoselski-Mann iterative process generates a sequence $\{x_n\}$ in the following manner:

$$x_1 \in C, \quad x_{n+1} = \alpha_n Tx_n + (1-\alpha_n)x_n, \quad \forall n \geq 1.$$

It is well known that the Krasnoselski-Mann iterative process only has weak convergence for nonexpansive mappings in infinite-dimensional Hilbert spaces; see [5–7] for more details and the references therein. In many disciplines, including economics, image recovery, quantum physics, and control theory, problems arise in infinite-dimensional spaces. In such problems, strong convergence (norm convergence) is often much more desirable than weak convergence, for it translates the physically tangible property that the energy $\|x_n - x\|$ of the error between the iterate x_n and the solution x eventually becomes arbitrarily small. To improve the weak convergence of a Krasnoselski-Mann iterative process, so-called hybrid projections have been considered; see [8–22] for more details and the references therein. The Halpern iterative process was initially introduced in [23]; see [23] for more details and the references therein. The Halpern iterative process generates a sequence $\{x_n\}$ in the following manner:

$$x_1 \in C, \quad x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad \forall n \geq 1,$$

where x_1 is an initial and u is a fixed element in C. Strong convergence of Halpern iterative process does not depend on metric projections. The Halpern iterative process has recently been extensively studied for treating accretive operators; see [24–31] and the references therein.

Let I denote the identity operator on E. An operator $A \subset E \times E$ with domain $D(A) = \{z \in E : Az \neq \emptyset\}$ and range $R(A) = \bigcup\{Az : z \in D(A)\}$ is said to be accretive if for each $x_i \in D(A)$ and $y_i \in Ax_i$, $i = 1, 2$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that $\langle y_1 - y_2, j(x_1 - x_2)\rangle \geq 0$. An accretive operator A is said to be m-accretive if $R(I + rA) = E$ for all $r > 0$. In this paper, we use $A^{-1}(0)$ to denote the set of zero points of A. For an accretive operator A, we can define a nonexpansive single valued mapping $J_r : R(I + rA) \to D(A)$ by $J_r = (I + rA)^{-1}$ for each $r > 0$, which is called the resolvent of A.

Now, we are in a position to give the lemmas to prove main results.

Lemma 1.1 [32] *Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ be four nonnegative real sequences satisfying $a_{n+1} \leq (1 - b_n)a_n + b_n c_n + d_n$, $\forall n \geq n_0$, where n_0 is some positive integer, $\{b_n\}$ is a number sequence in $(0, 1)$ such that $\sum_{n=n_0}^{\infty} b_n = \infty$, $\{c_n\}$ is a number sequence such that $\limsup_{n\to\infty} c_n \leq 0$, and $\{d_n\}$ is a positive number sequence such that $\sum_{n=n_0}^{\infty} d_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 1.2 [33] *Let C be a closed convex subset of a strictly convex Banach space E. Let $N \geq 1$ be some positive integer and let $T_i : C \to C$ be a nonexpansive mapping for each $i \in \{1, 2, \ldots, N\}$. Let $\{\delta_i\}$ be a real number sequence in $(0, 1)$ with $\sum_{i=1}^{N} \delta_i = 1$. Suppose that $\bigcap_{i=1}^{N} F(T_i)$ is nonempty. Then the mapping $\bigcap_{i=1}^{N} T_i$ is defined to be nonexpansive with $F(\bigcap_{i=1}^{N} T_i) = \bigcap_{i=1}^{N} F(T_i)$.*

Lemma 1.3 [34] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space E and let β_n be a sequence in $[0, 1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \geq 0$ and*

$$\limsup_{n\to\infty} \bigl(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\bigr) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.4 [35] *Let E be a real reflexive Banach space with the uniformly Gâteaux differentiable norm and the normal structure, and let C be a nonempty closed convex subset of E. Let $f : C \to C$ be α-contractive mapping and let $T : C \to C$ be a nonexpansive mapping with a fixed point. Let $\{x_t\}$ be a sequence generated by the following: $x_t = tf(x_t) + (1 - t)Tx_t$, where $t \in (0, 1)$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point x^* of T, which is the unique solution in $F(T)$ to the following variational inequality: $\langle f(x^*) - x^*, j(x^* - p)\rangle \geq 0$, $\forall p \in F(T)$.*

2 Main results

Theorem 2.1 *Let E be a real reflexive, strictly convex Banach space with the uniformly Gâteaux differentiable norm. Let $N \geq 1$ be some positive integer. Let A_m be an m-accretive operator in E for each $m \in \{1, 2, \ldots, N\}$. Assume that $C := \bigcap_{m=1}^{N} \overline{D(A_m)}$ is convex and has the normal structure. Let $f : C \to C$ be an α-contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0, 1)$ with the restriction $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\delta_{n,i}\}$ be a real number sequence in $(0, 1)$ with the restriction $\delta_{n,1} + \delta_{n,2} + \cdots + \delta_{n,N} = 1$. Let $\{r_m\}$ be a positive* *real numbers sequence and* $\{e_{n,i}\}$ *a sequence in* E *for each* $i \in \{1,2,\dots,N\}$. *Assume that* $\bigcap_{i=1}^N A_i^{-1}(0)$ *is not empty. Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^N \delta_{n,i} J_{r_i}(x_n + e_{n,i}), \quad \forall n \ge 1,$$

where $J_{r_i} = (I + r_i A_i)^{-1}$. *Assume that the control sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\delta_{n,i}\}$ *satisfy the following restrictions*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;
(c) $\sum_{n=1}^\infty \|e_{n,m}\| < \infty$;
(d) $\lim_{n\to\infty} \delta_{n,i} = \delta_i \in (0,1)$.

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x}$, *which is the unique solution to the following variational inequality*: $\langle f(\bar{x}) - \bar{x}, J(p - \bar{x})\rangle \le 0$, $\forall p \in \bigcap_{i=1}^N A_i^{-1}(0)$.

Proof Put $y_n = \sum_{i=1}^N \delta_{n,i} J_{r_i}(x_n + e_{n,i})$. Fixing $p \in \bigcap_{i=1}^N A_i^{-1}(0)$, we have

$$\begin{aligned} \|y_n - p\| &\le \sum_{i=1}^N \delta_{n,i} \big\|J_{r_i}(x_n + e_{n,i}) - p\big\| \\ &\le \sum_{i=1}^N \delta_{n,i} \big\|(x_n + e_{n,i}) - p\big\| \\ &\le \|x_n - p\| + \sum_{i=1}^N \|e_{n,i}\|. \end{aligned}$$

Hence, we have

$$\begin{aligned} \|x_{n+1} - p\| &\le \alpha_n \big\|f(x_n) - p\big\| + \beta_n \|x_n - p\| + \gamma_n \|y_n - p\| \\ &\le \alpha_n \alpha \|x_n - p\| + \alpha_n \big\|f(p) - p\big\| + \beta_n \|x_n - p\| + \gamma_n \|x_n - p\| + \gamma_n \sum_{i=1}^N \|e_{n,i}\| \\ &\le \big(1 - \alpha_n(1-\alpha)\big)\|x_n - p\| + \alpha_n(1-\alpha)\frac{\|f(p) - p\|}{1-\alpha} + \sum_{i=1}^N \|e_{n,i}\| \\ &\le \max\big\{\|x_n - p\|, \big\|f(p) - p\big\|\big\} + \sum_{i=1}^N \|e_{n,i}\| \\ &\ \ \vdots \\ &\le \max\big\{\|x_1 - p\|, \big\|f(p) - p\big\|\big\} + \sum_{j=1}^\infty \sum_{i=1}^N \|e_{j,i}\|. \end{aligned}$$

This proves that the sequence $\{x_n\}$ is bounded, and so is $\{y_n\}$. Since

$$\begin{aligned} y_n - y_{n-1} = {}& \sum_{i=1}^N \delta_{n,i}\big(J_{r_m}(x_n + e_{n,i}) - J_{r_i}(x_{n-1} + e_{n-1,i})\big) \\ &+ \sum_{i=1}^N (\delta_{n,i} - \delta_{n-1,i}) J_{r_i}(x_{n-1} + e_{n-1,i}), \end{aligned}$$

we have

$$\begin{aligned}
\|y_n - y_{n-1}\| &\le \sum_{i=1}^{N} \delta_{n,i} \bigl\|J_{r_i}(x_n + e_{n,i}) - J_{r_i}(x_{n-1} + e_{n-1,i})\bigr\| \\
&\quad + \sum_{i=1}^{N} |\delta_{n,i} - \delta_{n-1,i}| \bigl\|J_{r_i}(x_{n-1} + e_{n-1,i})\bigr\| \\
&\le \|x_n - x_{n-1}\| + \sum_{i=1}^{N} \|e_{n,i}\| + \sum_{i=1}^{N} \|e_{n-1,i}\| \\
&\quad + \sum_{i=1}^{N} |\delta_{n,i} - \delta_{n-1,i}| \bigl\|J_{r_i}(x_{n-1} + e_{n-1,i})\bigr\| \\
&\le \|x_n - x_{n-1}\| + \sum_{i=1}^{N} \|e_{n,i}\| + \sum_{i=1}^{N} \|e_{n-1,i}\| + M_1 \sum_{i=1}^{N} |\delta_{n,i} - \delta_{n-1,i}|,
\end{aligned}$$

where M_1 is an appropriate constant such that

$$M_1 = \max\Bigl\{\sup_{n\ge 1}\bigl\|J_{r_1}(x_n + e_{n,1})\bigr\|, \sup_{n\ge 1}\bigl\|J_{r_2}(x_n + e_{n,2})\bigr\|, \ldots, \sup_{n\ge 1}\bigl\|J_{r_N}(x_n + e_{n,N})\bigr\|\Bigr\}.$$

Define a sequence $\{z_n\}$ by $z_n := \frac{x_{n+1} - \beta_n x_n}{1-\beta_n}$, that is, $x_{n+1} = \beta_n x_n + (1-\beta_n) z_n$. It follows that

$$\begin{aligned}
\|yz_n - z_{n-1}\| &\le \frac{\alpha_n}{1-\beta_n}\bigl\|f(x_n) - y_n\bigr\| + \frac{\alpha_{n-1}}{1-\beta_{n-1}}\bigl\|f(x_{n-1}) - y_{n-1}\bigr\| + \|y_n - y_{n-1}\| \\
&\le \frac{\alpha_n}{1-\beta_n}\bigl\|f(x_n) - y_n\bigr\| + \frac{\alpha_{n-1}}{1-\beta_{n-1}}\bigl\|f(x_{n-1}) - y_{n-1}\bigr\| + \|x_n - x_{n-1}\| \\
&\quad + \sum_{i=1}^{N} |\delta_{n,i} - \delta_{n-1,i}| \|J_{r_i} x_{n-1}\| \\
&\le \frac{\alpha_n}{1-\beta_n}\bigl\|f(x_n) - y_n\bigr\| + \frac{\alpha_{n-1}}{1-\beta_{n-1}}\bigl\|f(x_{n-1}) - y_{n-1}\bigr\| + \|x_n - x_{n-1}\| \\
&\quad + M_2 \Biggl(\sum_{i=1}^{N} |\delta_{n,i} - \delta_i| + \sum_{i=1}^{N} |\delta_i - \delta_{n-1,i}|\Biggr),
\end{aligned}$$

where M_2 is an appropriate constant such that

$$M_2 = \max\Bigl\{\sup_{n\ge 1}\|J_{r_1} x_n\|, \sup_{n\ge 1}\|J_{r_2} x_n\|, \ldots, \sup_{n\ge 1}\|J_{r_N} x_n\|\Bigr\}.$$

This implies that

$$\begin{aligned}
&\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\| \\
&\quad \le \frac{\alpha_n}{1-\beta_n}\bigl\|f(x_n) - y_n\bigr\| + \frac{\alpha_{n-1}}{1-\beta_{n-1}}\bigl\|f(x_{n-1}) - y_{n-1}\bigr\| \\
&\qquad + M_2 \Biggl(\sum_{i=1}^{N} |\delta_{n,i} - \delta_i| + \sum_{i=1}^{N} |\delta_i - \delta_{n-1,i}|\Biggr).
\end{aligned}$$

From the restrictions (a), (b), (c), and (d), we find that

$$\limsup_{n\to\infty}\bigl(\|z_n - z_{n-1}\| - \|x_n - x_{n-1}\|\bigr) \le 0.$$

Using Lemma 1.4, we find that $\lim_{n\to\infty}\|z_n - x_n\| = 0$. This further shows that $\limsup_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Put $T = \sum_{i=1}^{N}\delta_i J_{r_i}$. It follows from Lemma 1.3 that T is nonexpansive with $F(T) = \bigcap_{i=1}^{N} F(J_{r_i}) = \bigcap_{i=1}^{N} A_i^{-1}(0)$. Note that

$$\begin{aligned}
&\|x_n - Tx_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \alpha_n\bigl\|f(x_n) - Tx_n\bigr\| + \beta_n\|x_n - Tx_n\| + \gamma_n\|y_n - Tx_n\| \\
&\quad \le \|x_n - x_{n+1}\| + \alpha_n\bigl\|f(x_n) - Tx_n\bigr\| + \beta_n\|x_n - Tx_n\| + M_2\sum_{i=1}^{N}|\delta_{n,i} - \delta_i|.
\end{aligned}$$

This implies that

$$(1-\beta_n)\|x_n - Tx_n\| \le \|x_n - x_{n+1}\| + \alpha_n\bigl\|f(x_n) - Tx_n\bigr\| + M_2\sum_{i=1}^{N}|\delta_{n,i} - \delta_i|.$$

It follows from the restrictions (a), (b), and (d) that

$$\lim_{n\to\infty}\|Tx_n - x_n\| = 0.$$

Now, we are in a position to prove that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, J(x_n - \bar{x})\rangle \le 0$, where $\bar{x} = \lim_{t\to 0} x_t$, and x_t solves the fixed point equation

$$x_t = tf(x_t) + (1-t)Tx_t, \quad \forall t \in (0,1).$$

It follows that

$$\begin{aligned}
\|x_t - x_n\|^2 &= t\bigl\langle f(x_t) - x_n, J(x_t - x_n)\bigr\rangle + (1-t)\bigl\langle Tx_t - x_n, j(x_t - x_n)\bigr\rangle \\
&= t\bigl\langle f(x_t) - x_t, J(x_t - x_n)\bigr\rangle + t\bigl\langle x_t - x_n, J(x_t - x_n)\bigr\rangle \\
&\quad + (1-t)\bigl\langle Tx_t - Tx_n, J(x_t - x_n)\bigr\rangle + (1-t)\bigl\langle Tx_n - x_n, J(x_t - x_n)\bigr\rangle \\
&\le t\bigl\langle f(x_t) - x_t, J(x_t - x_n)\bigr\rangle + \|x_t - x_n\|^2 + \|Tx_n - x_n\|\|x_t - x_n\|, \quad \forall t \in (0,1).
\end{aligned}$$

This implies that

$$\bigl\langle x_t - f(x_t), J(x_t - x_n)\bigr\rangle \le \frac{1}{t}\|Tx_n - x_n\|\|x_t - x_n\|, \quad \forall t \in (0,1).$$

Since $\lim_{n\to\infty}\|Tx_n - x_n\| = 0$, we find that $\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0$. Since J is strong to weak* uniformly continuous on bounded subsets of E, we find that

$$\begin{aligned}
&\bigl|\bigl\langle f(\bar{x}) - \bar{x}, J(x_n - \bar{x})\bigr\rangle - \bigl\langle x_t - f(x_t), J(x_t - x_n)\bigr\rangle\bigr| \\
&\quad \le \bigl|\bigl\langle f(\bar{x}) - \bar{x}, J(x_n - \bar{x})\bigr\rangle - \bigl\langle f(\bar{x}) - \bar{x}, J(x_n - x_t)\bigr\rangle\bigr|
\end{aligned}$$

$$\begin{aligned}
&+ \left|\langle f(\bar{x}) - \bar{x}, J(x_n - x_t)\rangle - \langle x_t - f(x_t), J(x_t - x_n)\rangle\right| \\
&\le \left|\langle f(\bar{x}) - \bar{x}, J(x_n - \bar{x}) - J(x_n - x_t)\rangle\right| + \left|\langle f(\bar{x}) - \bar{x} + x_t - f(x_t), J(x_n - x_t)\rangle\right| \\
&\le \left\|f(x_t) - \bar{x}\right\| \left\|J(x_n - \bar{x}) - J(x_n - x_t)\right\| + (1+\alpha)\|\bar{x} - x_t\|\|x_n - x_t\|.
\end{aligned}$$

Since $x_t \to \bar{x}$, as $t \to 0$, we have

$$\lim_{t\to 0}\left|\langle f(\bar{x}) - \bar{x}, J(x_n - \bar{x})\rangle - \langle f(x_t) - x_t, J(x_n - x_t)\rangle\right| = 0.$$

For $\epsilon > 0$, there exists $\delta > 0$ such that $\forall t \in (0, \delta)$, we have

$$\langle f(\bar{x}) - \bar{x}, J(x_n - \bar{x})\rangle \le \langle f(x_t) - x_t, J(x_n - x_t)\rangle + \epsilon.$$

This implies that $\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x}, J(x_n - \bar{x})\rangle \le 0$.

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. Since $\|\cdot\|^2$ is convex, we see that

$$\begin{aligned}
\|y_n - \bar{x}\|^2 &= \left\|\sum_{i=1}^N \delta_{n,i} J_{r_i}(x_n + e_{n,i}) - \bar{x}\right\|^2 \\
&\le \sum_{i=1}^N \delta_{n,i}\left\|J_{r_i}(x_n + e_{n,i}) - \bar{x}\right\|^2 \\
&\le \|x_n - \bar{x}\|^2 + \sum_{i=1}^N \|e_{n,i}\|\left(\|e_{n,i}\| + 2\|x_n - \bar{x}\|\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &= \alpha_n\langle f(x_n) - \bar{x}, J(x_{n+1} - \bar{x})\rangle + \beta_n\langle x_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\quad + \gamma_n\langle y_n - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\le \alpha_n\alpha\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \alpha_n\langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\quad + \beta_n\|x_n - \bar{x}\|\|x_{n+1} - \bar{x}\| + \gamma_n\|y_n - \bar{x}\|\|x_{n+1} - \bar{x}\| \\
&\le \frac{\alpha_n\alpha}{2}\left(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2\right) + \alpha_n\langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\quad + \frac{\beta_n}{2}\left(\|x_n - \bar{x}\|^2 + \|x_{n+1} - \bar{x}\|^2\right) + \frac{\gamma_n}{2}\|x_n - \bar{x}\|^2 \\
&\quad + \sum_{i=1}^N \|e_{n,i}\|\left(\|e_{n,i}\| + 2\|x_n - \bar{x}\|\right) + \frac{\gamma_n}{2}\|x_{n+1} - \bar{x}\|^2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|x_{n+1} - \bar{x}\|^2 &\le \left(1 - \alpha_n(1-\alpha)\right)\|x_n - \bar{x}\|^2 + 2\alpha_n\langle f(\bar{x}) - \bar{x}, J(x_{n+1} - \bar{x})\rangle \\
&\quad + \sum_{i=1}^N \|e_{n,i}\|\left(\|e_{n,i}\| + 2\|x_n - \bar{x}\|\right).
\end{aligned}$$

Using Lemma 1.1, we find $x_n \to \bar{x}$ as $n \to \infty$. This completes the proof. □

Remark 2.2 There are many spaces satisfying the restriction in Theorem 2.1, for example L^p, where $p > 1$.

Corollary 2.3 *Let E be a Hilbert space and let $N \geq 1$ be some positive integer. Let A_m be a maximal monotone operator in E for each $m \in \{1, 2, \dots, N\}$. Assume that $C := \bigcap_{m=1}^{N} \overline{D(A_m)}$ is convex and has the normal structure. Let $f : C \to C$ be an α-contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0,1)$ with the restriction $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\delta_{n,i}\}$ be a real number sequence in $(0,1)$ with the restriction $\delta_{n,1} + \delta_{n,2} + \cdots + \delta_{n,N} = 1$. Let $\{r_m\}$ be a positive real numbers sequence and $\{e_{n,i}\}$ a sequence in E for each $i \in \{1, 2, \dots, N\}$. Assume that $\bigcap_{i=1}^{N} A_i^{-1}(0)$ is not empty. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \delta_{n,i} J_{r_i}(x_n + e_{n,i}), \quad \forall n \geq 1,$$

where $J_{r_i} = (I + r_i A_i)^{-1}$. Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, and $\{\delta_{n,i}\}$ satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\sum_{n=1}^{\infty} \|e_{n,m}\| < \infty$;
(d) $\lim_{n\to\infty} \delta_{n,i} = \delta_i \in (0,1)$.

Then the sequence $\{x_n\}$ converges strongly to $\bar{x}$, which is the unique solution to the following variational inequality: $\langle f(\bar{x}) - \bar{x}, p - \bar{x} \rangle \leq 0$, $\forall p \in \bigcap_{i=1}^{N} A_i^{-1}(0)$.

3 Applications

In this section, we consider a variational inequality problem. Let $A : C \to E^*$ be a single valued monotone operator which is hemicontinuous; that is, continuous along each line segment in C with respect to the weak* topology of E^*. Consider the following variational inequality:

$$\text{find } x \in C \text{ such that } \langle y - x, Ax \rangle \geq 0, \quad \forall y \in C.$$

The solution set of the variational inequality is denoted by $\mathrm{VI}(C, A)$. Recall that the normal cone $N_C(x)$ for C at a point $x \in C$ is defined by

$$N_C(x) = \{x^* \in E^* : \langle y - x, x^* \rangle \leq 0, \forall y \in C\}.$$

Now, we are in a position to give the convergence theorem.

Theorem 3.1 *Let E be a real reflexive, strictly convex Banach space with the uniformly Gâteaux differentiable norm. Let $N \geq 1$ be some positive integer and let C be nonempty closed and convex subset of E. Let $A_i : C \to E^*$ a single valued, monotone and hemicontinuous operator. Assume that $\bigcap_{i=1}^{N} \mathrm{VI}(C, A_i)$ is not empty and C has the normal structure. Let $f : C \to C$ be an α-contractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ be real number sequences in $(0,1)$ with the restriction $\alpha_n + \beta_n + \gamma_n = 1$. Let $\{\delta_{n,i}\}$ be a real number sequence in $(0,1)$ with the restriction $\delta_{n,1} + \delta_{n,2} + \cdots + \delta_{n,N} = 1$. Let $\{r_m\}$ be a positive real numbers sequence* *and* $\{e_{n,i}\}$ *a sequence in* E *for each* $i \in \{1,2,\dots,N\}$. *Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \sum_{i=1}^{N} \delta_{n,i} \operatorname{VI}\left(C, A_i + \frac{1}{r_i}(I - x_n)\right), \quad \forall n \geq 1.$$

Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, *and* $\{\delta_{n,i}\}$ *satisfy the following restrictions*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(c) $\sum_{n=1}^{\infty} \|e_{n,m}\| < \infty$;
(d) $\lim_{n\to\infty} \delta_{n,i} = \delta_i \in (0,1)$.

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x}$, *which is the unique solution to the following variational inequality*: $\langle f(\bar{x}) - \bar{x}, J(p - \bar{x})\rangle \leq 0$, $\forall p \in \bigcap_{i=1}^{N} \operatorname{VI}(C, A_i)$.

Proof Define a mapping $T_i \subset E \times E^*$ by

$$T_i x := \begin{cases} A_i x + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

From Rockafellar [36], we find that T_i is maximal monotone with $T_i^{-1}(0) = \operatorname{VI}(C, A_i)$. For each $r_i > 0$, and $x_n \in E$, we see that there exists a unique $x_{r_i} \in D(T_i)$ such that $x_n \in x_{r_i} + r_i T_i(x_{r_i})$, where $x_{r_i} = (I + r_i T_i)^{-1} x_n$. Notice that

$$y_{n,i} = \operatorname{VI}\left(C, A_i + \frac{1}{r_i}(I - x_n)\right),$$

which is equivalent to

$$\left\langle y - y_{n,i}, A_i y_{n,i} + \frac{1}{r_i}(y_{n,i} - x_n)\right\rangle \geq 0, \quad \forall y \in C,$$

that is, $-A_i y_{n,i} + \frac{1}{r_i}(x_n - y_{n,i}) \in N_C(y_{n,i})$. This implies that $y_{n,i} = (I + r_i T_i)^{-1} x_n$. Using Theorem 2.1, we find the desired conclusion immediately. □

From Theorem 3.1, the following result is not hard to derive.

Corollary 3.2 *Let* E *be a real reflexive, strictly convex Banach space with the uniformly Gâteaux differentiable norm. Let* C *be nonempty closed and convex subset of* E. *Let* $A : C \to E^*$ *a single valued, monotone and hemicontinuous operator with* $\operatorname{VI}(C, A)$. *Assume that* C *has the normal structure. Let* $f : C \to C$ *be an* α*-contractive mapping. Let* $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *be real number sequences in* $(0,1)$ *with the restriction* $\alpha_n + \beta_n + \gamma_n = 1$. *Let* $\{x_n\}$ *be a sequence generated in the following manner*:

$$x_1 \in C, \quad x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \operatorname{VI}\left(C, A + \frac{1}{r}(I - x_n)\right), \quad \forall n \geq 1.$$

Assume that the control sequences $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *satisfy the following restrictions*:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(b) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x}$, *which is the unique solution to the following variational inequality*: $\langle f(\bar{x}) - \bar{x}, J(p - \bar{x})\rangle \leq 0$, $\forall p \in \mathrm{VI}(C, A_i)$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to this manuscript. All authors read and approved the final manuscript.

Author details

[1]School of Business and Administration, Henan University, Kaifeng, Henan, China. [2]School of Mathematics and Information Science, Shangqiu Normal University, Shangqiu, Henan, China. [3]Vietnam National University, Hanoi, Vietnam.

Acknowledgements

The authors are grateful to the editor and the reviewers for useful suggestions which improved the contents of the article.

Received: 16 January 2014 Accepted: 30 April 2014 Published: 14 May 2014

References

1. Bruck, RE: Nonexpansive projections on subsets of Banach spaces. Pac. J. Math. **47**, 341-355 (1973)
2. Reich, S: Asymptotic behavior of contractions in Banach spaces. J. Math. Anal. Appl. **44**, 57-70 (1973)
3. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Dekker, New York (1984)
4. Kopecká, EE, Reich, S: Nonexpansive retracts in Banach spaces. Banach Cent. Publ. **77**, 161-174 (2007)
5. Bauschke, HH, Matousková, E, Reich, S: Projection and proximal point methods: convergence results and counterexamples. Nonlinear Anal. **56**, 715-738 (2004)
6. Reich, S: Weak convergence theorems for nonexpansive mappings in Banach spaces. J. Math. Anal. Appl. **67**, 274-276 (1979)
7. Genel, A, Lindenstruss, J: An example concerning fixed points. Isr. J. Math. **22**, 81-86 (1975)
8. Cho, SY, Li, W, Kang, SM: Convergence analysis of an iterative algorithm for monotone operators. J. Inequal. Appl. **2013**, Article ID 199 (2013)
9. Cho, SY, Qin, X: On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems. Appl. Math. Comput. **235**, 430-438 (2014)
10. Yang, S, Zhang, MY: Strong convergence theorems for fixed points of generalized asymptotically quasi-ϕ-nonexpansive mappings. Adv. Fixed Point Theory **4**, 69-90 (2014)
11. Cho, SY, Qin, X, Kang, SM: Hybrid projection algorithms for treating common fixed points of a family of demicontinuous pseudocontractions. Appl. Math. Lett. **25**, 854-857 (2012)
12. Zhou, H: Convergence theorems of common fixed points for a family of Lipschitz quasi-pseudocontractions. Nonlinear Anal. **71**, 685-690 (2009)
13. Qin, X, Agarwal, RP: Shrinking projection methods for a pair of asymptotically quasi-ϕ-nonexpansive mappings. Numer. Funct. Anal. Optim. **31**, 1072-1089 (2010)
14. Chen, JH: Iterations for equilibrium and fixed point problems. J. Nonlinear Funct. Anal. **2013**, Article ID 4 (2013)
15. Hao, Y: Some results on a modified Mann iterative scheme in a reflexive Banach space. Fixed Point Theory Appl. **2013**, Article ID 227 (2013)
16. Kim, JK: Strong convergence theorems by hybrid projection methods for equilibrium problems and fixed point problems of the asymptotically quasi-ϕ-nonexpansive mappings. Fixed Point Theory Appl. **2011**, Article ID 10 (2011)
17. Qin, X, Cho, SY, Kang, SM: Strong convergence of shrinking projection methods for quasi-ϕ-nonexpansive mappings and equilibrium problems. J. Comput. Appl. Math. **234**, 750-760 (2010)
18. Wu, C, Sun, L: A monotone projection algorithm for fixed points of nonlinear operators. Fixed Point Theory Appl. **2013**, Article ID 318 (2013)
19. Wu, C, Lv, S: Bregman projection methods for zeros of monotone operators. J. Fixed Point Theory **2013**, Article ID 7 (2013)
20. Qin, X, Su, Y: Strong convergence theorems for relatively nonexpansive mappings in a Banach space. Nonlinear Anal. **67**, 1958-1965 (2007)
21. Ye, J, Huang, J: Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces. J. Math. Comput. Sci. **1**, 1-18 (2011)
22. Qin, X, Cho, YJ, Kang, SM: Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. J. Comput. Appl. Math. **225**, 20-30 (2009)
23. Halpern, B: Fixed points of nonexpanding maps. Bull. Am. Math. Soc. **73**, 957-961 (1967)
24. Qin, X, Cho, SY, Wang, L: A regularization method for treating zero points of the sum of two monotone operators. Fixed Point Theory Appl. **2014**, Article ID 75 (2014)
25. Takahashi, W: Viscosity approximation methods for resolvents of accretive operators in Banach spaces. J. Fixed Point Theory Appl. **1**, 135-147 (2007)
26. Song, J, Chen, M: A modified Mann iteration for zero points of accretive operators. Fixed Point Theory Appl. **2013**, Article ID 347 (2013)
27. Qin, X, Su, Y: Approximation of a zero point of accretive operator in Banach spaces. J. Math. Anal. Appl. **329**, 415-424 (2007)

28. Yang, S: Zero theorems of accretive operators in reflexive Banach spaces. J. Nonlinear Funct. Anal. **2013**, Article ID 2 (2013)
29. Qing, Y, Cho, SY: A regularization algorithm for zero points of accretive operators. Fixed Point Theory Appl. **2013**, Article ID 341 (2013)
30. Qing, Y, Cho, SY: Proximal point algorithms for zero points of nonlinear operators. Fixed Point Theory Appl. **2014**, Article ID 42 (2014)
31. Reich, S: Strong convergence theorems for resolvents of accretive operators in Banach spaces. J. Math. Anal. Appl. **75**, 287-292 (1980)
32. Liu, LS: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces. J. Math. Anal. Appl. **194**, 114-125 (1995)
33. Bruck, RE: Properties of fixed-point sets of nonexpansive mappings in Banach spaces. Trans. Am. Math. Soc. **179**, 251-262 (1973)
34. Chang, SS: Some problems and results in the study of nonlinear analysis. Nonlinear Anal. **30**, 4197-4208 (1997)
35. Qin, X, Cho, SY, Wang, L: Iterative algorithms with errors for zero points of *m*-accretive operators. Fixed Point Theory Appl. **2013**, Article ID 148 (2013)
36. Rockafellar, RT: On the maximality of sums of nonlinear monotone operators. Trans. Am. Math. Soc. **149**, 75-88 (1970)

10.1186/1687-1812-2014-118
Cite this article as: Wu et al.: Some results on zero points of *m*-accretive operators in reflexive Banach spaces. *Fixed Point Theory and Applications* 2014, 2014:118

Submit your manuscript to a SpringerOpen journal and benefit from:

- Convenient online submission
- Rigorous peer review
- Immediate publication on acceptance
- Open access: articles freely available online
- High visibility within the field
- Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com