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Weak- and strong-convergence theorems of solutions to split feasibility problem for nonspreading type mapping in Hilbert spaces

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Abstract

The purpose of this article is to study the weak- and strong-convergence theorems of solutions to split a feasibility problem for a family of nonspreading-type mapping in Hilbert spaces. The main result presented in this paper improves and extends some recent results of Censor *et al.*, Byrne, Yang, Moudafi, Xu, Censor and Segal, Masad and Reich, and others. As an application, we solve the hierarchical variational inequality problem by using the main theorem.

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1 Introduction

Throughout this paper, we assume that H is a real Hilbert space, D is a nonempty and closed convex subset of H. In the sequel, we denote by '$x_n \to x$' and '$x_n \rightharpoonup x$' the strong and weak convergence of $\{x_n\}$, respectively. Denote by $\mathcal{N}$ the set of all positive integers and by $F(T)$ the set of fixed points of a mapping $T : D \to D$.

Definition 1.1 Let $T : D \to D$ be a mapping.

(1) $T : D \to D$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in D.$$

(2) T is said to be quasi-nonexpansive if $F(T)$ is nonempty and

$$\|Tx - p\| \le \|x - p\|, \quad \forall x \in D, p \in F(T). \tag{1.1}$$

(3) T is said to be nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in D. \tag{1.2}$$

It is easy to prove that equation (1.2) is equivalent to

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in D. \tag{1.3}$$

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(4) T is said to be k-strictly pseudo-nonspreading [1], if there exists a constant $k \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty \rangle, \quad \forall x, y \in D. \quad (1.4)$$

Remark 1.2 It follows from Definition 1.1 that

(1) if T is nonspreading and $F(T) \ne \emptyset$, then T is quasi-nonexpansive;

(2) if T is nonspreading, then it is k-strictly pseudo-nonspreading with $k = 0$. But the converse is not true from the following example. Thus, we know that the class of k-strictly pseudo-nonspreading mappings is more general than the class of nonspreading mappings.

Example 1.3 [1] Let $\mathcal{R}$ denote the set of real numbers with the usual norm. Let $T : \mathcal{R} \to \mathcal{R}$ be a mapping defined by

$$Tx = \begin{cases} x, & x \in (-\infty, 0), \\ -2x, & x \in [0, \infty). \end{cases} \quad (1.5)$$

Then T is a k-strictly pseudo-nonspreading mapping, but it is not nonspreading.

In 2010, Kurokawa and Takahashi [2] obtained a weak mean ergodic theorem of Baillon's type [3] for nonspreading mappings in Hilbert spaces. They further proved a strong-convergence theorem somewhat related to Halpern's type [4] for this class of mappings using the idea of mean convergence in Hilbert spaces.

In 2011, Osilike and Isiogugu [1] first introduced the concept of k-strictly pseudo-nonspreading mappings and proved a weak mean convergence theorem of Baillon's type similar to the ones obtained in [2]. Furthermore, using the idea of mean convergence, a strong-convergence theorem similar to the one obtained in [2] is proved which extends and improves the main theorems of [2] and an affirmative answer given to an open problem posed by Kurokawa and Takahashi [2] for the case where the mapping T is averaged.

On the other hand, the split feasibility problem (SFP) in finitely dimensional spaces was first introduced by Censor and Elfving [5] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [6]. Recently, it has been found that the (SFP) can also be used in various disciplines such as image restoration, computer tomography and radiation therapy treatment planning [7–9].

The split feasibility problem in an infinitely dimensional Hilbert space can be found in [6, 8, 10–12].

The purpose of this paper is to introduce the following multiple-set split feasibility problem (MSSFP) for an infinite family of k-strictly pseudo-nonspreading mappings and a finite family of ρ-strictly pseudo-nonspreading mappings in infinitely dimensional Hilbert spaces, *i.e.*, to find $x^* \in C$ such that

$$Ax^* \in Q, \quad (1.6)$$

where H_1, H_2 are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator, $\{S_i\}_{i=1}^{\infty} : H_1 \to H_1$ is an infinite family of k_i-strictly pseudo-nonspreading mappings and

$\{T_i\}_{i=1}^N : H_2 \to H_2$ is a finite family of ρ_i-strictly pseudo-nonspreading mappings, $C :=$ $\bigcap_{i=1}^\infty F(S_i)$ and $Q := \bigcap_{i=1}^N F(T_i)$. Also we wish to study the weak and strong convergence of the following iterative sequence to a solution of problem (1.6):

$$\begin{cases} x_1 \in H_1 \quad \text{chosen arbitrarily}, \\ x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^\infty \alpha_{i,n} S_{i,\beta} y_n, \\ y_n = x_n + \gamma A^*(T_{n(\mathrm{mod}\, N)} - I) A x_n, \quad \forall n \geq 1, \end{cases}$$

where $S_{i,\beta} := \beta I + (1-\beta) S_i$, $\beta \in (0,1)$ is a constant.

In the sequel we denote Γ the set of solutions of (MSSFP) equation (1.6), *i.e.*,

$$\Gamma = \{x \in C, Ax \in Q\} = C \cap A^{-1}(Q). \tag{1.7}$$

2 Preliminaries

For this purpose, we first recall some definitions, notations and conclusions which will be needed in proving our main results.

Definition 2.1 Let E be a real Banach space, and $T : E \to E$ be a mapping.
(1) $I - T$ is said to be demiclosed at 0, if, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x^*$, $\|(I - T)x_n\| \to 0$, then $x^* = Tx^*$.
(2) T is said to be semicompact, if, for any bounded sequence $\{x_n\} \subset E$, $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x^* \in E$.

Lemma 2.2 [1] *Let H be a real Hilbert space, D be a nonempty and closed convex subset of H, and $T : D \to D$ be a k-strictly pseudo-nonspreading mapping.*
(1) *If $F(T) \neq \emptyset$, then $F(T)$ is closed and convex;*
(2) *$I - T$ is demiclosed at zero.*

Lemma 2.3 *Let H be a real Hilbert space. Then the following statements hold:*
(1) *For all $x, y \in H$ and for all $t \in [0,1]$,*

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2. \tag{2.1}$$

(2) *For all $x, y \in H$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle.$$

Lemma 2.4 [13] *Let E be a uniformly convex Banach space, $B_r(0) := \{x \in E : \|x\| \leq r\}$ be a closed ball with center 0 and radius $r > 0$. Then for any given sequence $\{x_1, x_2, \ldots, x_n, \ldots\} \subset B_r(0)$ and any given number sequence $\{\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots\}$ with $\lambda_i \geq 0$, $\sum_{i=1}^\infty \lambda_i = 1$, there exists a strictly increasing continuous and convex function $g : [0, 2r] \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \mathcal{N}$, $i < j$,*

$$\left\| \sum_{n=1}^\infty \lambda_n x_n \right\|^2 \leq \sum_{n=1}^\infty \lambda_n \|x_n\|^2 - \lambda_i \lambda_j g(\|x_i - x_j\|). \tag{2.2}$$

Lemma 2.5 [10] *Let* $\{a_n\}$, $\{b_n\}$ *and* $\{\delta_n\}$ *be sequences of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 + \delta_n)a_n + b_n, \quad \forall n \ge 1. \tag{2.3}$$

If $\sum_{i=1}^{\infty} \delta_n < \infty$ *and* $\sum_{i=1}^{\infty} b_n < \infty$, *then the limit* $\lim_{n \to \infty} a_n$ *exists.*

Lemma 2.6 *Let* D *be a nonempty and closed convex subset of* H *and* $T : D \to D$ *be a k-strictly pseudo-nonspreading mapping with* $F(T) \ne \emptyset$. *Let* $T_\beta = \beta I + (1 - \beta)T$, $\beta \in [k, 1)$. *Then the following conclusions hold*:

(1) $F(T) = F(T_\beta)$;

(2) $I - T_\beta$ *is demiclosed at zero*;

(3) $\|T_\beta x - T_\beta y\|^2 \le \|x - y\|^2 + \frac{2}{1-\beta}\langle x - T_\beta x, y - T_\beta y\rangle$;

(4) T_β *is a quasi-nonexpansive mapping.*

Proof Since $(I - T_\beta) = (1 - \beta)(I - T)$, the conclusions (1), (2) are obvious.

Now we prove the conclusion (3). In fact, since T is a k-strictly pseudo-nonspreading mapping, it follows from Lemma 2.3 that

$$
\begin{aligned}
\|T_\beta x - T_\beta y\|^2 &= \big\|\beta(x - y) + (1 - \beta)(Tx - Ty)\big\|^2 \\
&= \beta\|x - y\|^2 + (1 - \beta)\|Tx - Ty\|^2 \\
&\quad - \beta(1 - \beta)\big\|x - Tx - (y - Ty)\big\|^2 \\
&\le \beta\|x - y\|^2 + (1 - \beta)\big\{\|x - y\|^2 + k\big\|x - Tx - (y - Ty)\big\|^2 \\
&\quad + 2\langle x - Tx, y - Ty\rangle\big\} - \beta(1 - \beta)\big\|x - Tx - (y - Ty)\big\|^2 \\
&= \|x - y\|^2 + 2(1 - \beta)\langle x - Tx, y - Ty\rangle \\
&\quad - (1 - \beta)(\beta - k)\big\|x - Tx - (y - Ty)\big\|^2 \\
&\le \|x - y\|^2 + 2(1 - \beta)\langle x - Tx, y - Ty\rangle \\
&= \|x - y\|^2 + \frac{2}{(1 - \beta)}\langle x - T_\beta x, y - T_\beta y\rangle, \quad \forall x, y \in D. \tag{2.4}
\end{aligned}
$$

If $y \in F(T)$, then $y \in F(T_\beta)$. Hence from equation (2.4),

$$\|T_\beta x - y\| = \|T_\beta x - T_\beta y\| \le \|x - y\|, \quad \forall x \in D. \tag{2.5}$$

This completes the proof of Lemma 2.6. □

Lemma 2.7 [14] *Let* H *be a Hilbert space and* $\{u_n\}$ *be a sequence in* H *such that there exists a nonempty set* $W \subset H$ *satisfying*:

(1) *for every* $w \in W$, $\lim_{n \to \infty} \|u_n - w\|$ *exists*;

(2) *each weak-cluster point of the sequence* $\{w_n\}$ *is in* W.

Then there exists $w^* \in W$ *such that* $\{u_n\}$ *weakly converges to* w^*.

3 Weak- and strong-convergence theorems

For solving the multiple-set split feasibility problem (MSSFP) equation (1.6), we assume that the following conditions are satisfied:

(1) H_1 and H_2 are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator and $A^* : H_2 \to H_1$ is the adjoint of A;

(2) $\{S_i\}_{i=1}^\infty : H_1 \to H_1$ is an infinite family of k_i-strictly pseudo-nonspreading mappings with $k := \sup_{i \geq 1} k_i \in (0, 1)$;

(3) $\{T_i\}_{i=1}^N : H_2 \to H_2$ is a finite family of ρ_i-strictly pseudo-nonspreading mappings with $\rho = \max\{\rho_i : i = 1, 2, \ldots, N\} \in (0, 1)$;

(4) $C := \bigcap_{i=1}^\infty F(S_i) \neq \emptyset$ and $Q := \bigcap_{i=1}^N F(T_i) \neq \emptyset$.

Now we are in a position to give the following main theorem.

Theorem 3.1 *Let H_1, H_2, A, A^*, $\{S_i\}_{i=1}^\infty$, $\{T_i\}_{i=1}^N$, C, Q, k, ρ be the same as above. Let $\{x_n\}$ be a sequence generated by*

$$
\begin{cases}
x_1 \in H_1 & \text{chosen arbitrarily,} \\
x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^\infty \alpha_{i,n} S_{i,\beta} y_n, & \forall n \geq 1, \\
y_n = x_n + \gamma A^* (T_{n(\mathrm{mod}\,N)} - I) A x_n,
\end{cases}
\tag{3.1}
$$

where $S_{i,\beta} := \beta I + (1 - \beta) S_i$, $i \geq 1$, $\beta \in [k, 1)$ is a constant, $\{\alpha_{i,n}\} \subset (0, 1)$ and $\gamma > 0$ satisfy the following conditions:

(a) $\sum_{i=0}^\infty \alpha_{i,n} = 1$, *for each $n \geq 1$;*

(b) *for each $i \geq 1$*, $\liminf_{n \to \infty} \alpha_{0,n} \alpha_{i,n} > 0$;

(c) $\gamma \in (0, \frac{1-\rho}{\|A\|^2})$.

Let $\Gamma = \{x \in C, Ax \in Q\} \neq \emptyset$ (the set of solutions of (MSSFP) equation (1.6) defined by equation (1.7)). Then we have the following:

(I) *both $\{x_n\}$ and $\{y_n\}$ converge weakly to some point $x^* \in \Gamma$;*

(II) *in addition, if there exists some positive integer m such that S_m is semicompact, then both $\{x_n\}$ and $\{y_n\}$ converge strongly to $x^* \in \Gamma$.*

Proof First we prove the conclusion (I).

Step 1. We prove that the sequences $\{x_n\}$, $\{y_n\}$ and $\{S_{i,\beta} y_n\}$ are bounded and, for each $p \in \Gamma$, the following limits exist and

$$
\lim_{n \to \infty} \|x_n - p\| = \lim_{n \to \infty} \|y_n - p\|.
$$

In fact, for given $p \in \Gamma$, by the definition of Γ,

$$
p \in C = \bigcap_{i=1}^\infty F(S_i) = \bigcap_{i=1}^\infty F(S_{i,\beta})
$$

and

$$
Ap \in Q := \bigcap_{i=1}^N F(T_i).
$$

Therefore, we have

$$
Ap = T_{n(\mathrm{mod}\,N)} Ap.
\tag{3.2}
$$

Since $\{S_i\}_{i=1}^{\infty}$ is a family of k-strictly pseudo-nonspreading mappings, by Lemma 2.2, $C = \bigcap_{i=1}^{\infty} F(S_i)$ is closed and convex. It follows from Lemma 2.6 that, for each $n \geq 1$ and $p \in \Gamma$,

$$\|x_{n+1} - p\| = \left\| \alpha_{0,n}(y_n - p) + \sum_{i=1}^{\infty} \alpha_{i,n}(S_{i,\beta} y_n - p) \right\|$$

$$\leq \alpha_{0,n}\|y_n - p\| + \sum_{i=1}^{\infty} \alpha_{i,n}\|S_{i,\beta} y_n - p\|$$

$$= \|y_n - p\|, \tag{3.3}$$

$$\|y_n - p\|^2 = \left\| x_n - p + \gamma A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2$$

$$= \|x_n - p\|^2 + 2\gamma \left\langle x_n - p, A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\rangle$$

$$+ \gamma^2 \left\| A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2 \tag{3.4}$$

and

$$\gamma^2 \left\| A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2 = \gamma^2 \left\langle A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n, A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\rangle$$

$$= \gamma^2 \left\langle AA^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n, (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\rangle$$

$$\leq \gamma^2 \|A\|^2 \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2. \tag{3.5}$$

Further, since $\{T_i\}_{i=1}^{N}$ is a finite family of ρ-strictly pseudo-nonspreading mappings, we have

$$\left\langle x_n - p, A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\rangle$$

$$= \left\langle A(x_n - p), (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\rangle$$

$$= \left\langle A(x_n - p) + (T_{n(\mathrm{mod}\,N)} - I)Ax_n - (T_{n(\mathrm{mod}\,N)} - I)Ax_n, (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\rangle$$

$$= \left\langle T_{n(\mathrm{mod}\,N)}Ax_n - Ap, (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\rangle - \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2$$

$$= \frac{1}{2} \left\{ \|T_{n(\mathrm{mod}\,N)}Ax_n - Ap\|^2 + \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2 \right.$$

$$\left. - \|Ax_n - Ap\|^2 \right\} - \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2$$

$$= \frac{1}{2} \left\{ \|T_{n(\mathrm{mod}\,N)}Ax_n - T_{n(\mathrm{mod}\,N)}Ap\|^2 + \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2 \right.$$

$$\left. - \|Ax_n - Ap\|^2 \right\} - \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2$$

$$\leq \frac{1}{2} \left\{ \|Ax_n - Ap\|^2 + \rho \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2 \right\}$$

$$+ \frac{1}{2} \left\{ \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2 - \|Ax_n - Ap\|^2 \right\} - \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2$$

$$= \frac{\rho - 1}{2} \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2. \tag{3.6}$$

Substituting equations (3.5) and (3.6) into equation (3.4) and simplifying, we have

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 - \gamma \left(1 - \rho - \gamma \|A\|^2 \right) \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2. \tag{3.7}$$

By condition (c), $(1 - \rho - \gamma \|A\|^2) > 0$, therefore we have

$$\|y_n - p\|^2 \leq \|x_n - p\|^2. \tag{3.8}$$

Substituting equation (3.8) into equation (3.3), we have

$$\|x_{n+1} - p\| \leq \|x_n - p\|, \quad \forall n \geq 1.$$

This implies that the limit $\lim_{n \to \infty} \|x_n - p\|$ exists. It follows from equations (3.8) and (3.3) that the limit $\lim_{n \to \infty} \|y_n - p\|$ exists also, and

$$\lim_{n \to \infty} \|x_n - p\| = \lim_{n \to \infty} \|y_n - p\|, \quad \forall p \in \Gamma. \tag{3.9}$$

Therefore, $\{x_n\}$ and $\{y_n\}$ are bounded. Since for each $i \geq 1$, $S_{i,\beta}$ is quasi-nonexpansive, we have

$$\|S_{i,\beta} y_n - p\| \leq \|y_n - p\|.$$

Hence $\{S_{i,\beta} y_n\}$ is also bounded.

Step 2. Now we prove that for any given positive integer $l \geq 1$, the following conclusions hold:

$$\lim_{n \to \infty} \|y_n - S_{l,\beta} y_n\| = 0; \qquad \lim_{n \to \infty} \|T_{n(\mathrm{mod}\,N)} A x_n - A x_n\| = 0. \tag{3.10}$$

In fact, for any given $p \in \Gamma$, it follows from equation (3.1), Lemma 2.4, and equation (3.7) that

$$
\begin{aligned}
\|x_{n+1} - p\|^2 &= \left\| \alpha_{0,n}(y_n - p) + \sum_{i=1}^{\infty} \alpha_{i,n}(S_{i,\beta} y_n - p) \right\|^2 \\
&\leq \alpha_{0,n} \|y_n - p\|^2 + \sum_{i=1}^{\infty} \alpha_{i,n} \|S_{i,\beta} y_n - p\|^2 - \alpha_{0,n}\alpha_{l,n} g\big(\|y_n - S_{i,\beta} y_n\|\big) \\
&\leq \alpha_{0,n} \|y_n - p\|^2 + \sum_{i=1}^{\infty} \alpha_{i,n} \|y_n - p\|^2 - \alpha_{0,n}\alpha_{l,n} g\big(\|y_n - S_{i,\beta} y_n\|\big) \\
&= \|y_n - p\|^2 - \alpha_{0,n}\alpha_{l,n} g\big(\|y_n - S_{i,\beta} y_n\|\big) \\
&\leq \|x_n - p\|^2 - \gamma\big(1 - \rho - \gamma\|A\|^2\big)\big\|(T_{n(\mathrm{mod}\,N)} - I)A x_n\big\|^2 \\
&\quad - \alpha_{0,n}\alpha_{l,n} g\big(\|y_n - S_{i,\beta} y_n\|\big), \quad n \geq 1.
\end{aligned}
\tag{3.11}
$$

Therefore, we have

$$
\begin{aligned}
&\gamma\big(1 - \rho - \gamma\|A\|^2\big)\big\|(T_{n(\mathrm{mod}\,N)} - I)A x_n\big\|^2 + \alpha_{0,n}\alpha_{l,n} g\big(\|y_n - S_{l,\beta} y_n\|\big) \\
&\quad \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad \to 0 \quad (\text{as } n \to \infty).
\end{aligned}
$$

By conditions (b) and (c) we have

$$\lim_{n\to\infty} \left\| (T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\| = 0; \qquad \lim_{n\to\infty} g\left(\|y_n - S_{l,\beta}y_n\| \right) = 0. \tag{3.12}$$

Since g is continuous and strictly increasing with $g(0) = 0$, from equation (3.12) we have

$$\lim_{n\to\infty} \|y_n - S_{l,\beta}y_n\| = 0 \quad \text{for each } l \geq 1. \tag{3.13}$$

Hence conclusion (3.10) is proved.

Step 3. Now, we prove that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0; \qquad \lim_{n\to\infty} \|y_{n+1} - y_n\| = 0. \tag{3.14}$$

In fact, it follows from equation (3.1) that

$$
\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \left\| \alpha_{0,n}(y_n - x_n) + \sum_{i=1}^{\infty} \alpha_{i,n}(S_{i,\beta}y_n - x_n) \right\|^2 \\
&= \left\| \alpha_{0,n}\left(\gamma A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right) + \sum_{i=1}^{\infty} \alpha_{i,n}(S_{i,\beta}y_n - x_n) \right\|^2 \\
&\leq \alpha_{0,n} \left\| \gamma A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2 + \sum_{i=1}^{\infty} \alpha_{i,n} \|S_{i,\beta}y_n - x_n\|^2 \\
&\leq \alpha_{0,n} \left\| \gamma A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\|^2 \\
&\quad + \sum_{i=1}^{\infty} \alpha_{i,n}\left(\|S_{i,\beta}y_n - y_n\| + \|y_n - x_n\| \right)^2.
\end{aligned}
\tag{3.15}
$$

By virtue of equations (3.1) and (3.10), one has

$$
\begin{aligned}
\|y_n - x_n\| &= \gamma \left\| A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\| \\
&\to 0 \quad (\text{as } n \to \infty).
\end{aligned}
\tag{3.16}
$$

This together with equations (3.10) and (3.15) shows that

$$\|x_{n+1} - x_n\| \to 0 \quad (\text{as } n \to \infty).$$

Similarly, we have

$$
\begin{aligned}
\|y_{n+1} - y_n\| &= \left\| x_{n+1} + \gamma A^*(T_{(n+1)(\mathrm{mod}\,N)} - I)Ax_{n+1} \right. \\
&\quad \left. - \left[x_n + \gamma A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right] \right\| \\
&\leq \|x_{n+1} - x_n\| + \gamma \left\| A^*(T_{(n+1)(\mathrm{mod}\,N)} - I)Ax_{n+1} \right\| \\
&\quad + \gamma \left\| A^*(T_{n(\mathrm{mod}\,N)} - I)Ax_n \right\| \\
&\to 0 \quad (\text{as } n \to \infty).
\end{aligned}
$$

Step 4. Now we show that every weak-cluster point x^* of the sequence $\{x_n\}$ is in Γ.

Indeed, since $\{y_n\}$ is a bounded sequence in H_1, there exists a subsequence $\{y_{n_i}\} \subset \{y_n\}$ such that $y_{n_i} \rightharpoonup x^* \in H_1$. It follows from equation (3.10) that

$$\lim_{n \to \infty} \|y_{n_i} - S_{l,\beta} y_{n_i}\| = 0 \quad \text{for each } l \geq 1.$$

By Lemma 2.2, $(I - S_i)$ is demiclosed at zero. Since $(I - S_{l,\beta}) = (1 - \beta)(I - S_i)$, this implies that $(I - S_{l,\beta})$ is also demiclosed at zero. Hence $x^* \in F(S_{l,\beta}) = F(S_l)$. By the arbitrariness of $l \geq 1$, we have

$$x^* \in \bigcap_{i=1}^{\infty} F(S_i) = C.$$

On the other hand, it follows from equations (3.1) and (3.10) that

$$x_{n_i} = y_{n_i} - \gamma A^*(T_{n_i(\text{mod}\,N)} - I)Ax_{n_i} \rightharpoonup x^*. \tag{3.17}$$

Since A is a bounded linear operator, this implies that $Ax_{n_i} \rightharpoonup Ax^*$. Also, by equation (3.10)

$$\lim_{n_i \to \infty} \|T_{n_i(\text{mod}\,N)} Ax_{n_i} - Ax_{n_i}\| = 0. \tag{3.18}$$

Hence for any given positive integer $j = 1, 2, \ldots, N$, there exists a subsequence $\{n_{i_k}\} \subset \{n_i\}$ with $n_{i_k}(\text{mod}\,N) = j$ such that

$$\lim_{n_{i_k} \to \infty} \|T_j Ax_{n_{i_k}} - Ax_{n_{i_k}}\| = 0.$$

Since $Ax_{n_{i_k}} \rightharpoonup Ax^*$, and by Lemma 2.2, $I - T_j$ is demiclosed at 0. This implies that $Ax^* \in F(T_j)$. By the arbitrariness of $j = 1, 2, \ldots, N$,

$$Ax^* \in \bigcap_{j=1}^{N} F(T_j) = Q.$$

These show that $x^* \in \Gamma$.

Step 5. Summing up the above arguments, we have proved that: (i) for each $p \in \Gamma$, the limits $\lim_{n \to \infty} \|x_n - p\|$ and $\lim_{n \to \infty} \|y_n - p\|$ exist (see equation (3.2)); (ii) every weak-cluster point x^* of the sequence $\{x_n\}$ (or $\{y_n\}$) is in Γ. Taking $W = \Gamma$ and $\{u_n\} = \{x_n\}$ (or $\{y_n\}$) in Lemma 2.7, therefore all conditions in Lemma 2.7 are satisfied. By using Lemma 2.7, $x_n \rightharpoonup x^*$, $y_n \rightharpoonup x^*$ and $x^* \in \Gamma$. This completes the proof of the conclusion (I).

Next we prove the conclusion (II).

Without loss of generality, we may assume that S_1 is semicompact. Since $(I - S_{1,\beta}) = (1 - \beta)(I - S_1)$, this implies that $S_{1,\beta}$ is also semicompact. In view of equation (3.10), we have

$$\|y_n - S_{1,\beta} y_n\| \to 0 \quad (\text{as } n \to \infty). \tag{3.19}$$

Therefore, there exists a subsequence of $\{y_{n_i}\} \subset \{y_n\}$ such that $y_{n_i} \to u^* \in H_1$. Since $y_{n_i} \rightharpoonup x^*$, we have $x^* = u^*$ and so $y_{n_i} \to x^* \in \Gamma$. By virtue of equation (3.9), we have

$$\lim_{n \to \infty} \|y_n - x^*\| = 0, \qquad \lim_{n \to \infty} \|x_n - x^*\| = 0,$$

i.e., $\{y_n\}$ and $\{x_n\}$ both converge strongly to the point $x^* \in \Gamma$. This completes the proof of Theorem 3.1. □

Remark 3.2 Theorem 3.1 improves and extends the corresponding results of Censor *et al.* [5, 8, 9], Byrne [6], Yang [11], Moudafi [15], Xu [16], Censor and Segal [17], Masad and Reich [18], Deepho and Kumam [19, 20] and others in the following aspects:

(a) for the mappings, we extend the mappings from nonexpansive mappings, or demi-contractive mappings, to the more general family of k-strictly pseudo-nonspreading mappings;

(b) for the algorithms, we propose some new hybrid iterative algorithms which are different from the ones given in [5–7, 9, 17, 18, 21, 22]. Under suitable conditions, some weak- and strong-convergence results for the algorithms are proved.

If we put $\gamma = 0$ in Theorem 3.1, we immediately get the following.

Corollary 3.3 *Let* H, $\{S_i\}_{i=1}^{\infty}$, k *be the same as above. Let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} x_1 \in H \quad \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_{0,n}x_n + \sum_{i=1}^{\infty} \alpha_{i,n}S_{i,\beta}x_n, \end{cases} \quad \forall n \geq 1, \tag{3.20}$$

where $S_{i,\beta} := \beta I + (1 - \beta)S_i$, $i \geq 1$, $\beta \in [k, 1)$ *is a constant,* $\{\alpha_{i,n}\} \subset (0, 1)$ *satisfy the following conditions*:

(a) $\sum_{i=0}^{\infty} \alpha_{i,n} = 1$, *for each* $n \geq 1$;

(b) *for each* $i \geq 1$, $\liminf_{n \to \infty} \alpha_{0,n}\alpha_{i,n} > 0$. *Let*

$$\mathcal{F} := \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset.$$

Then we have the following:

(I) *the sequence* $\{x_n\}$ *converges weakly to some point* $x^* \in \mathcal{F}$;

(II) *in addition, if there exists some positive integer* m *such that* S_m *is semicompact, then the sequence* $\{x_n\}$ *converges strongly to* $x^* \in \mathcal{F}$.

4 Applications

In this section we utilize the results presented in Section 3 to study the hierarchical variational inequality problem.

Let H be a real Hilbert space, $\{S_i\}: H \to H$, $i = 1, 2, \ldots$ be a countable family of k_i-strictly pseudo-nonspreading mappings with $k = \sup_{i \geq 1} k_i \in (0, 1)$, and

$$\mathcal{F} := \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset.$$

Let $T : H \to H$ be a nonspreading mapping. The so-called hierarchical variational inequality problem for a countable family of mappings $\{S_i\}$ with respect to mapping T is to find an $x^* \in \mathcal{F}$ such that

$$\langle x^* - Tx^*, x^* - x \rangle \leq 0, \quad \forall x \in \mathcal{F}. \tag{4.1}$$

It is easy to see that equation (4.1) is equivalent to the following fixed point problem: to find $x^* \in \mathcal{F}$ such that

$$x^* = P_{\mathcal{F}} Tx^*, \tag{4.2}$$

where $P_{\mathcal{F}}$ is the metric projection from H onto $\mathcal{F}$. Letting $C = \mathcal{F}$ and $Q = F(P_{\mathcal{F}} T)$ (the fixed point set of $P_{\mathcal{F}} T$) and $A = I$ (the identity mapping on H), then the problem (4.2) is equivalent to the following multi-set split feasibility problem: to find $x^* \in C$ such that

$$x^* \in Q. \tag{4.3}$$

Hence from Theorem 3.1 we have the following theorem.

Theorem 4.1 *Let H, $\{S_i\}$, T, C, Q, k be the same as above. Let $\{x_n\}$, $\{y_n\}$ be the sequences defined by*

$$\begin{cases} x_1 \in H_1 & \text{chosen arbitrarily,} \\ x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} S_{i,\beta} y_n, \\ y_n = x_n + \gamma (T - I) x_n, & \forall n \geq 1, \end{cases} \tag{4.4}$$

where $S_{i,\beta} := \beta I + (1 - \beta) S_i$, $i \geq 1$, $\beta \in [k, 1)$, $\{\alpha_{i,n}\} \subset (0, 1)$ and $\gamma > 0$ satisfy the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{i,n} = 1$, *for each $n \geq 1$*;

(b) *for each $i \geq 1$, $\liminf_{n \to \infty} \alpha_{0,n} \alpha_{i,n} > 0$*;

(c) $\gamma \in (0, 1)$.

If $C \cap Q \neq \emptyset$, then $\{x_n\}$ converges weakly to a solution of the hierarchical variational inequality problem (4.1). *In addition, if one of the mappings S_i is semicompact, then both $\{x_n\}$ and $\{y_n\}$ converge strongly to a solution of the hierarchical variational inequality problem* (4.1).

Proof In fact, by the assumption that T is a nonspreading mapping, hence by Remark 1.2, T is a ρ-strictly pseudo-nonspreading with $\rho = 0$. Taking $N = 1$ and $A = I$ in Theorem 3.1, all conditions in Theorem 3.1 are satisfied. The conclusions of Theorem 4.1 can immediately be obtained from Theorem 3.1. □

Remark 4.2 If $T = I$ (the identity mapping), then we can get the results of Corollary 3.3.

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