Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the Ishikawa iteration processes for multivalued mappings in some CAT(κ) spaces

Bancha Panyanak*

This work is dedicated to Professor Wataru Takahashi on the occasion of his seventieth birthday

*Correspondence:
bancha.p@cmu.ac.th
Department of Mathematics, Faculty of Science, Chiang Mai University, Chiang Mai, 50200, Thailand

Abstract
The purpose of this paper is to prove the strong convergence of the Ishikawa iteration processes for some generalized multivalued nonexpansive mappings in the framework of CAT(1) spaces. Our results extend the corresponding results given by Shahzad and Zegeye (Nonlinear Anal. 71:838-844, 2009), Puttasontiphot (Appl. Math. Sci. 4:3005-3018, 2010), Song and Cho (Bull. Korean Math. Soc. 48:575-584, 2011) and many others.

Keywords: fixed point; multivalued mapping; strong convergence; Ishikawa iteration; CAT(κ) space

1 Introduction

Roughly speaking, a CAT(κ) space is a geodesic space of bounded curvature. The precise definition is given below. Here CAT means the initials of three mathematician's names (E Cartan, AD Alexandrov and A Toponogov) who have made important contributions to the understanding of curvature via inequalities for the distance function, and κ is a real number that we impose as the curvature bound of the space.

Fixed point theory in CAT(κ) spaces was first studied by Kirk [1, 2]. His works were followed by a series of new works by many authors (see, *e.g.*, [3–12]) mainly focusing on CAT(0) spaces. Since any CAT(κ) space is a CAT(κ') space for $\kappa' \geq \kappa$ (see [13, p.165]), all results for CAT(0) spaces immediately apply to any CAT(κ) space with $\kappa \leq 0$. Notice also that all CAT(κ) spaces (with appropriate sizes) are uniformly convex metric spaces in the sense of [14]. Thus, the results in [14] concerning uniformly convex metric spaces also hold in CAT(κ) spaces as well.

In 1974, Ishikawa [15] introduced an iteration process for approximating fixed points of a single-valued mapping t on a Hilbert space H by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n t\big((1-\beta_n)x_n + \beta_n t(x_n)\big), \quad n \geq 1,$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ satisfying some certain restrictions. For more details and literature on the convergence of the Ishikawa iteration process for single-valued mappings, see, *e.g.*, [16–28].

Springer

©2014 Panyanak; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

The first result concerning the convergence of an Ishikawa iteration process for multivalued mappings was proved by Sastry and Babu [29] in a Hilbert space. Panyanak [30] extended the result of Sastry and Babu to a uniformly convex Banach space. Since then the strong convergence of the Ishikawa iteration processes for multivalued mappings has been rapidly developed and many of papers have appeared (see, *e.g.*, [31–36]). Among other things, Shahzad and Zegeye [37] defined two types of Ishikawa iteration processes as follows.

Let E be a nonempty closed convex subset of a uniformly convex Banach space X, $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$, and $T : E \to 2^E$ be a multivalued mapping whose values are nonempty proximinal subsets of E. For each $x \in E$, let $P_T : E \to 2^E$ be a multivalued mapping defined by

$$P_T(x) := \Big\{ u \in T(x) : \|x - u\| = \inf_{y \in T(x)} \|x - y\| \Big\}.$$

(A): The sequence of Ishikawa iterates is defined by $x_1 \in E$,

$$y_n = \beta_n z_n + (1 - \beta_n) x_n, \quad n \geq 1,$$

where $z_n \in T(x_n)$, and

$$x_{n+1} = \alpha_n z'_n + (1 - \alpha_n) x_n, \quad n \geq 1,$$

where $z'_n \in T(y_n)$.

(B): The sequence of Ishikawa iterates is defined by $x_1 \in E$,

$$y_n = \beta_n z_n + (1 - \beta_n) x_n, \quad n \geq 1,$$

where $z_n \in P_T(x_n)$, and

$$x_{n+1} = \alpha_n z'_n + (1 - \alpha_n) x_n, \quad n \geq 1,$$

where $z'_n \in P_T(y_n)$.

They proved, under some suitable assumptions, that the sequence $\{x_n\}$ defined by (A) and (B) converges strongly to a fixed point of T. In 2010, Puttasontiphot [38] gave analogous results to those of Shahzad and Zegeye in complete CAT(0) spaces.

In this paper, we extend Puttasontiphot's results to the setting of CAT(κ) spaces with $\kappa \geq 0$.

2 Preliminaries

Let (X, d) be a metric space, and let $x \in X$, $E \subseteq X$. The *distance* from x to E is defined by

$$\operatorname{dist}(x, E) = \inf\big\{ d(x, y) : y \in E \big\}.$$

The *diameter* of E is defined by

$$\operatorname{diam}(E) = \sup\big\{ d(u, v) : u, v \in E \big\}.$$

The set E is called *proximinal* if for each $x \in X$, there exists an element $y \in E$ such that $d(x,y) = \operatorname{dist}(x,E)$. We shall denote by 2^E the family of nonempty subsets of E, by $\mathcal{P}(E)$ the family of nonempty proximinal subsets of E and by $\mathcal{C}(E)$ the family of nonempty closed subsets of E. Let $H(\cdot,\cdot)$ be the Hausdorff (generalized) distance on 2^E, *i.e.*,

$$H(A,B) = \max\Big\{\sup_{a\in A} \operatorname{dist}(a,B), \sup_{b\in B} \operatorname{dist}(b,A)\Big\}, \quad A,B \in 2^E.$$

Definition 2.1 Let E be a nonempty subset of a metric space (X,d) and $T : E \to 2^E$. Then T is said to

(i) be *nonexpansive* if $H(T(x),T(y)) \le d(x,y)$ for all $x,y \in E$;

(ii) be *quasi-nonexpansive* if $\operatorname{Fix}(T) \neq \emptyset$ and

$$H\big(T(x),T(p)\big) \le d(x,p) \quad \text{for all } x \in E \text{ and } p \in \operatorname{Fix}(T);$$

(iii) satisfy *condition* (I) if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0, f(r) > 0$ for $r \in (0,\infty)$ such that

$$\operatorname{dist}\big(x,T(x)\big) \ge f\big(\operatorname{dist}\big(x,\operatorname{Fix}(T)\big)\big) \quad \text{for all } x \in E;$$

(iv) be *hemicompact* if for any sequence $\{x_n\}$ in E such that

$$\lim_{n\to\infty} \operatorname{dist}\big(x_n,T(x_n)\big) = 0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in E$ such that $\lim_{k\to\infty} x_{n_k} = q$.

A point $x \in E$ is called a *fixed point* of T if $x \in T(x)$. We denote by $\operatorname{Fix}(T)$ the set of all fixed points of T.

The following lemma can be found in [39]. We observe that the boundedness of the images of T is superfluous.

Lemma 2.2 *Let E be a nonempty subset of a metric space (X,d) and $T : E \to \mathcal{P}(E)$ be a multivalued mapping. Then*

(i) $\operatorname{dist}(x,T(x)) = \operatorname{dist}(x,P_T(x))$ *for all* $x \in E$;

(ii) $x \in \operatorname{Fix}(T) \Longleftrightarrow x \in \operatorname{Fix}(P_T) \Longleftrightarrow P_T(x) = \{x\}$;

(iii) $\operatorname{Fix}(T) = \operatorname{Fix}(P_T)$.

Let (X,d) be a metric space. A *geodesic path* joining $x \in X$ to $y \in X$ (or, more briefly, a *geodesic* from x to y) is a map c from a closed interval $[0,l] \subset R$ to X such that $c(0) = x$, $c(l) = y$, and $d(c(t),c(t')) = |t - t'|$ for all $t,t' \in [0,l]$. In particular, c is an isometry and $d(x,y) = l$. The image $c([0,l])$ of c is called a *geodesic* (or *metric*) *segment* joining x and y. When it is unique, this geodesic segment is denoted by $[x,y]$. This means that $z \in [x,y]$ if and only if there exists $\alpha \in [0,1]$ such that

$$d(x,z) = (1-\alpha)d(x,y) \quad \text{and} \quad d(y,z) = \alpha d(x,y).$$

In this case, we write $z = \alpha x \oplus (1-\alpha)y$. The space (X,d) is said to be a *geodesic space* if every two points of X are joined by a geodesic, and X is said to be *uniquely geodesic* if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset E of X is said to be *convex* if E includes every geodesic segment joining any two of its points.

In a geodesic space (X, d), the metric $d : X \times X \to R$ is *convex* if for any $x, y, z \in X$ and $\alpha \in [0, 1]$, one has

$$d\bigl(x, \alpha y \oplus (1-\alpha)z\bigr) \leq \alpha d(x, y) + (1-\alpha)d(x, z).$$

Let $D \in (0, \infty]$, then (X, d) is called a *D-geodesic space* if any two points of X with their distance smaller than D are joined by a geodesic segment. Notice that (X, d) is a geodesic space if and only if it is a D-geodesic space.

Let $n \in N$, we denote by $\langle \cdot | \cdot \rangle$ the Euclidean scalar product in R^n, that is,

$$\langle x|y \rangle = x_1 y_1 + \cdots + x_n y_n, \quad \text{where } x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n).$$

Let S^n denote the n-dimensional sphere defined by

$$S^n = \bigl\{x = (x_1, \ldots, x_{n+1}) \in R^{n+1} : \langle x|x \rangle = 1\bigr\},$$

with metric $d(x, y) = \arccos\langle x|y \rangle$, $x, y \in S^n$ (see [13, Proposition 2.1]).

From now on, we assume that $\kappa \geq 0$ and define

$$D_\kappa := \frac{\pi}{\sqrt{\kappa}} \quad \text{if } \kappa > 0 \quad \text{and} \quad D_\kappa := \infty \quad \text{if } \kappa = 0.$$

We denote by M_κ^n the following metric spaces:

(i) if $\kappa = 0$ then M_0^n is the Euclidean space R^n;

(ii) if $\kappa > 0$ then M_κ^n is obtained from S^n by multiplying the distance function by the constant $1/\sqrt{\kappa}$.

A *geodesic triangle* $\triangle(x, y, z)$ in the metric space (X, d) consists of three points x, y, z in X (the *vertices* of $\triangle$) and three geodesic segments between each pair of vertices (the *edges* of $\triangle$). We write $p \in \triangle(x, y, z)$ when $p \in [x, y] \cup [y, z] \cup [z, x]$. For $\triangle(x, y, z)$ in a geodesic space X satisfying $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$, there exist points $\bar{x}, \bar{y}, \bar{z} \in M_\kappa^2$ such that

$$d(x, y) = d_{M_\kappa^2}(\bar{x}, \bar{y}), \qquad d(y, z) = d_{M_\kappa^2}(\bar{y}, \bar{z}), \quad \text{and} \quad d(z, x) = d_{M_\kappa^2}(\bar{z}, \bar{x})$$

(see [13, Lemma 2.14]). We call the triangle having vertices $\bar{x}$, $\bar{y}$, $\bar{z}$ in M_κ^2 a *comparison triangle* of $\triangle(x, y, z)$. Notice that it is unique up to an isometry of M_κ^2, and we denote it by $\overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$. A point $\bar{p} \in [\bar{x}, \bar{y}]$ is called a *comparison point* for $p \in [x, y]$ if $d(x, p) = d_{M_\kappa^2}(\bar{x}, \bar{p})$.

A geodesic triangle $\triangle(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ is said to satisfy the $\mathrm{CAT}(\kappa)$ *inequality* if for any $p, q \in \triangle(x, y, z)$ and for their comparison points $\bar{p}, \bar{q} \in \overline{\triangle}(\bar{x}, \bar{y}, \bar{z})$, one has

$$d(p, q) \leq d_{M_\kappa^2}(\bar{p}, \bar{q}).$$

Definition 2.3 A metric space (X, d) is called a $\mathrm{CAT}(\kappa)$ *space* if it is D_κ-geodesic and any geodesic triangle $\triangle(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_\kappa$ satisfies the $\mathrm{CAT}(\kappa)$ inequality.

It follows from [13, Proposition 1.4] that CAT(κ) spaces are uniquely geodesic spaces. In this paper, we consider CAT(κ) spaces with $\kappa \geq 0$. Since most of the results for such spaces are easily deduced from those for CAT(1) spaces, in what follows, we mainly focus on CAT(1) spaces. The following lemma is a consequence of Proposition 3.1 in [40].

Lemma 2.4 *If (X,d) is a CAT(1) space with $\operatorname{diam}(X) < \pi/2$, then there is a constant $K > 0$ such that*

$$d^2\big((1-\alpha)x \oplus \alpha y, z\big) \leq (1-\alpha)d^2(x,z) + \alpha d^2(y,z) - \frac{K}{2}\alpha(1-\alpha)d^2(x,y)$$

for any $\alpha \in [0,1]$ and any points $x, y, z \in X$.

The following lemma is also needed.

Lemma 2.5 [30] *Let $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences such that*

(i) $0 \leq \alpha_n, \beta_n < 1$;
(ii) $\beta_n \to 0$ *as* $n \to \infty$;
(iii) $\sum \alpha_n \beta_n = \infty$.

Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum_{n=1}^{\infty} \alpha_n \beta_n (1-\beta_n)\gamma_n$ is bounded. Then $\{\gamma_n\}$ has a subsequence which converges to zero.

3 Main results

We begin this section by proving a crucial lemma.

Lemma 3.1 *Let (X,d) be a CAT(1) space with convex metric, E be a nonempty closed convex subset of X, and $T : E \to 2^E$ be a quasi-nonexpansive mapping with $\operatorname{Fix}(T) \neq \emptyset$ and $T(p) = \{p\}$ for each $p \in \operatorname{Fix}(T)$. Let $\{x_n\}$ be the sequence of Ishikawa iterates defined by* (A) (*replacing* + *with* $\oplus$). *Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in \operatorname{Fix}(T)$.*

Proof Let $p \in \operatorname{Fix}(T)$. For each $n \geq 1$, we have

$$\begin{aligned} d(y_n, p) &= d\big(\beta_n z_n \oplus (1-\beta_n)x_n, p\big) \\ &\leq \beta_n d(z_n, p) + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n H\big(T(x_n), T(p)\big) + (1-\beta_n)d(x_n, p) \\ &\leq \beta_n d(x_n, p) + (1-\beta_n)d(x_n, p) \\ &\leq d(x_n, p) \end{aligned}$$

and

$$\begin{aligned} d(x_{n+1}, p) &= d\big(\alpha_n z_n' \oplus (1-\alpha_n)x_n, p\big) \\ &\leq \alpha_n d\big(z_n', p\big) + (1-\alpha_n)d(x_n, p) \\ &\leq \alpha_n H\big(T(y_n), T(p)\big) + (1-\alpha_n)d(x_n, p) \\ &\leq \alpha_n d(y_n, p) + (1-\alpha_n)d(x_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

This shows that the sequence $\{d(x_n,p)\}$ is decreasing and bounded below. Thus $\lim_{n\to\infty} d(x_n,p)$ exists for any $p \in \mathrm{Fix}(T)$. □

Now, we prove the strong convergence of the Ishikawa iteration process defined by (A).

Theorem 3.2 *Let* (X,d) *be a complete* $CAT(1)$ *space with convex metric and* $\mathrm{diam}(X) < \pi/2$, *E be a nonempty closed convex subset of X, and* $T: E \to \mathcal{C}(E)$ *be a quasi-nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$ *and* $T(p) = \{p\}$ *for each* $p \in \mathrm{Fix}(T)$. *Let* $\alpha_n, \beta_n \in [a,b] \subset (0,1)$ *and* $\{x_n\}$ *be the sequence of Ishikawa iterates defined by* (A) (*replacing* $+$ *with* $\oplus$). *If T satisfies condition* (I), *then* $\{x_n\}$ *converges strongly to a fixed point of T.*

Proof Let $p \in \mathrm{Fix}(T)$. By using Lemma 2.4, we have

$$\begin{aligned} d^2(x_{n+1},p) &= d^2\big(\alpha_n z'_n \oplus (1-\alpha_n)x_n, p\big) \\ &\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n d^2\big(z'_n,p\big) - \frac{K}{2}\alpha_n(1-\alpha_n)d^2\big(x_n,z'_n\big) \\ &\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n H^2\big(T(y_n),T(p)\big) - \frac{K}{2}\alpha_n(1-\alpha_n)d^2\big(x_n,z'_n\big) \\ &\leq (1-\alpha_n)d^2(x_n,p) + \alpha_n d^2(y_n,p) \end{aligned}$$

and

$$\begin{aligned} d^2(y_n,p) &= d^2\big(\beta_n z_n \oplus (1-\beta_n)x_n, p\big) \\ &\leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2(z_n,p) - \frac{K}{2}\beta_n(1-\beta_n)d^2(x_n,z_n) \\ &\leq (1-\beta_n)d^2(x_n,p) + \beta_n H^2\big(T(x_n),T(p)\big) - \frac{K}{2}\beta_n(1-\beta_n)d^2(x_n,z_n) \\ &\leq (1-\beta_n)d^2(x_n,p) + \beta_n d^2(x_n,p) - \frac{K}{2}\beta_n(1-\beta_n)d^2(x_n,z_n) \\ &\leq d^2(x_n,p) - \frac{K}{2}\beta_n(1-\beta_n)d^2(x_n,z_n). \end{aligned}$$

So that

$$d^2(x_{n+1},p) \leq (1-\alpha_n)d^2(x_n,p) + \alpha_n d^2(x_n,p) - \frac{K}{2}\alpha_n\beta_n(1-\beta_n)d^2(x_n,z_n).$$

This implies that

$$\frac{K}{2}a^2(1-b)d^2(x_n,z_n) \leq \frac{K}{2}\alpha_n\beta_n(1-\beta_n)d^2(x_n,z_n) \leq d^2(x_n,p) - d^2(x_{n+1},p) \tag{1}$$

and so

$$\sum_{n=1}^{\infty}\frac{K}{2}a^2(1-b)d^2(x_n,z_n) < \infty.$$

Thus, $\lim_{n\to\infty} d(x_n,z_n) = 0$. Also, $\mathrm{dist}(x_n,T(x_n)) \leq d(x_n,z_n) \to 0$ as $n \to \infty$. Since T satisfies condition (I), we have $\lim_{n\to\infty}\mathrm{dist}(x_n,\mathrm{Fix}(T)) = 0$. The proof of the remaining part follows the proof of Theorem 3.2 in [38], therefore we omit it. □

Theorem 3.3 *Let* (X,d) *be a complete* $CAT(1)$ *space with convex metric and* $\operatorname{diam}(X) < \pi/2$, *E be a nonempty closed convex subset of X, and* $T : E \to \mathcal{C}(E)$ *be a quasi-nonexpansive mapping with* $\operatorname{Fix}(T) \neq \emptyset$ *and* $T(p) = \{p\}$ *for each* $p \in \operatorname{Fix}(T)$. *Assume that* (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$; (iii) $\sum \alpha_n \beta_n = \infty$, *and let* $\{x_n\}$ *be the sequence of Ishikawa iterates defined by* (A) (*replacing* + *with* $\oplus$). *If T is hemicompact and continuous, then* $\{x_n\}$ *converges strongly to a fixed point of T.*

Proof Let $p \in \operatorname{Fix}(T)$. By (1) we have

$$\frac{K}{2} \sum_{n=1}^{\infty} \alpha_n \beta_n (1 - \beta_n) d^2(x_n, z_n) < \infty.$$

By Lemma 2.5, there exist subsequences $\{x_{n_k}\}$ and $\{z_{n_k}\}$ of $\{x_n\}$ and $\{z_n\}$ respectively such that $\lim_{k\to\infty} d(x_{n_k}, z_{n_k}) = 0$. Hence

$$\lim_{k\to\infty} \operatorname{dist}\bigl(x_{n_k}, T(x_{n_k})\bigr) \leq \lim_{k\to\infty} d(x_{n_k}, z_{n_k}) = 0.$$

Since T is hemicompact, by passing through a subsequence, we may assume that $x_{n_k} \to q$ for some $q \in E$. Since T is continuous,

$$\operatorname{dist}\bigl(q, T(q)\bigr) \leq d(q, x_{n_k}) + \operatorname{dist}\bigl(x_{n_k}, T(x_{n_k})\bigr) + H\bigl(T(x_{n_k}), T(q)\bigr) \to 0 \quad \text{as } k \to \infty.$$

This implies that $q \in \operatorname{Fix}(T)$ since $T(q)$ is closed. Thus $\lim_{n\to\infty} d(x_n, q)$ exists by Lemma 3.1 and hence q is the limit of $\{x_n\}$ itself. □

To avoid the restriction of T, that is, $T(p) = \{p\}$ for $p \in \operatorname{Fix}(T)$, we use the iteration process defined by (B).

Theorem 3.4 *Let* (X,d) *be a complete* $CAT(1)$ *space with convex metric and* $\operatorname{diam}(X) < \pi/2$, *E be a nonempty closed convex subset of X, and* $T : E \to \mathcal{P}(E)$ *be a multivalued mapping with* $\operatorname{Fix}(T) \neq \emptyset$ *and* P_T *is quasi-nonexpansive. Let* $\alpha_n, \beta_n \in [a,b] \subset (0,1)$ *and* $\{x_n\}$ *be the sequence of Ishikawa iterates defined by* (B) (*replacing* + *with* $\oplus$). *If T satisfies condition* (I), *then* $\{x_n\}$ *converges strongly to a fixed point of T.*

Proof It follows from Lemma 2.2 that $\operatorname{dist}(x, P_T(x)) = \operatorname{dist}(x, T(x))$ for all $x \in E$,

$$\operatorname{Fix}(P_T) = \operatorname{Fix}(T) \quad \text{and} \quad P_T(p) = \{p\} \quad \text{for each } p \in \operatorname{Fix}(P_T).$$

Since T satisfies condition (I), for each $x \in E$ we have

$$\operatorname{dist}\bigl(x, P_T(x)\bigr) = \operatorname{dist}\bigl(x, T(x)\bigr) \geq f\bigl(\operatorname{dist}\bigl(x, \operatorname{Fix}(T)\bigr)\bigr) = f\bigl(\operatorname{dist}\bigl(x, \operatorname{Fix}(P_T)\bigr)\bigr).$$

That is, P_T satisfies condition (I). Next, we show that $P_T(x)$ is closed for any $x \in E$. Let $\{y_n\} \subset P_T(x)$ and $\lim_{n\to\infty} y_n = y$ for some $y \in E$. Then

$$d(x, y_n) = \operatorname{dist}\bigl(x, T(x)\bigr) \quad \text{and} \quad \lim_{n\to\infty} d(x, y_n) = d(x, y).$$

It follows that $d(x, y) = \mathrm{dist}(x, T(x))$ and this implies $y \in P_T(x)$. Applying Theorem 3.2 to the map P_T, we can conclude that the sequence $\{x_n\}$ defined by (B) converges to a point $z \in \mathrm{Fix}(P_T) = \mathrm{Fix}(T)$. This completes the proof. □

The following theorem is an analogue of Theorem 3 in [39].

Theorem 3.5 *Let* (X, d) *be a complete* $CAT(1)$ *space with convex metric and* $\mathrm{diam}(X) < \pi/2$, E *be a nonempty closed convex subset of* X, *and* $T : E \to \mathcal{P}(E)$ *be a hemicompact mapping with* $\mathrm{Fix}(T) \neq \emptyset$ *and* P_T *is quasi-nonexpansive and continuous. Assume that* (i) $0 \leq \alpha_n, \beta_n < 1$; (ii) $\beta_n \to 0$; (iii) $\sum \alpha_n \beta_n = \infty$, *and let* $\{x_n\}$ *be the sequence of Ishikawa iterates defined by* (B) (*replacing* + *with* $\oplus$). *Then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof As in the proof of Theorem 3.4, we have

$$\mathrm{Fix}(P_T) = \mathrm{Fix}(T) \quad \text{and} \quad P_T(p) = \{p\} \quad \text{for each } p \in \mathrm{Fix}(P_T).$$

The hemicompactness of P_T follows from that of T. The conclusion follows from Theorem 3.3. □

Competing interests
The author declares that he has no competing interests.

Authors' contributions
The author completed the paper himself. The author read and approved the final manuscript.

Acknowledgements
The author thanks Chiang Mai University for financial support.

Received: 4 July 2013 Accepted: 9 October 2013 Published: 02 Jan 2014

References

1. Kirk, WA: Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003). Colecc. Abierta, vol. 64, pp. 195-225. Univ. Sevilla Secr. Publ., Seville (2003)
2. Kirk, WA: Geodesic geometry and fixed point theory II. In: International Conference on Fixed Point Theory and Applications, pp. 113-142. Yokohama Publ., Yokohama (2004)
3. Dhompongsa, S, Kaewkhao, A, Panyanak, B: Lim's theorems for multivalued mappings in CAT(0) spaces. J. Math. Anal. Appl. **312**, 478-487 (2005)
4. Chaoha, P, Phon-on, A: A note on fixed point sets in CAT(0) spaces. J. Math. Anal. Appl. **320**, 983-987 (2006)
5. Leustean, L: A quadratic rate of asymptotic regularity for CAT(0)-spaces. J. Math. Anal. Appl. **325**, 386-399 (2007)
6. Shahzad, N, Markin, J: Invariant approximations for commuting mappings in CAT(0) and hyperconvex spaces. J. Math. Anal. Appl. **337**, 1457-1464 (2008)
7. Espinola, R, Fernandez-Leon, A: CAT(κ)-spaces, weak convergence and fixed points. J. Math. Anal. Appl. **353**, 410-427 (2009)
8. Khan, AR, Khamsi, MA, Fukhar-ud-din, H: Strong convergence of a general iteration scheme in CAT(0) spaces. Nonlinear Anal. **74**, 783-791 (2011)
9. Abkar, A, Eslamian, M: Common fixed point results in CAT(0) spaces. Nonlinear Anal. **74**, 1835-1840 (2011)
10. Piatek, B: Halpern iteration in CAT(κ) spaces. Acta Math. Sin. Engl. Ser. **27**, 635-646 (2011)
11. He, JS, Fang, DH, Lopez, G, Li, C: Mann's algorithm for nonexpansive mappings in CAT(κ) spaces. Nonlinear Anal. **75**, 445-452 (2012)
12. Kimura, Y, Sato, K: Convergence of subsets of a complete geodesic space with curvature bounded above. Nonlinear Anal. **75**, 5079-5085 (2012)
13. Bridson, M, Haefliger, A: Metric Spaces of Non-positive Curvature. Springer, Berlin (1999)
14. Espinola, R, Lorenzo, P, Nicolae, A: Fixed points, selections and common fixed points for nonexpansive-type mappings. J. Math. Anal. Appl. **382**, 503-515 (2011)
15. Ishikawa, S: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**, 147-150 (1974)
16. Ishikawa, S: Fixed points and iteration of a nonexpansive mapping in a Banach space. Proc. Am. Math. Soc. **59**, 65-71 (1976)
17. Chidume, CE: On the Ishikawa fixed points iterations for quasicontractive mappings. J. Niger. Math. Soc. **4**, 1-11 (1985)
18. Kalinde, AK, Rhoades, BE: Fixed point Ishikawa iterations. J. Math. Anal. Appl. **170**, 600-606 (1992)

19. Tan, KK, Xu, HK: Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process. J. Math. Anal. Appl. **178**, 301-308 (1993)
20. Ghosh, MK, Debnath, L: Convergence of Ishikawa iterates of quasi-nonexpansive mappings. J. Math. Anal. Appl. **207**, 96-103 (1997)
21. Takahashi, W, Kim, GE: Approximating fixed points of nonexpansive mappings in Banach spaces. Math. Jpn. **48**, 1-9 (1998)
22. Osilike, MO: Strong and weak convergence of the Ishikawa iteration method for a class of nonlinear equations. Bull. Korean Math. Soc. **37**, 153-169 (2000)
23. Liu, Z, Kim, JK, Ume, JS: Characterizations for the convergence of Ishikawa iterative processes with errors in normed linear spaces. J. Nonlinear Convex Anal. **3**, 59-66 (2002)
24. Sahu, DR: On generalized Ishikawa iteration process and nonexpansive mappings in Banach spaces. Demonstr. Math. **36**, 721-734 (2003)
25. Kim, GE, Kiuchi, H, Takahashi, W: Weak and strong convergences of Ishikawa iterations for asymptotically nonexpansive mappings in the intermediate sense. Sci. Math. Jpn. **60**, 95-106 (2004)
26. Cho, YJ, Li, J, Huang, NJ: Random Ishikawa iterative sequence with errors for approximating random fixed points. Taiwan. J. Math. **12**, 51-61 (2008)
27. Peng, JW, Yao, JC: Ishikawa iterative algorithms for a generalized equilibrium problem and fixed point problems of a pseudo-contraction mapping. J. Glob. Optim. **46**, 331-345 (2010)
28. Zegeye, H, Shahzad, N, Alghamdi, MA: Convergence of Ishikawa's iteration method for pseudocontractive mappings. Nonlinear Anal. **74**, 7304-7311 (2011)
29. Sastry, KPR, Babu, GVR: Convergence of Ishikawa iterates for a multi-valued mapping with a fixed point. Czechoslov. Math. J. **55**, 817-826 (2005)
30. Panyanak, B: Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces. Comput. Math. Appl. **54**, 872-877 (2007)
31. Song, Y, Wang, H: Erratum to "Mann and Ishikawa iterative processes for multivalued mappings in Banach spaces". Comput. Math. Appl. **52**, 2999-3002 (2008)
32. Shahzad, N, Zegeye, H: Strong convergence results for nonself multimaps in Banach spaces. Proc. Am. Math. Soc. **136**, 539-548 (2008)
33. Laowang, W, Panyanak, B: Strong and Δ convergence theorems for multivalued mapping in CAT(0) spaces. J. Inequal. Appl. **2009**, Article ID 730132 (2009)
34. Sokhuma, K, Kaewkhao, A: Ishikawa iterative process for a pair of single-valued and multivalued nonexpansive mappings in Banach spaces. Fixed Point Theory Appl. **2010**, Article ID 618767 (2010)
35. Zegeye, H, Shahzad, N: Viscosity approximation methods for nonexpansive multimaps in Banach spaces. Acta Math. Sin. Engl. Ser. **26**, 1165-1176 (2010)
36. Cholamjiak, W, Suantai, S: Approximation of common fixed points of two quasi-nonexpansive multi-valued maps in Banach spaces. Comput. Math. Appl. **61**, 941-949 (2011)
37. Shahzad, N, Zegeye, H: On Mann and Ishikawa iteration schemes for multi-valued maps in Banach spaces. Nonlinear Anal. **71**, 838-844 (2009)
38. Puttasontiphot, T: Mann and Ishikawa iteration schemes for multivalued mappings in CAT(0) spaces. Appl. Math. Sci. **4**, 3005-3018 (2010)
39. Song, Y, Cho, YJ: Some notes on Ishikawa iteration for multi-valued mappings. Bull. Korean Math. Soc. **48**, 575-584 (2011)
40. Ohta, S: Convexities of metric spaces. Geom. Dedic. **125**, 225-250 (2007)

10.1186/1687-1812-2014-1
Cite this article as: Panyanak: **On the Ishikawa iteration processes for multivalued mappings in some CAT(κ) spaces.** *Fixed Point Theory and Applications* 2014, 2014:1

Submit your manuscript to a SpringerOpen journal and benefit from:

- ► Convenient online submission
- ► Rigorous peer review
- ► Immediate publication on acceptance
- ► Open access: articles freely available online
- ► High visibility within the field
- ► Retaining the copyright to your article

Submit your next manuscript at ► springeropen.com