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Some new fixed point results in partial ordered metric spaces via admissible mappings

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Abstract

The purpose of this paper is to discuss the existence of fixed points for new classes of mappings defined on an ordered metric space. The obtained results generalize and improve some fixed point results in the literature. Some examples show the usefulness of our results.

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1 Introduction and preliminaries

Over the last decades, the fixed point theory has become increasingly useful in the study of nonlinear phenomena. In fact, the fixed point theorems and techniques have been developed in pure and applied analysis, topology and geometry. It is well known that a fundamental result of this theory is Banach's contraction principle [1]. Consequently, in the last 50 years, it has been extensively studied and generalized to many settings; see for example [2–14].

In 2008, Dutta and Choudhury proved the following theorem.

Theorem 1.1 (See [15]) *Let (X,d) be a complete metric space and $f:X\to X$ be such that*

$$\psi\bigl(d(fx,fy)\bigr)\le\psi\bigl(d(x,y)\bigr)-\phi\bigl(d(x,y)\bigr),\quad\forall x,y\in X,$$

where $\psi,\phi:[0,+\infty[\,\to[0,+\infty[$ are continuous, non-decreasing, and $\psi(t)=\phi(t)=0$ if and only if $t=0$. Then f has a unique fixed point $x^*\in X$.

Note that the above theorem remains true if the hypothesis on ϕ is replaced by ϕ is lower semi-continuous and $\phi(t)=0$ if and only if $t=0$ (see *e.g.* [16, 17]).

Eslamian and Abkar stated the following theorem as a generalization of Theorem 1.1.

Theorem 1.2 *Let (X,d) be a complete metric space and $f:X\to X$ be such that*

$$\psi\bigl(d(fx,fy)\bigr)\le\alpha\bigl(d(x,y)\bigr)-\beta\bigl(d(x,y)\bigr),\quad\forall x,y\in X,$$

where $\psi, \alpha, \beta : [0, +\infty) \to [0, +\infty)$ are such that ψ is continuous and non-decreasing, α is continuous and β is lower semi-continuous,

$$\psi(t) = 0 \quad \text{if and only if} \quad t = 0, \qquad \alpha(0) = \beta(0) = 0 \quad \text{and}$$
$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for all } t > 0.$$

Then f has a unique fixed point $x^* \in X$.

Aydi *et al.* [18] proved that Theorem 1.2 is a consequence of Theorem 1.1

On the other hand, Ran and Reurings [19] initiate the fixed point theory in the metric spaces equipped with a partial order relation. Let X be a nonempty set equipped with a partial order relation $\preceq$ such that the function $d : X \times X \to [0, \infty)$ is a metric on X, then the triple $(X, d, \preceq)$ is called a partially ordered metric space. Two elements $x, y \in X$ are comparable if either $x \preceq y$ or $y \preceq x$. We write $x \prec y$ if $x \preceq y$ but $x \neq y$. A sequence $\{x_n\}$ in X is said to be non-decreasing with respect to $\preceq$ if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. A mapping $f : X \to X$ is said to be non-decreasing with respect to $\preceq$ if $x \preceq y$ implies $fx \preceq fy$. In further discussion, if there is no confusion, for the mappings on X and sequences in X, we use the phrase 'non-decreasing' instead 'non-decreasing with respect to $\preceq$'.

Harjani and Sadarangani [20] extended Theorem 1.1 in the framework of partially ordered metric spaces in the following way.

Theorem 1.3 *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $f : X \to X$ be a continuous non-decreasing mapping such that*

$$\psi\big(d(fx, fy)\big) \le \psi\big(d(x, y)\big) - \phi\big(d(x, y)\big), \quad \forall x \preceq y,$$

where $\psi, \phi : [0, +\infty[\to [0, +\infty[$ are continuous and non-decreasing and $\psi(t) = \phi(t) = 0$ if and only if $t = 0$. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point $x^* \in X$.

Choudhury and Kundu [21] generalized Theorems 1.2 and 1.3 as follows.

Theorem 1.4 *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let $f : X \to X$ be a non-decreasing mapping such that*

$$\psi\big(d(fx, fy)\big) \le \alpha\big(d(x, y)\big) - \beta\big(d(x, y)\big), \quad \forall x \preceq y,$$

where $\psi, \alpha, \beta : [0, +\infty[\to [0, +\infty[$ are such that ψ is continuous and non-decreasing, α is continuous, β is lower semi-continuous,

$$\psi(t) = 0 \quad \text{if and only if} \quad t = 0, \qquad \alpha(0) = \beta(0) = 0 \quad \text{and}$$
$$\psi(t) - \alpha(t) + \beta(t) > 0 \quad \text{for all } t > 0.$$

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a unique fixed point $x^* \in X$.

Aydi *et al.* [18] proved that Theorem 1.4 is a consequence of Theorem 1.3.

Karapinar and Salimi [22] proved the following theorem as a generalization of Theorems 1.2 and 1.3 where the approach of Aydi *et al.* [18] cannot be modified for it.

Theorem 1.5 *Let $(X, d, \preceq)$ be an ordered metric space such that (X, d) is complete and let $f : X \to X$ be a non-decreasing self mappings. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$, and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - \alpha(s) + \beta(s) > 0 \quad \textit{for all } t > 0 \textit{ and } s = t \textit{ or } s = 0$$

and

$$\psi\big(d(fx, fy)\big) \leq \alpha\big(d(x, y)\big) - \beta\big(d(x, y)\big)$$

for all comparable $x, y \in X$ where

$$\Psi = \big\{\psi : [0, \infty) \to [0, \infty) \textit{ such that } \psi \textit{ is non-decreasing and lower semicontinuous}\big\},$$
$$\Phi_\alpha = \big\{\alpha : [0, \infty) \to [0, \infty) \textit{ such that } \alpha \textit{ is upper semicontinuous}\big\}$$

and

$$\Phi_\beta = \big\{\beta : [0, \infty) \to [0, \infty) \textit{ such that } \beta \textit{ is lower semicontinuous}\big\}.$$

Suppose that either

(a) *f is continuous, or*

(b) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

On the other hand, in 2012, Samet *et al.* [23] introduced the concepts of α-ψ-contractive and α-admissible mappings and established various fixed point theorems for such mappings in complete metric spaces. More recently, Salimi *et al.* [24] modified the notions of α-ψ-contractive and α-admissible mappings and established fixed point theorems which are proper generalizations of the recent results in [22, 23]. For more on α-admissible mappings, see [25–27] and the references therein.

Samet *et al.* [23] defined the notion of α-admissible mappings as follows.

Definition 1.6 Let T be a self-mapping on X and $\alpha : X \times X \to [0, +\infty)$ be a function. We say that T is an α-admissible mapping if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq 1.$$

In [23] the authors consider the family Ψ of non-decreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$, where ψ^n is the nth iterate of ψ and give the following theorem.

Theorem 1.7 *Let (X, d) be a complete metric space and T be an α-admissible mapping. Assume that*

$$\alpha(x, y)d(Tx, Ty) \leq \psi\big(d(x, y)\big) \tag{1.1}$$

for all $x, y \in X$, where $\psi \in \Psi$. Also, suppose that the following assertions hold:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0, Tx_0) \geq 1$,
(ii) *either* T *is continuous or for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $x_n \to x$ *as* $n \to +\infty$, *we have* $\alpha(x_n, x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then T *has a fixed point.*

Recently, Hussain *et al.* [28] obtained the following Geraghty type [29] fixed point theorems via α-admissible mappings.

Theorem 1.8 *Let* (X,d) *be a complete metric space and* $f : X \to X$ *be an* α*-admissible mapping. Assume that there exists a function* $\beta : [0,\infty) \to [0,1]$ *such that for any bounded sequence* $\{t_n\}$ *of positive reals,* $\beta(t_n) \to 1$ *implies* $t_n \to 0$ *and*

$$\big(d(fx,fy) + \ell\big)^{\alpha(x,fx)\alpha(y,fy)} \leq \beta\big(d(x,y)\big)d(x,y) + \ell$$

for all $x, y \in X$ *where* $\ell \geq 1$. *Suppose that either*

(a) f *is continuous, or*
(b) *if* $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n, *then* $\alpha(x, fx) \geq 1$.

If there exists $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$, *then* f *has a fixed point.*

Theorem 1.9 *Let* (X,d) *be a complete metric space and* $f : X \to X$ *be an* α*-admissible mapping. Assume that there exists a function* $\beta : [0,\infty) \to [0,1]$ *such that for any bounded sequence* $\{t_n\}$ *of positive reals,* $\beta(t_n) \to 1$ *implies* $t_n \to 0$ *and*

$$\big(\alpha(x,fx)\alpha(y,fy) + \ell\big)^{d(fx,fy)} \leq 2^{\beta(d(x,y))d(x,y)}$$

for all $x, y \in X$ *where* $0 < \ell \leq 1$. *Suppose that either*

(a) f *is continuous, or*
(b) *if* $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$ *and* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n, *then* $\alpha(x, fx) \geq 1$.

If there exists $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$, *then* f *has a fixed point.*

Theorem 1.10 *Let* (X,d) *be a metric space such that* (X,d) *is complete and* $f : X \to X$ *be an* α*-admissible mapping. Assume that there exists a function* $\beta : [0,\infty) \to [0,1]$ *such that for any bounded sequence* $\{t_n\}$ *of positive reals,* $\beta(t_n) \to 1$ *implies* $t_n \to 0$ *and*

$$\alpha(x,fx)\alpha(y,fy)d(fx,fy) \leq \beta\big(d(x,y)\big)d(x,y)$$

for all $x, y \in X$. *Suppose that either*

(a) f *is continuous, or*
(b) *if* $\{x_n\}$ *is a sequence in* X *such that* $x_n \to x$ *and* $\alpha(x_n, fx_n) \geq 1$ *for all* n, *then* $\alpha(x, fx) \geq 1$.

If there exists $x_0 \in X$ *such that* $\alpha(x_0, fx_0) \geq 1$, *then* f *has a fixed point.*

For more details on α-admissible mappings and related fixed point results we refer the reader to [30–32].

More recently, Salimi *et al.* [24] modified and generalized the notions of α-ψ-contractive mappings and α-admissible mappings by the following ways.

Definition 1.11 [24] Let T be a self-mapping on X and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions. We say that T is an α-admissible mapping with respect to η if

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad \alpha(Tx, Ty) \geq \eta(Tx, Ty).$$

Note that if we take $\eta(x, y) = 1$ then this definition reduces to Definition 1.6. Also, if we take, $\alpha(x, y) = 1$ then we say that T is an η-subadmissible mapping.

The following result properly contains Theorem 1.7, and Theorems 2.3 and 2.4 of [22].

Theorem 1.12 [24] *Let (X, d) be a complete metric space and T be an α-admissible mapping with respect to η. Assume that*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \quad \Longrightarrow \quad d(Tx, Ty) \leq \psi\big(M(x, y)\big), \tag{1.2}$$

where $\psi \in \Psi$ and

$$M(x, y) = \max\left\{d(x, y), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$,*

(ii) *either T is continuous or for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, we have $\alpha(x_n, x) \geq \eta(x_n, x)$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then T has a fixed point.

For more details on modified α-ψ-contractive mappings and related fixed point results we refer the reader to [33, 34].

2 Main results

In this section, motivated by the work of Hussain *et al.* [28] and Salimi *et al.* [24] we state and prove the following fixed point results in the setting of partially ordered metric spaces.

Theorem 2.1 *Let $(X, d, \preceq)$ be a partially ordered metric space such that (X, d) is complete. Assume $f : X \to X$ and $\gamma : X \times X \to [0, \infty)$ be two mappings such that f is a non-decreasing and γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$, and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - \alpha(s) + \beta(s) > 0 \quad \textit{for all } t > 0 \textit{ and } s = t \textit{ or } s = 0 \tag{2.1}$$

and

$$\begin{aligned} &\gamma(x, fx)\gamma(y, fy) \geq 1 \\ &\quad \Longrightarrow \quad \big(\psi\big(d(fx, fy)\big) + \ell\big)^{\gamma(x,x)\gamma(y,y)} \leq \alpha\big(d(x, y)\big) - \beta\big(d(x, y)\big) + \ell \end{aligned} \tag{2.2}$$

for all comparable $x, y \in X$ where $\ell \geq 1$. Suppose that either

(i) *f is continuous, or*

(ii) *if a non-decreasing sequence* $\{x_n\}$ *is such that* $x_n \to x$ *as* $n \to \infty$, $\gamma(x_n, fx_n) \geq 1$, *and* $\gamma(x_n, x_n) \geq 1$ *for all* n, *then* $\gamma(x, x) \geq 1$, $\gamma(x, fx) \geq 1$, *and* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

If there exists $x_0 \in X$ *such that* $\gamma(x_0, x_0) \geq 1$, $\gamma(x_0, fx_0) \geq 1$, *and* $x_0 \preceq fx_0$, *then* f *has a fixed point.*

Proof Let $x_0 \preceq fx_0$. We define an iterative sequence $\{x_n\}$ in the following way:

$$x_n = f^n x_0 = fx_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Since f is non-decreasing and $x_0 \preceq fx_0$, we have

$$x_0 \preceq x_1 \preceq x_2 \preceq \cdots, \tag{2.3}$$

and hence $\{x_n\}$ is a non-decreasing sequence. Let $\gamma(x_0, x_0) \geq 1$. Since f is a γ-admissible mapping and $\gamma(x_0, x_0) \geq 1$, we deduce that $\gamma(x_1, x_1) = \gamma(fx_0, fx_0) \geq 1$. By continuing this process, we get $\gamma(x_n, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Also, assume $\gamma(x_0, fx_0) \geq 1$. Similarly we get $\gamma(x_n, fx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. If $x_{n_0} = x_{n_0+1} = fx_{n_0}$ for some $n_0 \in \mathbb{N}$, then the point x_0 is the desired fixed point of f which completes the proof. Hence, we suppose that $x_n \neq x_{n+1}$, that is, $d(x_{n-1}, x_n) > 0$ for all n. Hence, (2.3) implies

$$x_0 \prec x_1 \prec x_2 \prec \cdots. \tag{2.4}$$

We want to show that the sequence $\{d_n := d(x_n, x_{n+1})\}$ is non-increasing sequence of reals. Suppose, to the contrary, that there exists some $n_0 \in \mathbb{N}$ such that

$$d(x_{n_0-1}, x_{n_0}) \leq d(x_{n_0}, x_{n_0+1}). \tag{2.5}$$

Since ψ is non-decreasing, we obtain

$$\psi\big(d(x_{n_0-1}, x_{n_0})\big) \leq \psi\big(d(x_{n_0}, x_{n_0+1})\big). \tag{2.6}$$

Taking $x = x_{n-1}$ and $y = x_n$ in (2.2) we derive

$$\begin{aligned}
\psi\big(d(x_n, x_{n+1})\big) + \ell &= \psi\big(d(fx_{n-1}, fx_n)\big) + \ell \\
&\leq \big(\psi\big(d(fx_{n-1}, fx_n)\big) + \ell\big)^{\gamma(x_{n-1}, x_{n-1})\gamma(x_n, x_n)} \\
&\leq \alpha\big(d(x_{n-1}, x_n)\big) - \beta\big(d(x_{n-1}, x_n)\big) + \ell.
\end{aligned}$$

Hence

$$\psi\big(d(x_n, x_{n+1})\big) \leq \alpha\big(d(x_{n-1}, x_n)\big) - \beta\big(d(x_{n-1}, x_n)\big) \tag{2.7}$$

for all $n \in \mathbb{N}$. Now, by taking $x = x_{n_0-1}$ and $y = x_{n_0}$ in (2.7) and applying (2.6) we have

$$\psi\big(d(x_{n_0-1}, x_{n_0})\big) \leq \alpha\big(d(x_{n_0-1}, x_{n_0})\big) - \beta\big(d(x_{n_0-1}, x_{n_0})\big),$$

which contradicts (2.12). Therefore, we conclude that $d_n < d_{n-1}$ holds for all $n \in \mathbb{N}$. Hence $\{d_n\}$ is a non-increasing sequence of positive real numbers. Thus, there exists $r \geq 0$ such that $\lim_{n\to\infty} d_n = r$. We shall show that $r = 0$ by method of *reductio ad absurdum*. For this purpose, we assume that $r > 0$. By (2.7) together with the properties of α, β, ψ we have

$$\begin{aligned}\psi(r) &\leq \liminf_{n\to\infty} \psi(d_n) \leq \limsup_{n\to\infty} \psi(d_n)\\ &\leq \limsup_{n\to\infty}\bigl[\alpha(d_{n-1}) - \beta(d_{n-1})\bigr] \leq \alpha(r) - \beta(r),\end{aligned}$$

which is a contradiction. Hence

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.8}$$

We shall show that the sequence $\{x_n\}$ is a Cauchy sequence. Suppose that it is not. Then there are $\varepsilon > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k with $n(k) > m(k) > k$

$$d(x_{n(k)}, x_{m(k)}) \geq \varepsilon. \tag{2.9}$$

Additionally, corresponding to $m(k)$, we may choose $n(k)$ such that it is the smallest integer satisfying (2.9) and $n(k) > m(k) \geq k$. Thus,

$$d(x_{n(k)}, x_{m(k)-1}) < \varepsilon.$$

Now, for all $k \in \mathbb{N}$ we have

$$\varepsilon \leq d(x_{n(k)}, x_{m(k)}) \leq d(x_{n(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{m(k)}) < \varepsilon + d_{m(k)-1}.$$

So

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon. \tag{2.10}$$

Again, we have

$$d(x_{n(k)}, x_{m(k)}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)+1}, x_{n(k)+1}) + d(x_{n(k)+1}, x_{n(k)})$$

and

$$d(x_{n(k)+1}, x_{m(k)+1}) \leq d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) + d(x_{n(k)+1}, x_{n(k)}).$$

By taking the limit as $k \to +\infty$ in the above inequalities and applying (2.8) and (2.10), we deduce

$$\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{2.11}$$

Now, from (2.2) with $x = x_{m(k)}$ and $y = x_{n(k)}$ we have

$$\begin{aligned}&\psi\bigl(d(x_{m(k)+1}, fx_{n(k)+1})\bigr) + \ell\\ &\quad\leq \bigl(\psi\bigl(d(x_{m(k)+1}, fx_{n(k)+1})\bigr) + \ell\bigr)^{\gamma(x_{m(k)}, x_{m(k)})\gamma(x_{n(k)}, x_{n(k)})}\end{aligned}$$

$$\begin{aligned}&= \bigl(\psi\bigl(d(fx_{m(k)},fx_{n(k)})\bigr)+\ell\bigr)^{\gamma(x_{m(k)},x_{m(k)})\gamma(x_{n(k)},x_{n(k)})}\\&\le \alpha\bigl(d(x_{m(k)},x_{n(k)})\bigr)-\beta\bigl(d(x_{m(k)},x_{n(k)})\bigr)+\ell.\end{aligned}$$

Then

$$\psi\bigl(d(x_{m(k)+1},fx_{n(k)+1})\bigr)\le \alpha\bigl(d(x_{m(k)},x_{n(k)})\bigr)-\beta\bigl(d(x_{m(k)},x_{n(k)})\bigr).$$

Taking the lim inf as $k\to+\infty$ in the above inequality, we have

$$\begin{aligned}\psi(\varepsilon)&\le \liminf \psi\bigl(d(x_{n(k)+1},x_{m(k)+1})\bigr)\le \limsup \psi\bigl(d(x_{n(k)+1},x_{m(k)+1})\bigr)\\&\le \limsup\bigl(\alpha\bigl(d(x_{n(k)},x_{m(k)})\bigr)-\beta\bigl(d(x_{n(k)},x_{m(k)})\bigr)\bigr)\\&= \limsup \alpha\bigl(d(x_{n(k)},x_{m(k)})\bigr)-\liminf \beta\bigl(d(x_{n(k)},x_{m(k)})\bigr)\\&\le \alpha(\varepsilon)-\beta(\varepsilon).\end{aligned}$$

So we have

$$\psi(\varepsilon)\le \alpha(\varepsilon)-\beta(\varepsilon),$$

which contradicts the fact that $\psi(t)-\alpha(t)+\beta(t)>0$ for all $t>0$. Hence

$$\lim_{n\to\infty} d(x_n,x_m)=0,$$

that is, the sequence $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete, then there exists $x^*\in X$ such that $x_n\to x^*$ as $n\to\infty$. Suppose that (i) holds. Then

$$x^*=\lim_{n\to\infty} x_{n+1}=f\Bigl(\lim_{n\to\infty} x_n\Bigr)=f\bigl(x^*\bigr).$$

Hence, x^* is a fixed point of f. Suppose that (ii) holds, that is, $\gamma(x^*,x^*)\ge 1$, $\gamma(x^*,fx^*)\ge 1$, and $x_n\preceq x^*$ for all $n\ge 0$. We claim that x^* is a fixed point of f, that is, $\lim_{n\to\infty} d(x_{n+1},fx^*)=0$. Suppose, to the contrary, that $\lim_{n\to\infty} d(x_{n+1},fx^*)=d(x^*,fx^*)>0$. Due to condition (2.2), we have

$$\begin{aligned}\psi\bigl(d\bigl(fx^*,x_{n+1}\bigr)\bigr)+\ell&=\psi\bigl(d\bigl(fx^*,fx_n\bigr)\bigr)+\ell\\&\le \bigl(\psi\bigl(d\bigl(fx^*,fx_n\bigr)\bigr)+\ell\bigr)^{\gamma(x^*,x^*)\gamma(x_n,x_n)}\\&\le \alpha\bigl(d\bigl(x^*,x_n\bigr)\bigr)-\beta\bigl(d\bigl(x^*,x_n\bigr)\bigr)+\ell.\end{aligned}$$

Taking the lim inf as $n\to\infty$ in the above inequality, we obtain

$$\begin{aligned}\psi\bigl(d\bigl(x^*,fx^*\bigr)\bigr)&\le \liminf_{n\to\infty}\psi\bigl(d\bigl(x_{n+1},fx^*\bigr)\bigr)\\&=\liminf_{n\to\infty}\psi\bigl(d\bigl(fx_n,fx^*\bigr)\bigr)\le \limsup_{n\to\infty}\psi\bigl(d\bigl(x_n,fx^*\bigr)\bigr)\\&\le \limsup_{n\to\infty}\bigl(\alpha\bigl(d\bigl(x_n,x^*\bigr)\bigr)-\beta\bigl(d\bigl(x_n,x^*\bigr)\bigr)\bigr)\\&\le \alpha(0)-\beta(0),\end{aligned}$$

which is a contradiction. Hence $\lim_{n\to\infty} d(x_{n+1},fx^*)=d(x^*,fx^*)=0$ and so, $x^*=fx^*$. □

Example 2.2 Let $X = [0,\infty)$ be endowed with the usual metric $d(x,y) = |x - y|$ for all $x, y \in X$ and $f : X \to X$ be defined by

$$fx = \begin{cases} \frac{x}{2(x+1)} & \text{if } x \in [0,1], \\ 3x^2 & \text{if } x \in (1,\infty). \end{cases}$$

Define also $\gamma : X \times X \to [0,+\infty)$, $\psi : [0,+\infty) \to [0,+\infty)$, $\alpha : [0,+\infty) \to [0,+\infty)$, and $\beta : [0,+\infty) \to [0,+\infty)$ by

$$\gamma(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ 0 & \text{otherwise}, \end{cases} \qquad \psi(t) = t + 1/2,$$

$$\alpha(t) = t + 1 \quad \text{and} \quad \beta(t) = t/2 + 1/2.$$

We prove that Theorem 2.1 can be applied to f. But Theorem 1.5 cannot be applied.

Clearly, (X,d) is a complete metric space. We show that f is a γ-admissible mapping. Let $x, y \in X$. If $\gamma(x,y) \geq 1$ then $x, y \in [0,1]$. On the other hand, for all $x \in [0,1]$ we have $fx \leq 1$. It follows that $\gamma(fx,fy) \geq 1$. Thus the assertion holds. Because of the above arguments, $\gamma(0,0) \geq 1$. Now, if $\{x_n\}$ is a sequence in X such that $\gamma(x_n,x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\{x_n\} \subset [0,1]$ and hence $x \in [0,1]$. This implies that $\gamma(x,x) \geq 1$. Also $\psi(t) = t + 1/2 > t/2 + 1/2 = \alpha(t) - \beta(t)$ and $\psi(t) = t + 1/2 > 1/2 = \alpha(0) - \beta(0)$ for all $t > 0$. Let $\gamma(x,fx)\gamma(y,fy) \geq 1$. Then $x, y \in [0,1]$. Indeed, if $x \notin [0,1]$ or $y \notin [0,1]$. So, $\gamma(x,fx) = 0$ or $\gamma(y,fy)$. That is, $\gamma(x,fx)\gamma(y,fy) = 0 < 1$ which is a contradiction. Without any loss of generality we assume that $y \geq x$. We get

$$\begin{aligned} \bigl(\psi\bigl(d(fx,fy)\bigr) + \ell\bigr)^{\gamma(x,x)\gamma(y,y)} &= fy - fx + 1/2 + \ell \\ &= \frac{y}{2(y+1)} - \frac{x}{2(x+1)} + 1/2 + \ell \\ &= \frac{y-x}{2(1+x)(1+y)} + 1/2 + \ell \\ &\leq \frac{y-x}{2} + 1/2 + \ell \\ &= \alpha\bigl(d(x,y)\bigr) - \beta\bigl(d(x,y)\bigr) + \ell. \end{aligned}$$

Then the condition of Theorem 2.1 holds and f has a fixed point. Let $x = 2$ and $y = 3$, then

$$\psi\bigl(d(f2,f3)\bigr) = 15 + 1/2 > 1 = \alpha\bigl(d(2,3)\bigr) - \beta\bigl(d(2,3)\bigr).$$

That is, the contractive condition of Theorem 1.5 does not hold for this example.

Corollary 2.3 *Let $(X,d,\preceq)$ be a partially ordered metric space such that (X,d) is complete. Assume $f : X \to X$ and $\gamma : X \times X \to [0,\infty)$ be two mappings such that f is a non-decreasing γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$, and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - \alpha(s) + \beta(s) > 0 \quad \textit{for all } t > 0 \textit{ and } s = t \textit{ or } s = 0. \tag{2.12}$$

Suppose that either

(i) *f is continuous, or*

(ii) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$, $\gamma(x_n, fx_n) \geq 1$, and $\gamma(x_n, x_n) \geq 1$ for all n, then $\gamma(x,x) \geq 1$, $\gamma(x,fx) \geq 1$, and $x_n \preceq x$ for all $n \in \mathbb{N}$,*

(iii)

$$\gamma(x,fx)\gamma(y,fy)\big(\psi\big(d(fx,fy)\big) + \ell\big)^{\gamma(x,x)\gamma(y,y)} \leq \alpha\big(d(x,y)\big) - \beta\big(d(x,y)\big) + \ell$$

for all comparable $x, y \in X$ where $\ell \geq 1$.

If there exists $x_0 \in X$ such that $\gamma(x_0, x_0) \geq 1$, $\gamma(x_0, fx_0) \geq 1$, and $x_0 \preceq fx_0$, then f has a fixed point.

Proof Let $\gamma(x,fx)\gamma(y,fy) \geq 1$. Then from (iii) we have

$$\begin{aligned}\big(\psi\big(d(fx,fy)\big) + \ell\big)^{\gamma(x,x)\gamma(y,y)} &\leq \gamma(x,fx)\gamma(y,fy)\big(\psi\big(d(fx,fy)\big) + \ell\big)^{\gamma(x,x)\gamma(y,y)} \\ &\leq \alpha\big(d(x,y)\big) - \beta\big(d(x,y)\big) + \ell.\end{aligned}$$

That is,

$$\begin{aligned}&\gamma(x,fx)\gamma(y,fy) \geq 1 \\ &\quad \Longrightarrow \quad \big(\psi\big(d(fx,fy)\big) + \ell\big)^{\gamma(x,x)\gamma(y,y)} \leq \alpha\big(d(x,y)\big) - \beta\big(d(x,y)\big) + \ell.\end{aligned}$$

Hence, all conditions of Theorem 2.1 hold and f has a fixed point. □

Now, we prove our second main result as follows.

Theorem 2.4 *Let $(X, d, \preceq)$ be a partially ordered metric space such that (X, d) is complete. Assume $f : X \to X$ and $\gamma : X \times X \to [0, \infty)$ are two mappings such that f is a non-decreasing γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$, and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - \alpha(s) + \beta(s) > 0 \quad \text{for all } t > 0 \text{ and } s = t \text{ or } s = 0$$

and

$$\begin{aligned}&\gamma(x,fx)\gamma(y,fy) \geq 1 \\ &\quad \Longrightarrow \big(\gamma(x,x)\gamma(y,y) + 1\big)^{\psi(d(fx,fy))} \leq 2^{\alpha(d(x,y)) - \beta(d(x,y))}\end{aligned} \tag{2.13}$$

for all comparable $x, y \in X$. Suppose that either

(i) *f is continuous, or*

(ii) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$, $\gamma(x_n, x_n) \geq 1$, and $\gamma(x_n, fx_n) \geq 1$ for all n, then $\gamma(x,x) \geq 1$, $\gamma(x,fx) \geq 1$, and $x_n \preceq x$ for all $n \in \mathbb{N}$.*

If there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, fx_0) \geq 1$, and $x_0 \preceq fx_0$, then f has a fixed point.

Proof Let $x_0 \preceq fx_0$. We define an iterative sequence $\{x_n\}$ in the following way:

$$x_n = f^n x_0 = fx_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

From Theorem 2.1 we know that $\{x_n\}$ is a non-decreasing sequence, $\gamma(x_n,x_n) \geq 1$ and $\gamma(x_n,fx_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Also, similarly, we suppose that $d(x_{n-1},x_n) > 0$ for all n. We shall show that the sequence $\{d_n := d(x_n,x_{n+1})\}$ is non-increasing sequence of reals. Assume that there exists some $n_0 \in \mathbb{N}$ such that

$$d(x_{n_0-1},x_{n_0}) \leq d(x_{n_0},x_{n_0+1}).$$

Hence

$$\psi\big(d(x_{n_0-1},x_{n_0})\big) \leq \psi\big(d(x_{n_0},x_{n_0+1})\big). \tag{2.14}$$

Taking $x = x_{n-1}$ and $y = x_n$ in (2.13) and applying (2.6) we get

$$\begin{aligned} 2^{\psi(d(x_n,x_{n+1}))} &= 2^{\psi(d(fx_{n-1},fx_n))} \\ &\leq \big(\gamma(x_{n-1},x_{n-1})\gamma(x_n,x_n)+1\big)^{\psi(d(fx_{n-1},fx_n))} \\ &\leq 2^{\alpha(d(x_{n-1},x_n))-\beta(d(x_{n-1},x_n))}. \end{aligned}$$

Hence

$$\psi\big(d(x_n,x_{n+1})\big) \leq \alpha\big(d(x_{n-1},x_n)\big) - \beta\big(d(x_{n-1},x_n)\big) \tag{2.15}$$

for all $n \in \mathbb{N}$. Now, by taking $x = x_{n_0-1}$ and $y = x_{n_0}$ in (2.15) and using (2.14) we deduce

$$\psi\big(d(x_{n_0-1},x_{n_0})\big) \leq \alpha\big(d(x_{n_0-1},x_{n_0})\big) - \beta\big(d(x_{n_0-1},x_{n_0})\big),$$

which is a contradiction. Then $d_n < d_{n-1}$ holds for all $n \in \mathbb{N}$ and so there exists $r \geq 0$ such that $\lim_{n\to\infty} d_n = r$. Reviewing the proof of Theorem 2.1 we can show that $r = 0$. Now, suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon > 0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k with $n(k) > m(k) > k$

$$\lim_{k\to\infty} d(x_{n(k)},x_{m(k)}) = \varepsilon \tag{2.16}$$

and

$$\lim_{k\to\infty} d(x_{n(k)+1},x_{m(k)+1}) = \varepsilon. \tag{2.17}$$

By (2.13) with $x = x_{m(k)}$ and $y = x_{n(k)}$ we have

$$\begin{aligned} &2^{\psi(d(x_{m(k)+1},fx_{n(k)+1}))} \\ &\quad\leq \big(\gamma(x_{m(k)},x_{m(k)})\gamma(x_{n(k)},x_{n(k)})+1\big)^{\psi(d(x_{m(k)+1},fx_{n(k)+1}))} \\ &\quad= \big(\gamma(x_{m(k)},x_{m(k)})\gamma(x_{n(k)},x_{n(k)})+1\big)^{\psi(d(fx_{m(k)},fx_{n(k)}))} \\ &\quad\leq 2^{\alpha(d(x_{m(k)},x_{n(k)}))-\beta(d(x_{m(k)},x_{n(k)}))} \end{aligned}$$

and so

$$\psi\big(d(x_{m(k)+1},fx_{n(k)+1})\big) \leq \alpha\big(d(x_{m(k)},x_{n(k)})\big) - \beta\big(d(x_{m(k)},x_{n(k)})\big).$$

By taking the lim inf as $k \to +\infty$ in the above inequality, we have

$$\begin{aligned}\psi(\varepsilon) &\le \liminf \psi\bigl(d(x_{n(k)+1}, x_{m(k)+1})\bigr) \le \limsup \psi\bigl(d(x_{n(k)+1}, x_{m(k)+1})\bigr)\\ &\le \limsup\bigl(\alpha\bigl(d(x_{n(k)}, x_{m(k)})\bigr) - \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr)\bigr)\\ &= \limsup \alpha\bigl(d(x_{n(k)}, x_{m(k)})\bigr) - \liminf \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr)\\ &\le \alpha(\varepsilon) - \beta(\varepsilon).\end{aligned}$$

Therefore

$$\psi(\varepsilon) \le \alpha(\varepsilon) - \beta(\varepsilon),$$

which is a contradiction. Hence,

$$\lim_{n\to\infty} d(x_n, x_m) = 0.$$

Then $\{x_n\}$ is a Cauchy sequence. Since (X,d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Let (i) hold. Then

$$x^* = \lim_{n\to\infty} x_{n+1} = f\Bigl(\lim_{n\to\infty} x_n\Bigr) = f\bigl(x^*\bigr).$$

So, x^* is a fixed point of f. Now, we assume that (ii) holds, that is, $\gamma(x^*,x^*) \ge 1$, $\gamma(x^*,fx^*) \ge 1$, and $x_n \preceq x^*$ for all $n \ge 0$. We claim that x^* is a fixed point of f, equivalently, $\lim_{n\to\infty} d(x_{n+1},fx^*) = 0$. Suppose, to the contrary, that $\lim_{n\to\infty} d(x_{n+1},fx^*) = d(x^*,fx^*) > 0$. From (2.13), we have

$$\begin{aligned}2^{\psi(d(fx^*,x_{n+1}))} &= 2^{\psi(d(fx^*,fx_n))}\\ &\le \bigl(\gamma\bigl(x^*,x^*\bigr)\gamma(x_n,x_n) + 1\bigr)^{\psi(d(fx^*,fx_n))}\\ &\le 2^{\alpha(d(x^*,x_n))-\beta(d(x^*,x_n)).}\end{aligned}$$

Taking the lim inf as $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned}\psi\bigl(d\bigl(x^*,fx^*\bigr)\bigr) &\le \liminf_{n\to\infty} \psi\bigl(d\bigl(x_{n+1},fx^*\bigr)\bigr)\\ &= \liminf_{n\to\infty} \psi\bigl(d\bigl(fx_n,fx^*\bigr)\bigr) \le \limsup_{n\to\infty} \psi\bigl(d\bigl(x_n,fx^*\bigr)\bigr)\\ &\le \limsup_{n\to\infty}\bigl(\alpha\bigl(d\bigl(x_n,x^*\bigr)\bigr) - \beta\bigl(d\bigl(x_n,x^*\bigr)\bigr)\bigr)\\ &\le \alpha(0) - \beta(0),\end{aligned}$$

which is a contradiction. Then $\lim_{n\to\infty} d(x_{n+1},fx^*) = d(x^*,fx^*) = 0$ and hence, $x^* = fx^*$. □

Example 2.5 Let X and d be as in Example 2.2. Define $f : X \to X$ by

$$fx = \begin{cases}\frac{1}{4}(1 - x^2) & \text{if } x \in [0,1],\\ e^x & \text{if } x \in (1,\infty).\end{cases}$$

Define also γ, ψ, α, and β as in Example 2.2. We shall show that Theorem 2.4 can be applied to f, but Theorem 1.5 cannot be applied. Proceeding as in the proof of Example 2.2 f is a γ-admissible mapping, $\alpha(0,0) \geq 1$, and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\gamma(x,x) \geq 1$. Let $\gamma(x,fx)\gamma(y,fy) \geq 1$. Then $x, y \in [0,1]$. Assume $y \geq x$. We get

$$\begin{aligned}
\bigl(\gamma(x,x)\gamma(y,y)+1\bigr)^{\psi(d(fx,fy))} &= 2^{fx-fy+1/2} \\
&= 2^{[\frac{1}{4}(y^2-x^2)+1/2]} \\
&= 2^{[\frac{1}{4}(y+x)(y-x)+1/2]} \\
&\leq 2^{[\frac{1}{2}(y-x)+1/2]} \\
&= 2^{\alpha(d(x,y))-\beta(d(x,y))}.
\end{aligned}$$

Then the condition of Theorem 2.4 holds and so f has a fixed point. Let $x = \ln 2$ and $y = \ln 4$, then

$$\begin{aligned}
\psi\bigl(d\bigl(f(\ln 2), f(\ln 4)\bigr)\bigr) = 2 + 1/2 &> \frac{1}{2}\ln 2 + 1/2 \\
&= \alpha\bigl(d(\ln 2, \ln 4)\bigr) - \beta\bigl(d(\ln 2, \ln 4)\bigr).
\end{aligned}$$

Hence, the condition of Theorem 1.5 does not hold for this example.

Corollary 2.6 *Let $(X, d, \preceq)$ be a partially ordered metric space such that (X, d) is complete. Assume $f : X \to X$ and $\gamma : X \times X \to [0,\infty)$ are two mappings such that f is a non-decreasing γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$, and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - \alpha(s) + \beta(s) > 0 \quad \textit{for all } t > 0 \textit{ and } s = t \textit{ or } s = 0$$

and

$$\gamma(x,fx)\gamma(y,fy)\bigl(\gamma(x,x)\gamma(y,y)+1\bigr)^{\psi(d(fx,fy))} \leq 2^{\alpha(d(x,y))-\beta(d(x,y))}$$

for all comparable $x, y \in X$. Suppose that either

(i) *f is continuous, or*
(ii) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$ as $n \to \infty$, $\gamma(x_n, x_n) \geq 1$, and $\gamma(x_n, fx_n) \geq 1$ for all n, then $\gamma(x,x) \geq 1$, $\gamma(x,fx) \geq 1$, and $x_n \preceq x$ for all $n \in \mathbb{N}$.*

If there exists $x_0 \in X$ such that $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, fx_0) \geq 1$, and $x_0 \preceq fx_0$, then f has a fixed point.

Theorem 2.7 *Let $(X, d, \preceq)$ be a partially ordered metric space such that (X, d) is complete. Assume that $f : X \to X$ and $\gamma : X \times X \to [0,\infty)$ are two mappings such that f is a non-decreasing γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$, and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - \alpha(s) + \beta(s) > 0 \quad \textit{for all } t > 0 \textit{ and } s = t \textit{ or } s = 0$$

and

$$\gamma(x,fx)\gamma(y,fy)\geq 1 \quad\Longrightarrow\quad \gamma(x,x)\gamma(y,y)\psi\bigl(d(fx,fy)\bigr)\leq\alpha\bigl(d(x,y)\bigr)-\beta\bigl(d(x,y)\bigr) \tag{2.18}$$

for all comparable $x,y\in X$. Suppose that either

(i) *f is continuous, or*

(ii) *if a non-decreasing sequence $\{x_n\}$ is such that $x_n\to x$ as $n\to\infty$, $\gamma(x_n,x_n)\geq 1$, and $\gamma(x_n,fx_n)\geq 1$ for all n, then $\gamma(x,x)\geq 1$, $\gamma(x,fx)\geq 1$, and $x_n\preceq x$ for all $n\in\mathbb{N}$.*

If there exists $x_0\in X$ such that $\alpha(x_0,x_0)\geq 1$, $\alpha(x_0,fx_0)\geq 1$, and $x_0\preceq fx_0$, then f has a fixed point.

Proof Let $x_0\preceq fx_0$. We define an iterative sequence $\{x_n\}$ in the following way:

$$x_n=f^n x_0=fx_{n-1}\quad\text{for all } n\in\mathbb{N}.$$

From Theorem 2.1 we know that $\{x_n\}$ is a non-decreasing sequence, $\gamma(x_n,x_n)\geq 1$, and $\gamma(x_n,fx_n)\geq 1$ for all $n\in\mathbb{N}\cup\{0\}$. Also, similarly, we suppose that $d(x_{n-1},x_n)>0$ for all n. We shall show that the sequence $\{d_n:=d(x_n,x_{n+1})\}$ is non-increasing. Assume that there exists some $n_0\in\mathbb{N}$ such that

$$d(x_{n_0-1},x_{n_0})\leq d(x_{n_0},x_{n_0+1}).$$

Hence

$$\psi\bigl(d(x_{n_0-1},x_{n_0})\bigr)\leq\psi\bigl(d(x_{n_0},x_{n_0+1})\bigr). \tag{2.19}$$

Taking $x=x_{n-1}$ and $y=x_n$ in (2.18) and applying (2.19) we get

$$\begin{aligned}\psi\bigl(d(x_n,x_{n+1})\bigr) &= \psi\bigl(d(fx_{n-1},fx_n)\bigr)\\ &\leq\gamma(x_{n-1},x_{n-1})\gamma(x_n,x_n)\psi\bigl(d(fx_{n-1},fx_n)\bigr)\\ &\leq\alpha\bigl(d(x_{n-1},x_n)\bigr)-\beta\bigl(d(x_{n-1},x_n)\bigr).\end{aligned}$$

Hence

$$\psi\bigl(d(x_n,x_{n+1})\bigr)\leq\alpha\bigl(d(x_{n-1},x_n)\bigr)-\beta\bigl(d(x_{n-1},x_n)\bigr) \tag{2.20}$$

for all $n\in\mathbb{N}$. Now, by taking $x=x_{n_0-1}$ and $y=x_{n_0}$ in (2.20) and using (2.19) we deduce

$$\psi\bigl(d(x_{n_0-1},x_{n_0})\bigr)\leq\alpha\bigl(d(x_{n_0-1},x_{n_0})\bigr)-\beta\bigl(d(x_{n_0-1},x_{n_0})\bigr),$$

which is a contradiction. Then $d_n<d_{n-1}$ holds for all $n\in\mathbb{N}$ and so there exists $r\geq 0$ such that $\lim_{n\to\infty}d_n=r$. Proceeding as in the proof of Theorem 2.1 we conclude that $r=0$. Now, suppose, to the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exist $\varepsilon>0$ and sequences $m(k)$ and $n(k)$ such that for all positive integers k with $n(k)>m(k)>k$

$$\lim_{k\to\infty}d(x_{n(k)},x_{m(k)})=\varepsilon \tag{2.21}$$

and

$$\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon. \tag{2.22}$$

By (2.18) with $x = x_{m(k)}$ and $y = x_{n(k)}$ we have

$$\begin{aligned}
&\psi\bigl(d(x_{m(k)+1}, fx_{n(k)+1})\bigr)\\
&\quad\le \gamma(x_{m(k)}, x_{m(k)})\gamma(x_{n(k)}, x_{n(k)})\psi\bigl(d(x_{m(k)+1}, fx_{n(k)+1})\bigr)\\
&\quad= \gamma(x_{m(k)}, x_{m(k)})\gamma(x_{n(k)}, x_{n(k)})\psi\bigl(d(fx_{m(k)}, fx_{n(k)})\bigr)\\
&\quad\le \alpha\bigl(d(x_{m(k)}, x_{n(k)})\bigr) - \beta\bigl(d(x_{m(k)}, x_{n(k)})\bigr)
\end{aligned}$$

and so

$$\psi\bigl(d(x_{m(k)+1}, fx_{n(k)+1})\bigr) \le \alpha\bigl(d(x_{m(k)}, x_{n(k)})\bigr) - \beta\bigl(d(x_{m(k)}, x_{n(k)})\bigr).$$

Taking the lim inf as $k \to +\infty$ in the above inequality, we have

$$\begin{aligned}
\psi(\varepsilon) &\le \liminf \psi\bigl(d(x_{n(k)+1}, x_{m(k)+1})\bigr) \le \limsup \psi\bigl(d(x_{n(k)+1}, x_{m(k)+1})\bigr)\\
&\le \limsup\bigl(\alpha\bigl(d(x_{n(k)}, x_{m(k)})\bigr) - \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr)\bigr)\\
&= \limsup \alpha\bigl(d(x_{n(k)}, x_{m(k)})\bigr) - \liminf \beta\bigl(d(x_{n(k)}, x_{m(k)})\bigr)\\
&\le \alpha(\varepsilon) - \beta(\varepsilon).
\end{aligned}$$

Therefore

$$\psi(\varepsilon) \le \alpha(\varepsilon) - \beta(\varepsilon),$$

which is a contradiction. Hence

$$\lim_{n\to\infty} d(x_n, x_m) = 0,$$

that is, $\{x_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Let (i) hold. Then

$$x^* = \lim_{n\to\infty} x_{n+1} = f\Bigl(\lim_{n\to\infty} x_n\Bigr) = f(x^*).$$

So, x^* is a fixed point of f. Now, we assume that (ii) holds, that is, $\gamma(x^*, x^*) \ge 1$, $\gamma(x^*, fx^*) \ge 1$, and $x_n \preceq x^*$ for all $n \ge 0$. We claim that x^* is a fixed point of f, or equivalently, $\lim_{n\to\infty} d(x_{n+1}, fx^*) = 0$. Suppose, to the contrary, that $\lim_{n\to\infty} d(x_{n+1}, fx^*) = d(x^*, fx^*) > 0$. From (2.18), we have

$$\begin{aligned}
\psi\bigl(d(fx^*, x_{n+1})\bigr) &= \psi\bigl(d(fx^*, fx_n)\bigr)\\
&\le \gamma(x^*, x^*)\gamma(x_n, x_n)\psi\bigl(d(fx^*, fx_n)\bigr)\\
&\le \alpha\bigl(d(x^*, x_n)\bigr) - \beta\bigl(d(x^*, x_n)\bigr).
\end{aligned}$$

Taking the $\liminf$ as $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned}\psi\bigl(d\bigl(x^*,fx^*\bigr)\bigr) &\leq \liminf_{n\to\infty} \psi\bigl(d\bigl(x_{n+1},fx^*\bigr)\bigr)\\ &= \liminf_{n\to\infty} \psi\bigl(d\bigl(fx_n,fx^*\bigr)\bigr) \leq \limsup_{n\to\infty} \psi\bigl(d\bigl(x_n,fx^*\bigr)\bigr)\\ &\leq \limsup_{n\to\infty}\bigl(\alpha\bigl(d\bigl(x_n,x^*\bigr)\bigr) - \beta\bigl(d\bigl(x_n,x^*\bigr)\bigr)\bigr)\\ &\leq \alpha(0) - \beta(0),\end{aligned}$$

which is a contradiction. Then $\lim_{n\to\infty} d(x_{n+1},fx^*) = d(x^*,fx^*) = 0$, and hence $x^* = fx^*$. □

Example 2.8 Let X and d be as in Example 2.2. Define $f : X \to X$ by

$$fx = \begin{cases} \frac{1}{8}x^4 & \text{if } x \in [0,1],\\ e^{\sin x} + x & \text{if } x \in (1,\infty).\end{cases}$$

Define also γ, ψ, α, and β as in Example 2.2. We shall show that Theorem 2.7 can be applied for f, but Theorem 1.5 cannot be applied. Reviewing the proof of Example 2.2, f is a γ-admissible mapping, $\alpha(0,0) \geq 1$ and if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_n) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $x_n \to x$ as $n \to +\infty$, then $\gamma(x,x) \geq 1$. Let $\gamma(x,fx)\gamma(y,fy) \geq 1$. Then $x, y \in [0,1]$. Assume $y \geq x$. We get

$$\begin{aligned}\gamma(x,x)\gamma(y,y)\psi\bigl(d(fx,fy)\bigr) &= fy - fx + 1/2\\ &= \frac{1}{8}(y+x)(y-x)\bigl(y^2+x^2\bigr) + 1/2\\ &\leq \frac{1}{2}(y-x) + 1/2\\ &= \alpha\bigl(d(x,y)\bigr) - \beta\bigl(d(x,y)\bigr).\end{aligned}$$

Then the condition of Theorem 2.7 holds and f has a fixed point. Clearly, the condition of Theorem 1.5 does not hold for this example.

Corollary 2.9 *Let $(X,d,\preceq)$ be a partially ordered metric space such that (X,d) is complete. Assume $f : X \to X$ and $\gamma : X \times X \to [0,\infty)$ are two mappings such that f is a nondecreasing γ-admissible mapping. Assume that there exist $\psi \in \Psi$, $\alpha \in \Phi_\alpha$, and $\beta \in \Phi_\beta$ such that*

$$\psi(t) - \alpha(s) + \beta(s) > 0 \quad \textit{for all } t > 0 \textit{ and } s = t \textit{ or } s = 0$$

and

$$\gamma(x,fx)\gamma(y,fy)\gamma(x,x)\gamma(y,y)\psi\bigl(d(fx,fy)\bigr) \leq \alpha\bigl(d(x,y)\bigr) - \beta\bigl(d(x,y)\bigr)$$

for all comparable $x, y \in X$. Suppose that either

(i) *f is continuous, or*

(ii) *if a non-decreasing sequence* $\{x_n\}$ *is such that* $x_n \to x$ *as* $n \to \infty$, $\gamma(x_n, x_n) \geq 1$, *and* $\gamma(x_n, fx_n) \geq 1$ *for all* n, *then* $\gamma(x, x) \geq 1$, $\gamma(x, fx) \geq 1$, *and* $x_n \preceq x$ *for all* $n \in \mathbb{N}$.

If there exists $x_0 \in X$ *such that* $\alpha(x_0, x_0) \geq 1$, $\alpha(x_0, fx_0) \geq 1$, *and* $x_0 \preceq fx_0$, *then* f *has a fixed point*.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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