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Uniformly closed replaced AKTT or *AKTT condition to get strong convergence theorems for a countable family of relatively quasi-nonexpansive mappings and systems of equilibrium problems

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Abstract

The purpose of this paper is to construct a new iterative scheme and to get a strong convergence theorem for a countable family of relatively quasi-nonexpansive mappings and a system of equilibrium problems in a uniformly convex and uniformly smooth real Banach space using the properties of generalized f-projection operator. The notion of uniformly closed mappings is presented and an example will be given which is a countable family of uniformly closed relatively quasi-nonexpansive mappings but not a countable family of relatively nonexpansive mappings. Another example shall be given which is uniformly closed but does not satisfy condition AKTT and *AKTT. Our results can be applied to solve a convex minimization problem. In addition, this paper clarifies an ambiguity in a useful lemma. The results of this paper modify and improve many other important recent results.

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1 Introduction and preliminaries

Let E be a real Banach space and C be a nonempty closed convex subset of E. A mapping $T: C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

Let E be a real Banach space and C be a nonempty closed convex subset of E. A point $p \in C$ is said to be an asymptotic fixed point of T if there exists a sequence $\{x_n\}_{n=0}^{\infty} \subset C$ such that $x_n \rightharpoonup p$ and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed point is denoted by $\hat{F}(T)$. We say that a mapping T is relatively nonexpansive (see [1–4]) if the following conditions are satisfied:

(I) $F(T) \neq \emptyset$;

(II) $\phi(p, Tx) \leq \phi(p, x)$, $\forall x \in C$, $p \in F(T)$;

(III) $F(T) = \hat{F}(T)$.

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If T satisfies (I) and (II), then T is said to be relatively quasi-nonexpansive. It is easy to see that the class of relatively quasi-nonexpansive mappings contains the class of relatively nonexpansive mappings.

Let E be a real Banach space. The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}\bigl(\|x+y\| + \|x-y\|\bigr) - 1 : \|x\| \leq 1, \|y\| \leq \tau\right\}.$$

E is uniformly smooth if and only if

$$\lim_{\tau\to 0} \frac{\rho_E \tau}{\tau} = 0.$$

Let $\dim E \geq 2$. The modulus of convexity of E is the function $\delta_E(\epsilon) := \inf\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\}$. E is uniformly convex if for any $\epsilon \in (0,2]$, there exists $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon$, then $\|\frac{1}{2}(x+y)\| \leq 1 - \delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. A normed space E is called strictly convex if for all $x, y \in E$, $x \neq y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$, $\forall \lambda \in (0,1)$.

Let E^* be the dual space of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \bigl\{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\bigr\}.$$

The following properties of J are well known (see [5–7] for more details):

(1) If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E.
(2) If E is reflexive, then J is a mapping from E onto E^*.
(3) If E is smooth, then J is single valued.

Throughout this paper, we denote by ϕ the functional on $E \times E$ defined by

$$\phi(x,y) = \|x\|^2 - 2\bigl\langle x, J(y)\bigr\rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.1}$$

Let E be a smooth, strictly convex, and reflexive real Banach space and let C be a nonempty closed convex subset of E. Following Alber [8], the generalized projection Π_C from E onto C is defined by

$$\Pi_C(x) = \arg\min_{y\in C} \phi(y,x), \quad \forall x \in E.$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. It is obvious that

$$\bigl(\|x\| - \|y\|\bigr)^2 \leq \phi(x,y) \leq \bigl(\|x\| + \|y\|\bigr)^2, \quad \forall x, y \in E. \tag{1.2}$$

Next, we recall the notion of generalized f-projection operator and its properties. Let $G : C \times E^* \to R \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi\rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{1.3}$$

where $\xi \in C$, $\varphi \in E^*$, ρ is a positive number and $f : C \to R \cup \{+\infty\}$ is proper, convex, and lower semi-continuous. From the definitions of G and f, it is easy to see the following properties:

(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;

(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed.

Definition 1.1 [9] Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed, and convex subset of E. We say that $\Pi_C^f : E^* \to 2^C$ is *a generalized f-projection operator* if

$$\Pi_C^f \varphi = \Big\{u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi)\Big\}, \quad \forall \varphi \in E^*.$$

For the generalized f-projection operator, Wu and Huang [9] proved in the following theorem some basic properties.

Lemma 1.2 [9] *Let E be a real reflexive Banach space with its dual E^*. Let C be a nonempty, closed, and convex subset of E. Then the following statements hold*:

(i) Π_C^f *is a nonempty closed convex subset of C for all $\varphi \in E^*$.*

(ii) *If E is smooth, then for all $\varphi \in E^*$, $x \in \Pi_C^f \varphi$ if and only if*

$$\langle x - y, \varphi - Jx\rangle + \rho f(y) - \rho f(x) \ge 0, \quad \forall y \in C.$$

(iii) *If E is strictly convex and $f : C \to R \cup \{+\infty\}$ is positive homogeneous* (*i.e.*, *$f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$*), *then Π_C^f is a single-valued mapping.*

Fan *et al.* [10] showed that the condition f is positive homogeneous which appeared in Lemma 1.2 can be removed.

Lemma 1.3 [10] *Let E be a real reflexive Banach space with its dual E^* and C a nonempty, closed, and convex subset of E. Then if E is strictly convex, then Π_C^f is a single-valued mapping.*

Recall that J is a single-valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution in (1.3) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx\rangle + \|x\|^2 + 2\rho f(\xi). \tag{1.4}$$

Now, we consider the second generalized f-projection operator in a Banach space.

Definition 1.4 [11] Let E be a real Banach space and C a nonempty, closed, and convex subset of E. We say that $\Pi_C^f : E \to 2^C$ is *a generalized f-projection operator* if

$$\Pi_C^f x = \Big\{u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx)\Big\}, \quad \forall x \in E.$$

Obviously, the definition of relatively quasi-nonexpansive mapping T is equivalent to

(1) $F(T) \neq \emptyset$;

(2) $G(p, JTx) \le G(p, Jx)$, $\forall x \in C$, $p \in F(T)$.

Lemma 1.5 [12] *Let E be a Banach space and $f : E \to R \cup \{+\infty\}$ be a lower semi-continuous convex functional. Then there exist $x \in E^*$ and $\alpha \in R$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha, \quad \forall x \in E.$$

We know that the following lemmas hold for operator Π_C^f.

Lemma 1.6 [13] *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E. Then the following statements hold*:

(i) Π_C^f *is a nonempty, closed, and convex subset of C for all $x \in E$;*

(ii) *for all $x \in E$, $\hat{x} \in \Pi_C^f x$ if and only if*

$$\langle \hat{x} - y, Jx - J\hat{x} \rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C;$$

(iii) *if E is strictly convex, then $\Pi_C^f x$ is a single-valued mapping.*

Lemma 1.7 [13] *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E. Let $x \in E$ and $\hat{x} \in \Pi_C^f x$. Then*

$$\phi(y, \hat{x}) + G(\hat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C.$$

The fixed points set $F(T)$ of a relatively quasi-nonexpansive mapping is closed convex as given in the following lemma.

Lemma 1.8 [14, 15] *Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E. Let T be a closed relatively quasi-nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Also, this following lemma will be used in the sequel.

Lemma 1.9 [16] *Let C be a nonempty closed convex subset of a smooth, uniformly convex Banach space E. Let $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ be sequences in E such that either $\{x_n\}_{n=0}^{\infty}$ or $\{y_n\}_{n=0}^{\infty}$ is bounded. If $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 1.10 [17] *Let $p > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space X is uniformly convex if and only if there is a continuous, strictly increasing and convex function $g : R^+ \to R^+$, $g(0) = 0$, such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - W_p(\lambda) g(\|x - y\|)$$

for all $x, y \in B_r$ and $0 \leq \lambda \leq 1$, where $W_p(\lambda) = \lambda(1-\lambda)^p + \lambda^p(1-\lambda)$.

Remark We can see from the Lemma 1.10 that the function g has no relation with the selection of x, y and λ. However, the key point above, in the process of generalization and application about this lemma, has been ambiguous gradually. For instance, the following lemma states that the function g has something to do with λ, which always leads to failure in the proof.

Lemma (stated in [11, Lemma 2.10]) *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$ and $\lambda \in [0,1]$. Then there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \to R, \qquad g(0) = 0$$

such that for every $x, y \in B_r(0)$, the following inequality holds:

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x-y\|).$$

Let F be a bifunction of $C \times C$ into R. The equilibrium problem is to find $x^* \in C$ such that $F(x^*, y) \geq 0$, for all $y \in C$. We shall denote the solutions set of the equilibrium problem by $EP(F)$. Numerous problems in physics, optimization, and economics reduce to find a solution of equilibrium problem. The equilibrium problems include fixed point problems, optimization problems, and variational inequality problems as special cases.

For solving the equilibrium problem for a bifunction $F : C \times C \to R$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;
(A2) F is monotone, *i.e.*, $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x,y)$;
(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semi-continuous.

Lemma 1.11 [18] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E and let F be a bifunction of $C \times C$ into R satisfying* (A1)-(A4). *Let $r > 0$ and $x \in E$. Then there exists $z \in C$ such that*

$$F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in K.$$

Lemma 1.12 [19] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : C \times C \to R$ satisfies* (A1)-(A4). *For $r > 0$ and $x \in E$, define a mapping $T_r^F : E \to C$ as follows*:

$$T_r^F(x) = \left\{z \in C : F(z,y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \forall y \in C\right\}$$

for all $z \in E$. Then the following hold:

(1) *T_r^F is single valued*;
(2) *T_r^F is a firmly nonexpansive-type mapping, i.e., for any $x, y \in E$,*

$$\langle T_r^F x - T_r^F y, JT_r^F x - JT_r^F y\rangle \leq \langle T_r^F x - T_r^F y, Jx - Jy\rangle;$$

(3) $F(T_r^F) = EP(F)$;
(4) *$EP(F)$ is closed and convex.*

Lemma 1.13 [19] *Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : C \times C \to R$ satisfies* (A1)-(A4) *and let $r > 0$. Then for each $x \in E$ and $q \in F(T_r^F)$,*

$$\phi(q, T_r^F x) + \phi(T_r^F x, x) \leq \phi(q, x).$$

Let $\{T_n\}$ be a sequence of mappings from C into E, where C is a nonempty closed convex subset of a real Banach space E. For a subset B of C, we say that

(i) $(\{T_n\}, B)$ satisfies condition AKTT (see [15]) if

$$\sum_{n=1}^{\infty} \sup\bigl\{\|T_{n+1}x - T_n x\| : x \in B\bigr\} < \infty;$$

(ii) $(\{T_n\}, B)$ satisfies condition *AKTT (see [15]) if

$$\sum_{n=1}^{\infty} \sup\bigl\{\|JT_{n+1}x - JT_n x\| : x \in B\bigr\} < \infty.$$

Recently, Shehu [11] proved strong convergence theorems for approximation of common element of set of common fixed points of countably infinite family of relatively quasi-nonexpansive mappings and set of common solutions to a system of equilibrium problems in a uniformly convex and uniformly smooth real Banach space using the properties of generalized f-projection operator. The author obtained the following theorem.

Theorem 1.14 [11] *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty closed convex subset of E. For each $k = 1, 2, \ldots, m$, let F_k be a bifunction from $C \times C$ satisfying* (A1)-(A4) *and let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of relatively quasi-nonexpansive mappings of C into itself such that $F := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{k=1}^{m} EP(F_k)) \neq \emptyset$. Let $f : E \to R$ be a convex and lower semi-continuous mapping with $C \subset \operatorname{int}(D(f))$ and suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}^{f} x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n), \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n, \\ C_{n+1} = \{w \in C_n : G(w, Ju_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_0, \quad n \geq 1, \end{cases} \tag{1.5}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ $\{r_{k,n}\}_{n=1}^{\infty} \subset (0,\infty)$ $(k = 1, 2, \ldots, m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$ $(k = 1, 2, \ldots, m)$. Suppose that for each bounded subset B of C, the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Let T be the mapping from C into E defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that T is closed and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F^f x_0$.

In this paper we will construct a new iterative scheme and will get strong convergence theorem for a countable family of relatively quasi-nonexpansive mappings and a system of equilibrium problems in a uniformly convex and uniformly smooth real Banach space using the properties of generalized f-projection operator. The notion of uniformly closed mappings is presented and an example will be given which is a countable family of uniformly closed relatively quasi-nonexpansive mappings but not a countable family of relatively nonexpansive mappings. Another example shall be given which is uniformly closed but not satisfy condition AKTT and *AKTT.

2 Main results

Now, we shall first introduce the notion of uniformly closed mappings and give an example which is a countable family of uniformly closed relatively quasi-nonexpansive mappings but not a countable family of relatively nonexpansive mappings in the sense of G. Another example shall be given which is uniformly closed but not satisfy condition AKTT and *AKTT.

Definition 2.1 Let E be a Banach space, C be a nonempty closed convex subset of E. Let $\{T_n\}_{n=1}^{\infty} : C \to E$ be a sequence of mappings of C into E such that $\bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. $\{T_n\}_{n=1}^{\infty}$ is said to be *uniformly closed*, if $p \in \bigcap_{n=1}^{\infty} F(T_n)$, whenever $\{x_n\} \subset C$ converges strongly to p and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$.

Example 1 Let $E = l^2$, where

$$l^2 = \left\{ \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots) : \sum_{n=1}^{\infty} |\xi_n|^2 < \infty \right\},$$

$$\|\xi\| = \left(\sum_{n=1}^{\infty} |\xi_n|^2 \right)^{\frac{1}{2}}, \quad \forall \xi \in l^2,$$

$$\langle \xi, \eta \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n, \quad \forall \xi = (\xi_1, \xi_2, \xi_3, \ldots, \xi_n, \ldots), \eta = (\eta_1, \eta_2, \eta_3, \ldots, \eta_n, \ldots) \in l^2.$$

It is well known that l^2 is a Hilbert space, so that $(l^2)^* = l^2$. Let $\{x_n\} \subset E$ be a sequence defined by

$$\begin{aligned} x_0 &= (1, 0, 0, 0, \ldots), \\ x_1 &= (1, 1, 0, 0, \ldots), \\ x_2 &= (1, 0, 1, 0, 0, \ldots), \\ x_3 &= (1, 0, 0, 1, 0, 0, \ldots), \\ &\cdots \\ x_n &= (\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \ldots, \xi_{n,k}, \ldots) \\ &\cdots, \end{aligned}$$

where

$$\xi_{n,k} = \begin{cases} 1, & \text{if } k = 1, n+1, \\ 0, & \text{if } k \neq 1, k \neq n+1, \end{cases}$$

for all $n \geq 1$.

Define a countable family of mappings $T_n : E \to E$ as follows:

$$T_n(x) = \begin{cases} \frac{n}{n+1} x_n, & \text{if } x = x_n, \\ -x, & \text{if } x \neq x_n, \end{cases}$$

for all $n \geq 0$.

Conclusion 2.2 $\{T_n\}_{n=0}^{\infty}$ *has a unique fixed point* 0, *that is*, $F(T_n) = \{0\} \neq \emptyset$, $\forall n \geq 0$.

Proof The conclusion is obvious. □

Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of quasi-relatively quasi-nonexpansive mappings, if

$$\bigcap_{n=0}^{\infty} F(T_n) = \widehat{F}\bigl(\{T_n\}_{n=0}^{\infty}\bigr),$$

the $\{T_n\}_{n=1}^{\infty}$ is said to be a countable family of relatively nonexpansive mappings in the sense of functional G, where

$$\widehat{F}\bigl(\{T_n\}_{n=0}^{\infty}\bigr) = \bigl\{p \in C : \exists x_n \rightharpoonup p, \|x_n - T_n x_n\| \to 0, x_n \in C\bigr\}$$

is said to be the asymptotic fixed point set of $\{T_n\}_{n=1}^{\infty}$.

Conclusion 2.3 $\{T_n\}_{n=0}^{\infty}$ *is a countable family of relatively quasi-nonexpansive mappings but not a countable family of relatively nonexpansive mappings in the sense of functional* G.

Proof By Conclusion 2.2, we only need to show that $G(0, JT_n x) \leq G(0, Jx)$, $\forall x \in E$. Note that $E = l^2$ is a Hilbert space, for any $n \geq 0$ we can derive

$$\begin{aligned} & G(0, JT_n x) \leq G(0, Jx) \quad \forall x \in E \\ & \quad \Leftrightarrow \quad \phi(0, T_n x) \leq \phi(0, x) \\ & \quad \Leftrightarrow \quad \|0 - T_n x\|^2 \leq \|0 - x\|^2 \\ & \quad \Leftrightarrow \quad \|T_n x\|^2 \leq \|x\|^2. \end{aligned}$$

It is obvious that $\{x_n\}$ converges weakly to $x_0 = (1, 0, 0, \ldots)$, and

$$\|x_n - T_n x_n\| = \left\| \frac{n}{n+1} x_n - x_n \right\| = \frac{1}{n+1} \|x_n\| \to 0,$$

as $n \to \infty$, so x_0 is an asymptotic fixed point of $\{T_n\}_{n=0}^{\infty}$. Joining with Conclusion 2.2, we can obtain $\bigcap_{n=0}^{\infty} F(T_n) \neq \widehat{F}(\{T_n\}_{n=0}^{\infty})$.

Thus, $\{T_n\}_{n=0}^{\infty}$ is a countable family of relatively quasi-nonexpansive mappings but not a countable family of relatively nonexpansive mappings in the sense of G. □

Conclusion 2.4 $\{T_n\}_{n=0}^{\infty}$ *is a countable family of uniformly closed relatively quasi-nonexpansive mappings in the sense of functional* G.

Proof In fact, for any strong convergent sequence $\{z_n\} \subset E$ such that $z_n \to z_0$ and $\|z_n - T_n z_n\| \to 0$ as $n \to \infty$, there exists a sufficiently large nature number N, such that $z_n \neq x_m$ for any $n, m > N$ (since x_n is not a Cauchy sequence it cannot converge to any element in E). Then $T_n z_n = -z_n$ for $n > N$, it follows from $\|z_n - T_n z_n\| \to 0$ that $2z_n \to 0$ and hence $z_n \to z_0 = 0$.

Therefore, $\{T_n\}_{n=0}^{\infty}$ is a countable family of uniformly closed relatively quasi-nonexpansive mappings but not a countable family of relatively nonexpansive mappings in the sense of functional G. □

Now, we give an example which is a countable family of uniformly closed quasi-nonexpansive mappings but not satisfied condition AKTT and *AKTT.

Example 2 Let $X = \Re^2$. For any complex number $x = re^{i\theta} \in X$, define a countable family of quasi-nonexpansive mappings as follows:

$$T_n : re^{i\theta} \to re^{i(\theta+n\frac{\pi}{2})}, \quad n = 1, 2, 3, \ldots.$$

Proof It is easy to see that $\bigcap_{n=1}^{\infty} F(T_n) = \{0\}$. We first prove that $\{T_n\}$ is uniformly closed. In fact, for any strong convergent sequence $\{x_n\} \subset X$ such that $x_n \to x_0$ and $\|x_n - T_n x_n\| \to 0$ as $n \to \infty$, there must be $x_0 = 0 \in \bigcap_{n=1}^{\infty} F(T_n)$. Otherwise, if $x_n \to x_0 \neq 0$, and

$$\|x_{4n+1} - T_{4n+1} x_{4n+1}\| \to 0,$$

since T_1 is continuous, we have

$$\begin{aligned}&\|x_{4n+1} - T_{4n+1} x_{4n+1}\| \\ &\quad = \|x_{4n+1} - T_1 x_{4n+1}\| \to \|x_0 - T_1 x_0\| \neq 0.\end{aligned}$$

This is a contradiction. Therefore, $\{T_n\}$ is uniformly closed.

Besides, take any $x = re^{i\theta} \neq 0$. For any n by the definition of T_n, we have

$$\|T_n x - T_{n+1} x\| = \left\| re^{\frac{\pi i}{2}} \right\| = r > 0$$

and

$$\|JT_n x - JT_{n+1} x\| = \left\| re^{\frac{\pi i}{2}} \right\| = r > 0.$$

That is to say, $\{T_n\}$ does not satisfied condition AKTT and *AKTT. □

Now we are in a position to present our main theorems.

Theorem 2.5 *Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of uniformly closed relatively quasi-nonexpansive mappings of C into itself and other conditions are the same as Theorem* 1.14 *except for condition AKTT, *AKTT and condition 'Let T be the mapping from C into E defined by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and suppose that T is closed and $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$'. Then the sequence $\{x_n\}_{n=0}^{\infty}$ generated by* (1.5) *converges strongly to $\Pi_F^f x_0$.*

Proof We first show that C_n, $\forall n \geq 1$, is closed and convex. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_n is closed convex for some $n > 1$. From the definition of C_{n+1}, we have $z \in C_{n+1}$ implies $G(z, Ju_n) \leq G(z, Jx_n)$. This is equivalent to

$$2\big(\langle z, Jx_n\rangle - \langle z, Ju_n\rangle\big) \leq \|x_n\|^2 - \|u_n\|^2.$$

This implies that C_{n+1} is closed convex for the same $n > 1$. Hence, C_n is closed and convex for all $n \geq 1$. This shows that $\Pi^f_{C_{n+1}} x_0$ is well defined for all $n \geq 0$.

By taking $\theta_n^k = T^{F_k}_{r_{k,n}} T^{F_{k-1}}_{r_{k-1,n}} \cdots T^{F_2}_{r_{2,n}} T^{F_1}_{r_{1,n}}$, $k = 1, 2, \ldots, m$ and $\theta_n^0 = I$ for all $n \geq 1$, we obtain $u_n = \theta_n^m y_n$.

We next show that $F \subset C_n$, $\forall n \geq 1$. From Lemma 1.12, one sees that $T^{F_k}_{r_{k,n}}$, $k = 1, 2, \ldots, m$, is relatively nonexpansive mapping. For $n = 1$, we have $F \subset C = C_1$. Now, assume that $F \subset C_n$ for some $n \geq 2$. Then for each $x^* \in F$, we obtain

$$\begin{aligned}
G\big(x^*, Ju_n\big) &= G\big(x^*, J\theta_n^m y_n\big) \leq G\big(x^*, Jy_n\big) \\
&= G\big(x^*, \big(\alpha_n Jx_n + (1-\alpha_n) JT_n x_n\big)\big) \\
&= \big\|x^*\big\|^2 - 2\alpha_n \big\langle x^*, Jx_n\big\rangle - 2(1-\alpha_n)\big\langle x^*, JT_n x_n\big\rangle \\
&\quad + \big\|\alpha_n Jx_n + (1-\alpha_n) JT_n x_n\big\|^2 + 2\rho f\big(x^*\big) \\
&\leq \big\|x^*\big\|^2 - 2\alpha_n \big\langle x^*, Jx_n\big\rangle - 2(1-\alpha_n)\big\langle x^*, JT_n x_n\big\rangle \\
&\quad + \alpha_n \|Jx_n\|^2 + (1-\alpha_n)\|JT_n x_n\|^2 + 2\rho f\big(x^*\big) \\
&= \alpha_n G\big(x^*, Jx_n\big) + (1-\alpha_n) G\big(x^*, JT_n x_n\big) \leq G\big(x^*, Jx_n\big).
\end{aligned} \tag{2.1}$$

So, $x^* \in C_n$. This implies that $F \subset C_n$, $\forall n \geq 1$ and the sequence $\{x_n\}_{n=0}^\infty$ generated by (1.5) is well defined.

We now show that $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. Since $f : E \to R$ is a convex and lower semicontinuous, applying Lemma 1.5, we see that there exist $u^* \in E^*$ and $\alpha \in R$ such that

$$f(y) \geq \big\langle y, u^*\big\rangle + \alpha, \quad \forall y \in E.$$

It follows that

$$\begin{aligned}
G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho f(x_n) \\
&\geq \|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2 + 2\rho\big\langle x_n, u^*\big\rangle + 2\rho\alpha \\
&= \|x_n\|^2 - 2\big\langle x_n, Jx_0 - \rho u^*\big\rangle + \|x_0\|^2 + 2\rho\alpha \\
&\geq \|x_n\|^2 - 2\|x_n\| \big\|Jx_0 - \rho u^*\big\| + \|x_0\|^2 + 2\rho\alpha \\
&= \big(\|x_n\| - \big\|Jx_0 - \rho u^*\big\|\big)^2 + \|x_0\|^2 - \big\|Jx_0 - \rho u^*\big\|^2 + 2\rho\alpha.
\end{aligned} \tag{2.2}$$

Since $x_n = \Pi^f_{C_n} x_0$, it follows from (2.2) that

$$G\big(x^*, Jx_0\big) \geq G(x_n, Jx_0) \geq \big(\|x_n\| - \big\|Jx_0 - \rho u^*\big\|\big)^2 + \|x_0\|^2 - \big\|Jx_0 - \rho u^*\big\|^2 + 2\rho\alpha$$

for each $x^* \in F(T)$. This implies that $\{x_n\}_{n=1}^\infty$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^\infty$. By the construction of C_n, we have $C_m \subset C_n$ and $x_m = \Pi^f_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$. It then follows from Lemma 1.7 that

$$\phi(x_m, x_n) + G(x_n, Jx_0) \leq G(x_m, Jx_0). \tag{2.3}$$

It is obvious that

$$\phi(x_m, x_n) \geq \left(\|x_m\| - \|x_n\|\right)^2 \geq 0.$$

In particular,

$$\phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0)$$

and

$$\phi(x_{n+1}, x_n) \geq \left(\|x_{n+1}\| - \|x_n\|\right)^2 \geq 0,$$

and so $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^{\infty}$ exists.

By the fact that $C_m \subset C_n$ and $x_m = \Pi^f_{C_m} x_0 \in C_n$ for any positive integer $m \geq n$, we obtain

$$\phi(x_m, u_n) \leq \phi(x_m, x_n).$$

Now, (2.3) implies that

$$\phi(x_m, u_n) \leq \phi(x_m, x_n) \leq G(x_m, Jx_0) - G(x_n, Jx_0). \tag{2.4}$$

Taking the limit as $m, n \to \infty$ in (2.4), we obtain

$$\lim_{n\to\infty} \phi(x_m, x_n) = 0.$$

It then follows from Lemma 1.9 that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since E is a Banach space and C is closed and convex, there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now since $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$ we have in particular that $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$ and this further implies that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. Since $x_{n+1} = \Pi^f_{C_{n=1}} x_0 \in C_{n+1}$ we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 0.$$

Then we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0.$$

Since E is uniformly convex and smooth, we have from Lemma 1.9

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 = \lim_{n\to\infty} \|x_{n+1} - u_n\|.$$

So,

$$\|x_n - u_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - u_n\|.$$

Hence,

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{2.5}$$

Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty} \|x_n - u_n\| = 0$, we obtain

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0. \tag{2.6}$$

Let $r = \sup_{n\ge 1}\{\|x_n\|, \|T_n x_n\|\}$. Since E is uniformly smooth, we know that E^* is uniformly convex. Then from Lemma 1.10, we have

$$\begin{aligned}
G(x^*, Ju_n) &= G(x^*, J\theta_n^m y_n) \le G(x^*, Jy_n) \\
&= G\big(x^*, \big(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n\big)\big) \\
&= \|x^*\|^2 - 2\alpha_n\langle x^*, Jx_n\rangle - 2(1-\alpha_n)\langle x^*, JT_n x_n\rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)JT_n x_n\|^2 + 2\rho f(x^*) \\
&\le \|x^*\|^2 - 2\alpha_n\langle x^*, Jx_n\rangle - 2(1-\alpha_n)\langle x^*, JT_n x_n\rangle \\
&\quad + \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\|JT_n x_n\|^2 \\
&\quad - \alpha_n(1-\alpha_n)g\big(\|Jx_n - JT_n x_n\|\big) + 2\rho f(x^*) \\
&= \alpha_n G(x^*, Jx_n) + (1-\alpha_n)G(x^*, JT_n x_n) \\
&\quad - \alpha_n(1-\alpha_n)g\big(\|Jx_n - JT_n x_n\|\big) \\
&\le G(x^*, Jx_n) - \alpha_n(1-\alpha_n)g\big(\|Jx_n - JT_n x_n\|\big).
\end{aligned}$$

It then follows that

$$\alpha_n(1-\alpha_n)g\big(\|Jx_n - JT_n x_n\|\big) \le G(x^*, Jx_n) - G(x^*, Ju_n).$$

But

$$\begin{aligned}
G(x^*, Jx_n) - G(x^*, Ju_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle x^*, Jx_n - Ju_n\rangle \\
&\le \|x_n\|^2 - \|u_n\|^2 + 2\big|\langle x^*, Jx_n - Ju_n\rangle\big| \\
&\le \big|\|x_n\| - \|u_n\|\big|\big(\|x_n\| + \|u_n\|\big) + 2\|x^*\|\|Jx_n - Ju_n\| \\
&\le \|x_n - u_n\|\big(\|x_n\| + \|u_n\|\big) + 2\|x^*\|\|Jx_n - Ju_n\|.
\end{aligned}$$

From (2.5) and (2.6), we obtain

$$G(x^*, Jx_n) - G(x^*, Ju_n) \to 0, \quad n \to \infty.$$

Using the condition $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have

$$\lim_{n\to\infty} g\big(\|Jx_n - JT_n x_n\|\big) = 0.$$

By the properties of g, we have $\lim_{n\to\infty}\|Jx_n - JT_nx_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_n - T_nx_n\| = 0.$$

Since $\{T_n\}_{n=1}^{\infty}$ are uniformly closed, and $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Then $p \in F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

Next, we show that $p \in \bigcap_{k=1}^{m} EP(F_k)$. From (2.1), we obtain

$$\begin{aligned}\phi(x^*, u_n) &= \phi(x^*, \theta_n^m y_n) = \phi(x^*, T_{r_{m,n}}^{F_m}\theta_n^{m-1}y_n) \\ &\le \phi(x^*, \theta_n^{m-1}y_n) \le \phi(x^*, x_n).\end{aligned} \tag{2.7}$$

Since $x^* \in EP(F_m) = F(T_{r_{m,n}}^{F_m})$ for all $n \ge 1$, it follows from (2.7) and Lemma 1.13 that

$$\begin{aligned}\phi(u_n, \theta_n^{m-1}y_n) &= \phi(T_{r_{m,n}}^{F_m}\theta_n^{m-1}y_n, \theta_n^{m-1}y_n) \\ &\le \phi(x^*, \theta_n^{m-1}y_n) - \phi(x^*, u_n) \le \phi(x^*, x_n) - \phi(x^*, u_n).\end{aligned}$$

From (2.5) and (2.6), we obtain $\lim_{n\to\infty}\phi(\theta_n^m y_n, \theta_n^{m-1}y_n) = \lim_{n\to\infty}\phi(u_n, \theta_n^{m-1}y_n) = 0$. From Lemma 1.9, we have

$$\lim_{n\to\infty}\|\theta_n^m y_n - \theta_n^{m-1}y_n\| = \lim_{n\to\infty}\|u_n - \theta_n^{m-1}y_n\| = 0. \tag{2.8}$$

Hence, we have from (2.8) that

$$\lim_{n\to\infty}\|J\theta_n^m y_n - J\theta_n^{m-1}y_n\| = 0. \tag{2.9}$$

Again, since $x^* \in EP(F_{m-1}) = F(T_{r_{m-1,n}}^{F_{m-1}})$ for all $n \ge 1$, it follows from (2.7) and Lemma 1.13 that

$$\begin{aligned}\phi(\theta_n^{m-1}y_n, \theta_n^{m-2}y_n) &= \phi(T_{r_{m-1,n}}^{F_{m-1}}\theta_n^{m-2}y_n, \theta_n^{m-2}y_n) \\ &\le \phi(x^*, \theta_n^{m-2}y_n) - \phi(x^*, \theta_n^{m-1}y_n) \le \phi(x^*, x_n) - \phi(x^*, u_n).\end{aligned}$$

Again, from (2.5) and (2.6), we obtain $\lim_{n\to\infty}\phi(\theta_n^{m-1}y_n, \theta_n^{m-2}y_n) = 0$. From Lemma 1.9, we have

$$\lim_{n\to\infty}\|\theta_n^{m-1}y_n - \theta_n^{m-2}y_n\| = 0 \tag{2.10}$$

and hence,

$$\lim_{n\to\infty}\|J\theta_n^{m-1}y_n - J\theta_n^{m-2}y_n\| = 0. \tag{2.11}$$

In a similar way, we can verify that

$$\lim_{n\to\infty}\|\theta_n^{m-2}y_n - \theta_n^{m-3}y_n\| = \cdots = \lim_{n\to\infty}\|\theta_n^1 y_n - y_n\| = 0. \tag{2.12}$$

From (2.8), (2.10), and (2.12), we can conclude that

$$\lim_{n\to\infty}\left\|\theta_n^k y_n - \theta_n^{k-1} y_n\right\| = 0, \quad k = 1, 2, \dots, m. \tag{2.13}$$

Since $x_n \to p$, $n \to \infty$, we obtain from (2.5) that $u_n \to p$, $n \to \infty$. Again, from (2.8), (2.10), (2.12), and $u_n \to p$, $n \to \infty$, we have that $\theta_n^k y_n \to p$, $n \to \infty$ for each $k = 1, 2, \dots, m$. Also, using (2.13), we obtain

$$\lim_{n\to\infty}\left\|J\theta_n^k y_n - J\theta_n^{k-1} y_n\right\| = 0, \quad k = 1, 2, \dots, m.$$

Since $\liminf_{n\to\infty} r_{k,n} > 0$, $k = 1, 2, \dots, m$,

$$\lim_{n\to\infty}\frac{\|J\theta_n^k y_n - J\theta_n^{k-1} y_n\|}{r_{k,n}} = 0. \tag{2.14}$$

By Lemma 1.12, we have for each $k = 1, 2, \dots, m$

$$F_k\bigl(\theta_n^k y_n, y\bigr) + \frac{1}{r_{k,n}}\bigl\langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n\bigr\rangle \ge 0, \quad \forall y \in C.$$

Furthermore, using (A2) we obtain

$$\frac{1}{r_{k,n}}\bigl\langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n\bigr\rangle \ge F_k\bigl(y, \theta_n^k y_n\bigr). \tag{2.15}$$

By (A4), (2.14), and $\theta_n^k y_n \to p$, we have for each $k = 1, 2, \dots, m$

$$F_k(y, p) \le 0, \quad y \in C.$$

For fixed $y \in C$, let $z_t = ty + (1-t)p$ for all $t \in (0, 1]$. This implies that $z_t \in C$. This yields $F_k(z_t, p) \le 0$. It follows from (A1) and (A4) that

$$0 = F_k(z_t, z_t) \le tF_k(z_t, y) + (1-t)F_k(z_t, p) \le tF_k(z_t, y)$$

and hence

$$0 \le F_k(z_t, y).$$

From condition (A3), we obtain

$$F_k(p, y) \ge 0, \quad y \in C.$$

This implies that $p \in EP(F_k)$, $k = 1, 2, \dots, m$. Thus, $p \in \bigcap_{k=1}^m EP(F_k)$. Hence, we have $p \in F = \bigcap_{k=1}^m EP(F_k) \cap (\bigcap_{n=1}^\infty F(T_n))$.

Finally, we show that $p = \Pi_F^f x_0$. Since $F = \bigcap_{k=1}^m EP(F_k) \cap (\bigcap_{n=1}^\infty F(T_n))$ is a closed and convex set, from Lemma 1.6, we know that $\Pi_F^f x_0$ is single valued and denote $w = \Pi_F^f x_0$. Since $x_n = \Pi_{c_n}^f x_0$ and $w \in F \subset C_n$, we have

$$G(x_n, Jx_0) \le G(w, Jx_0), \quad \forall n \ge 0.$$

We know that $G(\xi, J\varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed. This implies that

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(w, Jx_0).$$

From the definition of $\Pi_F^f x_0$ and $p \in F$, we see that $p = w$. This completes the proof. □

Corollary 2.6 *Let E be a uniformly convex and uniformly smooth real Banach space, and let C be a nonempty closed convex subset of E. For each $k = 1, 2, \ldots, m$, let F_k be a bifunction from $C \times C$ satisfying* (A1)-(A4) *and let $\{T_n\}_{n=1}^{\infty}$ be a countable family of uniformly closed relatively quasi-nonexpansive mappings of C into itself such that $F := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{k=1}^{m} EP(F_k)) \neq \emptyset$. Suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}^f x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT_n x_n), \\ u_n = T_{r_{m,n}}^{F_m} T_{r_{m-1,n}}^{F_{m-1}} \cdots T_{r_{2,n}}^{F_2} T_{r_{1,n}}^{F_1} y_n, \\ C_{n+1} = \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, and $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$ $(k = 1, 2, \ldots, m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$ $(k = 1, 2, \ldots, m)$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

Proof Take $f(x) = 0$ for all $x \in E$ in Theorem 2.5, then $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi_C^f x_0 = \Pi_C x_0$. Then Corollary 2.6 holds. □

Take $F_k \equiv 0$ $(k = 1, 2, \ldots, m)$, it is obvious that the following holds.

Corollary 2.7 *Let E be a uniformly convex and uniformly smooth real Banach space, and let C be a nonempty closed convex subset of E. Let $\{T_n\}_{n=1}^{\infty}$ be a countable family of uniformly closed relatively quasi-nonexpansive mappings of C into itself such that $F = (\bigcap_{n=1}^{\infty} F(T_n)) \neq \emptyset$. Let $f : E \to R$ be a convex and lower semi-continuous mapping with $C \subset \operatorname{int}(D(f))$ and suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}^f x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n) JT_n x_n), \\ C_{n+1} = \{w \in C_n : G(w, Jy_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^f x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ such that $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, and $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$ $(k = 1, 2, \ldots, m)$ satisfying $\liminf_{n\to\infty} r_{k,n} > 0$ $(k = 1, 2, \ldots, m)$. Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F x_0$.

3 Applications

Let $\varphi : C \to R$ be a real-valued function. The convex minimization problem is to find $x^* \in C$ such that

$$\varphi(x^*) \leq \varphi(y), \tag{3.1}$$

$\forall y \in C$. The set of solutions of (3.1) is denoted by $CMP(\varphi)$. For each $r > 0$ and $x \in E$, define the mapping

$$T_r^{\varphi}(x) = \left\{ z \in C : \varphi(y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq \varphi(z), \forall y \in C \right\}.$$

Theorem 3.1 *Let E be a uniformly convex and uniformly smooth real Banach space, and let C be a nonempty closed convex subset of E. For each $k = 1, 2, \ldots, m$, let φ_k be a bifunction from $C \times C$ satisfying* (A1)-(A4) *and let $\{T_n\}_{n=1}^{\infty}$ be a countable family of uniformly closed relatively quasi-nonexpansive mappings of C into itself such that $F := (\bigcap_{n=1}^{\infty} F(T_n)) \cap (\bigcap_{k=1}^{m} CMP(\varphi_k)) \neq \emptyset$. Let $f : E \to R$ be a convex and lower semi-continuous mapping with $C \subset \operatorname{int}(D(f))$ and suppose $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1}^{f} x_0$,*

$$\begin{cases} y_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T_n x_n), \\ u_n = T_{r_{m,n}}^{\varphi_m} T_{r_{m-1,n}}^{\varphi_{m-1}} \cdots T_{r_{2,n}}^{\varphi_2} T_{r_{1,n}}^{\varphi_1} y_n, \\ C_{n+1} = \{w \in C_n : G(w, Ju_n) \leq G(w, Jx_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}^{f} x_0, \quad n \geq 1, \end{cases}$$

where J is the duality mapping on E. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in $(0,1)$ *such that* $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$ *and* $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$ $(k = 1, 2, \ldots, m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$ $(k = 1, 2, \ldots, m)$. *Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_F^f x_0$.*

Proof Define $F_k(x, y) = \varphi_k(y) - \varphi_k(x)$, $x, y \in C$ and $k = 1, 2, \ldots, m$. Then $F(T_{r_k}^{F_k}) = EP(F_k) = CMP(\varphi_k) = F(T_{r_k}^{\varphi_k})$ for each $k = 1, 2, \ldots, m$, and therefore $\{F_k\}_{k=1}^{m}$ satisfies conditions (A1) and (A2). Furthermore, one can easily show that $\{F_k\}_{k=1}^{m}$ satisfies (A3) and (A4). Therefore, from Theorem 2.5, we obtain Theorem 3.1. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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References

1. Butnariu, D, Reich, S, Zaslavski, AJ: Asymptotic behaviour of relatively nonexpansive operators in Banach spaces. J. Appl. Anal. **7**, 151-174 (2001)
2. Butnariu, D, Reich, S, Zaslavski, AJ: Weak convergence of orbits of nonlinear operator in reflexive Banach spaces. Numer. Funct. Anal. Optim. **24**, 489-508 (2003)
3. Censor, Y, Reich, S: Iterations of paracontractions and firmly nonexpansive operators with applications to feasibility and optimization. Optimization **37**, 323-339 (1996)
4. Matsushita, S, Takahashi, W: A strong convergence theorem for relatively nonexpansive mappings in Banach spaces. J. Approx. Theory **134**, 257-266 (2005)
5. Chidume, CE: Geometric Properties of Banach Spaces and Nonlinear Iterations. Lecture Notes in Mathematics, vol. 1965. Springer, Berlin (2009). ISBN:978-1-84882-189-7. xviii+326 pp.
6. Takahashi, W: Nonlinear Functional Analysis-Fixed Point Theory and Applications. Yokohama Publishers, Yokohama (2000) (in Japanese)
7. Takahashi, W: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
8. Alber, YI: Metric and generalized projection operator in Banach spaces: properties and applications. In: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type. Lecture Notes in Pure and Applied Mathematics, vol. 178, pp. 15-50. Dekker, New York (1996)

9. Wu, KQ, Huang, NJ: The generalized f-projection operator with application. Bull. Aust. Math. Soc. **73**, 307-317 (2006)
10. Fan, JH, Liu, X, Li, JL: Iterative schemes for approximating solutions of generalized variational inequalities in Banach spaces. Nonlinear Anal. **70**, 3997-4007 (2009)
11. Shehu, Y: Strong convergence theorems for infinite family of relatively quasi-nonexpansive mappings and systems of equilibrium problems. Appl. Math. Comput. **218**, 5146-5156 (2012)
12. Deimling, K: Nonlinear Functional Analysis. Springer, Berlin (1985)
13. Li, X, Huang, N, O'Regan, D: Strong convergence theorems for relatively nonexpansive mappings in Banach spaces with applications. Comput. Math. Appl. **60**, 1322-1331 (2010)
14. Qin, X, Cho, YJ, Kang, SM: Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. J. Comput. Appl. Math. **225**, 20-30 (2009)
15. Nilsrakoo, W, Saejung, S: Strong convergence to common fixed points of countable relatively quasi-nonexpansive mappings. Fixed Point Theory Appl. **2008**, Article ID 312454 (2008)
16. Kamimura, S, Takahashi, W: Strong convergence of a proximal-type algorithm in a Banach space. SIAM J. Optim. **13**, 938-945 (2002)
17. Xu, HK: Inequalities in Banach spaces with applications. Nonlinear Anal. **16**(2), 1127-1138 (1991)
18. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**, 123-145 (1994)
19. Takahashi, W, Zembayashi, K: Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces. Nonlinear Anal. **70**, 45-57 (2009)

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