Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems of generalized cyclic orbital Meir-Keeler contractions

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Abstract
In this paper, we introduce two class of generalized cyclic orbital Meir-Keeler contractions and we study the existence and uniqueness of fixed points for these mappings. Our results in this paper extend and generalize several existing fixed-point theorems in the literature.
MSC: 47H10; 54C60; 54H25; 55M20

Keywords: fixed-point theorem; cyclic map; generalized cyclic orbital Meir-Keeler contraction

1 Introduction and preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all non-negative numbers, while $\mathbb{N}$ is the set of all natural numbers. It is well known and easy to prove that if (X,d) is a complete metric space, and if $f: X \to X$ is continuous and f satisfies

$$d(fx, f^2x) \leq k \cdot d(x, fx), \quad \text{for all } x \in X \text{ and } k \in (0,1),$$

then f has a fixed point in X. Using the above conclusion, Kirk, Srinivasan and Veeramani [1] proved the following fixed-point theorem.

Theorem 1 [1] *Let A and B be two nonempty closed subsets of a complete metric space (X,d), and suppose $f: A \cup B \to A \cup B$ satisfies*
(i) *$f(A) \subset B$ and $f(B) \subset A$,*
(ii) *$d(fx, fy) \leq k \cdot d(x,y)$ for all $x \in A$, $y \in B$ and $k \in (0,1)$.*
Then $A \cap B$ is nonempty and f has a unique fixed point in $A \cap B$.

The following definitions and results will be needed in the sequel. Let A and B be two nonempty subsets of a metric space (X,d). A mapping $f: A \cup B \to A \cup B$ is called a cyclic map if $f(A) \subseteq B$ and $f(B) \subseteq A$. In 2010, Karpagam and Agrawal [2] introduced the notion of cyclic orbital contraction, and obtained a unique fixed point theorem for such a map.

Definition 1 [2] Let A and B be nonempty subsets of a metric space (X,d), $f: A \cup B \to A \cup B$ be a cyclic map such that for some $x \in A$, there exists a $\kappa_x \in (0,1)$ such that

$$d(f^{2n}x, fy) \leq \kappa_x \cdot d(f^{2n-1}x, y), \quad n \in \mathbb{N}, y \in A.$$

Then f is called a cyclic orbital contraction.

Theorem 2 [2] *Let A and B be two nonempty closed subsets of a complete metric space (X, d), and let $f : A \cup B \to A \cup B$ be a cyclic orbital contraction. Then f has a fixed point in $A \cap B$.*

Further, many results dealing with cyclic contractions have appeared in the literature (see, *e.g.*, [3–16]).

In 2012, Chen [17] introduced the below notion of cyclic orbital stronger Meir-Keeler contraction, and obtained a unique fixed-point theorem for such class of mappings.

Definition 2 [17] Let (X, d) be a metric space. We call $\psi : \mathbb{R}^+ \to [0, 1)$ a stronger Meir-Keeler type mapping in X if the mapping ψ satisfies the following condition:

$$\forall \eta > 0, \exists \delta > 0, \exists \gamma_\eta \in [0, 1), \forall x, y \in X \quad \bigl(\eta \le d(x, y) < \delta + \eta \Rightarrow \psi\bigl(d(x, y)\bigr) < \gamma_\eta\bigr).$$

Definition 3 [17] Let A and B be nonempty subsets of a metric space (X, d). Suppose $f : A \cup B \to A \cup B$ is a cyclic map such that for some $x \in A$, there exists a stronger Meir-Keeler type mapping $\psi_x : \mathbb{R}^+ \to [0, 1)$ in X such that

$$d\bigl(f^{2n}x, fy\bigr) \le \psi_x\bigl(d\bigl(f^{2n-1}x, y\bigr)\bigr) \cdot d\bigl(f^{2n-1}x, y\bigr),$$

for all $n \in \mathbb{N}$ and $y \in A$. Then f is called a cyclic orbital stronger Meir-Keeler ψ_x-contraction.

Clearly, if $f : A \cup B \to A \cup B$ is a cyclic orbital contraction, then f is a cyclic orbital stronger Meir-Keeler ψ_x-contraction, where $\psi_x(t) = k_x$ for all $t \in \mathbb{R}^+$.

Theorem 3 [17] *Let A and B be two nonempty closed subsets of a complete metric space (X, d), and let $\psi_x : \mathbb{R}^+ \to [0, 1)$ be a stronger Meir-Keeler type mapping in X. Suppose $f : A \cup B \to A \cup B$ is a cyclic orbital stronger Meir-Keeler ψ_x-contraction. Then $A \cap B$ is nonempty and f has a unique fixed point in $A \cap B$.*

Chen [17] also introduced the below notion of cyclic orbital weaker Meir-Keeler contraction, and obtained a unique fixed-point theorem for such class of mappings.

Definition 4 [17] Let (X, d) be a metric space, and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$. Then ψ is called a weaker Meir-Keeler type mapping in X, if the mapping ψ satisfies the following condition:

$$\forall \eta > 0, \exists \delta > 0, \forall x, y \in X \quad \bigl(\eta \le d(x, y) < \delta + \eta \Rightarrow \exists n_0 \in \mathbb{N}\ \psi^{n_0}\bigl(d(x, y)\bigr) < \eta\bigr).$$

Definition 5 [17] Let (X, d) be a metric space. We call $f : \mathbb{R}^+ \to \mathbb{R}^+$ a ψ-mapping in X if the function f satisfies the following conditions:

(ψ_1) f is a weaker Meir-Keeler type mapping in X with $f(0) = 0$;
(ψ_2) (a) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} f(t_n) \le \gamma$, and
(b) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} f(t_n) = 0$;
(ψ_3) $\{f^n(t)\}_{n\in\mathbb{N}}$ is decreasing, for each $t \in \mathbb{R}^+ \setminus \{0\}$.

Definition 6 [17] Let A and B be nonempty subsets of a metric space (X,d). Suppose $f : A \cup B \to A \cup B$ is a cyclic map such that for some $x \in A$, there exists a ψ-mapping $\psi_x : \mathbb{R}^+ \to \mathbb{R}^+$ in X such that

$$d\big(f^{2n}x,fy\big) \leq \psi_x\big(d\big(f^{2n-1}x,y\big)\big),$$

for all $n \in \mathbb{N}$ and $y \in A$. Then f is called a cyclic orbital weaker Meir-Keeler ψ_x-contraction.

Theorem 4 [17] *Let A and B be two nonempty closed subsets of a complete metric space (X,d), and let $\psi_x : \mathbb{R}^+ \to \mathbb{R}^+$ be a ψ-mapping in X. Suppose $f : A \cup B \to A \cup B$ is a cyclic orbital weaker Meir-Keeler ψ_x-contraction. Then $A \cap B$ is nonempty and f has a unique fixed point in $A \cap B$.*

2 Fixed-point theorems (I)

In this section, we will introduce the class of generalized cyclic orbital stronger Meir-Keeler (ψ_x,φ)-contraction and we study the existence and uniqueness of fixed points for such mappings. Our results in this section extend and generalize several existing fixed-point theorems in the literature, including Theorem 2 and Theorem 3.

In the sequel, we denote by Θ the class of functions $\varphi : \mathbb{R}^{+5} \to \mathbb{R}^+$ satisfying the following conditions:

(φ_1) φ is a strictly increasing, continuous function in each coordinate;
(φ_2) for all $t > 0$, $\varphi(t,t,t,0,2t) < t$, $\varphi(t,t,t,2t,0) < t$, $\varphi(t,0,0,t,t) < t$, $\varphi(0,0,t,t,0) < t$, and $\varphi(0,0,0,0,0) = 0$.

Example 1 Let $\varphi : \mathbb{R}^{+5} \to \mathbb{R}^+$ denote

$$\varphi(t_1,t_2,t_3,t_4,t_5) = \frac{2}{3} \cdot \max\left\{t_1,t_2,t_3,\frac{1}{2}t_4,\frac{1}{2}t_5\right\}.$$

Then φ satisfies the above conditions (φ_1) and (φ_2).

We now denote the below notion of generalized cyclic orbital stronger Meir-Keeler (ψ_x,φ)-contraction.

Definition 7 Let A and B be nonempty subsets of a metric space (X,d). Suppose $f : A \cup B \to A \cup B$ is a cyclic map such that for some $x \in A$, there exist a stronger Meir-Keeler type mapping $\psi_x : \mathbb{R}^+ \to [0,1)$ in X and $\varphi \in \Theta$ such that

$$d\big(f^{2n}x,fy\big) \leq \psi_x\big(d\big(f^{2n-1}x,y\big)\big) \cdot \theta,$$

where

$$\theta = \varphi\big(d\big(f^{2n-1}x,y\big),d\big(f^{2n-1}x,f^{2n}x\big),d(fy,y),d\big(f^{2n-1}x,fy\big),d\big(f^{2n}x,y\big)\big)$$

for all $n \in \mathbb{N}$ and $y \in A$. Then f is called a generalized cyclic orbital stronger Meir-Keeler (ψ_x,φ)-contraction.

Our main result is the following.

Theorem 5 *Let A and B be two nonempty closed subsets of a complete metric space (X,d), and let $\psi_x:\mathbb{R}^+\to[0,1)$ be a stronger Meir-Keeler type mapping in X and $\varphi\in\Theta$. Suppose $f:A\cup B\to A\cup B$ is a generalized cyclic orbital stronger Meir-Keeler (ψ_x,φ)-contraction. Then $A\cap B$ is nonempty and f has a unique fixed point in $A\cap B$.*

Proof Since $f:A\cup B\to A\cup B$ is a generalized cyclic orbital stronger Meir-Keeler (ψ_x,φ)-contraction and for $x\in A$, we have $f^{2n}x\in A$. Put $y=f^{2n}x$, for $n\in\mathbb{N}$. Then we have that for each $n\in\mathbb{N}$

$$\begin{aligned}
&d\bigl(f^{2n}x,f^{2n+1}x\bigr)\le\psi_x\bigl(d\bigl(f^{2n-1}x,f^{2n}x\bigr)\bigr)\cdot\theta,\\
&\theta=\varphi\bigl(d\bigl(f^{2n-1}x,f^{2n}x\bigr),d\bigl(f^{2n-1}x,f^{2n}x\bigr),d\bigl(f^{2n+1}x,f^{2n}x\bigr),d\bigl(f^{2n-1}x,f^{2n+1}x\bigr),d\bigl(f^{2n}x,f^{2n}x\bigr)\bigr)\\
&\quad=\varphi\bigl(d\bigl(f^{2n-1}x,f^{2n}x\bigr),d\bigl(f^{2n-1}x,f^{2n}x\bigr),d\bigl(f^{2n+1}x,f^{2n}x\bigr),d\bigl(f^{2n-1}x,f^{2n}x\bigr)\\
&\qquad+d\bigl(f^{2n}x,f^{2n+1}x\bigr),0\bigr)
\end{aligned}$$

and by the conditions of the function φ, we get

$$\theta<d\bigl(f^{2n-1}x,f^{2n}x\bigr),$$

and

$$\begin{aligned}
d\bigl(f^{2n}x,f^{2n+1}x\bigr)&<\psi_x\bigl(d\bigl(f^{2n-1}x,f^{2n}x\bigr)\bigr)\cdot d\bigl(f^{2n-1}x,f^{2n}x\bigr)\\
&\le d\bigl(f^{2n-1}x,f^{2n}x\bigr).
\end{aligned}\tag{2.1}$$

Similarly, we put $y=f^{2n}x$ and for each $n\in\mathbb{N}$

$$\begin{aligned}
&d\bigl(f^{2n+1}x,f^{2n+2}x\bigr)=d\bigl(f^{2n+2}x,f^{2n+1}x\bigr)\\
&\qquad\qquad\le\psi_x\bigl(d\bigl(f^{2n+1}x,f^{2n}x\bigr)\bigr)\cdot\theta,\\
&\theta=\varphi\bigl(d\bigl(f^{2n+1}x,f^{2n}x\bigr),d\bigl(f^{2n+1}x,f^{2n+2}x\bigr),d\bigl(f^{2n+1}x,f^{2n}x\bigr),\\
&\qquad d\bigl(f^{2n+1}x,f^{2n+1}x\bigr),d\bigl(f^{2n+2}x,f^{2n}x\bigr)\bigr)\\
&\quad=\varphi\bigl(d\bigl(f^{2n+1}x,f^{2n}x\bigr),d\bigl(f^{2n+1}x,f^{2n+2}x\bigr),d\bigl(f^{2n+1}x,f^{2n}x\bigr),0,d\bigl(f^{2n}x,f^{2n+1}x\bigr)\\
&\qquad+d\bigl(f^{2n+1}x,f^{2n+2}x\bigr)\bigr)
\end{aligned}$$

and by the conditions of the function φ, we get

$$\theta<d\bigl(f^{2n}x,f^{2n+1}x\bigr),$$

and

$$\begin{aligned}
d\bigl(f^{2n+1}x,f^{2n+2}x\bigr)&<\psi_x\bigl(d\bigl(f^{2n+1}x,f^{2n}x\bigr)\bigr)\cdot d\bigl(f^{2n}x,f^{2n+1}x\bigr)\\
&\le d\bigl(f^{2n}x,f^{2n+1}x\bigr).
\end{aligned}\tag{2.2}$$

Using inequalities (2.1) and (2.2), we deduce that $\{d(f^n x, f^{n+1}x)\}$ is a decreasing sequence and hence it is convergent. Let $\lim_{n\to\infty} d(f^n x, f^{n+1}x) = \eta$. Then there exists $\kappa_0 \in \mathbb{N}$ and $\delta > 0$ such that for all $n \geq \kappa_0$,

$$\eta \leq d(f^n x, f^{n+1}x) < \eta + \delta.$$

Taking into account the above inequality and the definition of stronger Meir-Keeler type mapping ψ_x in X, corresponding to η use, there exists $\gamma_\eta \in [0,1)$ such that

$$\psi_x\bigl(d(f^n x, f^{n+1}x)\bigr) < \gamma_\eta \quad \text{for all } n \geq \kappa_0. \tag{2.3}$$

Put $n_0 = [\frac{\kappa_0+3}{2}]$, where $[\frac{\kappa_0+3}{2}]$ is the integer part of $\frac{\kappa_0+3}{2}$. It follows from (2.1), (2.2) and (2.3) that we deduce that for all $n \geq n_0$,

$$\begin{aligned} d(f^{2n}x, f^{2n+1}x) &< \psi_x\bigl(d(f^{2n-1}x, f^{2n}x)\bigr) \cdot d(f^{2n-1}x, f^{2n}x) \\ &< \gamma_\eta \cdot d(f^{2n-1}x, f^{2n}x), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(f^{2n+1}x, f^{2n+2}x) &< \psi_x\bigl(d(f^{2n+1}x, f^{2n}x)\bigr) \cdot d(f^{2n}x, f^{2n+1}x) \\ &< \gamma_\eta \cdot d(f^{2n}x, f^{2n+1}x). \end{aligned} \tag{2.5}$$

It follows from (2.4) and (2.5) that for each $n \in \mathbb{N} \cup \{0\}$

$$d(f^{2n_0+n}x, f^{2n_0+n+1}x) < \gamma_\eta^n \cdot d(f^{2n_0-1}x, f^{2n_0}x). \tag{2.6}$$

Since $\gamma_\eta < 1$, we get

$$\lim_{n\to\infty} d(f^{2n_0+n}x, f^{2n_0+n+1}x) = 0.$$

For $m, n \in \mathbb{N}$ with $m > n$, we have

$$d(f^{2n_0+n}x, f^{2n_0+m}x) \leq \sum_{i=n}^{m-1} d(f^{2n_0+i}x, f^{2n_0+i+1}x) < \frac{\gamma_\eta^{m-1}}{1-\gamma_\eta} d(f^{2n_0}x, f^{2n_0+1}x),$$

and hence $d(f^n x, f^m x) \to 0$, since $0 < \gamma_\eta < 1$. So, $\{f^n x\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, A and B are closed, $\{f^n x\} \subset A \cup B$, there exists $\nu \in A \cup B$ such that $\lim_{n\to\infty} f^n x = \nu$. Now $\{f^{2n}x\}$ is a sequence in A and $\{f^{2n+1}x\}$ is a sequence in B, and also both converge to ν. Since A and B are closed, $\nu \in A \cap B$, and so $A \cap B$ is nonempty. Next, we want to show that ν is a fixed point of f. Suppose that ν is not a fixed point of f. Then $d(\nu, f\nu) > 0$. Since $\lim_{n\to\infty} d(f^{2n-1}x, \nu) = 0$ and

$$d(f^{2n}x, f\nu) \leq \psi_x\bigl(d(f^{2n-1}x, \nu)\bigr) \cdot \theta,$$

where

$$\theta = \varphi\bigl(d(f^{2n-1}x, \nu), d(f^{2n-1}x, f^{2n}x), d(f\nu, \nu), d(f^{2n-1}x, f\nu), d(f^{2n}x, \nu)\bigr),$$

we obtain that

$$\begin{aligned}
d(\nu, f\nu) &= \lim_{n\to\infty} d\big(f^{2n}x, f\nu\big) \\
&\le \gamma_\eta \cdot \varphi\big(d(\nu,\nu), d(\nu,\nu), d(f\nu,\nu), d(\nu, f\nu), d(\nu,\nu)\big) \\
&\le \varphi\big(0, 0, d(\nu, f\nu), d(\nu, f\nu), 0\big) \\
&< d(\nu, f\nu).
\end{aligned}$$

This leads to a contradiction. So, $d(\nu, f\nu) = 0$, that is, ν is a fixed point of f.

Finally, we want to show the uniqueness of the fixed point. Let μ be another fixed point of f. By the cyclic character of f, we have $\nu, \mu \in A \cap B$. Since f is a generalized cyclic orbital stronger Meir-Keeler (ψ_x, φ)-contraction, we have

$$d(\nu, \mu) = d(\nu, f\mu) = \lim_{n\to\infty} d\big(f^{2n}x, f\mu\big), \tag{2.7}$$

and

$$d\big(f^{2n}x, f\mu\big) \le \psi_x\big(d\big(f^{2n-1}x, \mu\big)\big) \cdot \theta < \gamma_\eta \cdot \theta, \tag{2.8}$$

where

$$\theta = \varphi\big(d\big(f^{2n-1}x, \mu\big), d\big(f^{2n-1}x, f^{2n}x\big), d(f\mu, \mu), d\big(f^{2n-1}x, f\mu\big), d\big(f^{2n}x, \mu\big)\big).$$

It follows from (2.7), (2.8) and the condition (φ_2) of the mapping φ that

$$\begin{aligned}
d(\nu, \mu) &< \gamma_\eta \cdot \varphi\big(d(\nu,\mu), d(\nu,\nu), d(f\mu,\mu), d(\nu, f\mu), d(\nu,\mu)\big) \\
&\le \varphi\big(d(\nu,\mu), 0, 0, d(\nu,\mu), d(\nu,\mu)\big) \\
&< d(\nu, \mu).
\end{aligned}$$

This leads to a contradiction. Therefore, $\nu = \mu$, and so ν is the unique fixed point of f. □

We give the following example to illustrate Theorem 5.

Example 2 Let $A = B = X = \mathbb{R}^+$ and we define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = |x - y|, \quad \text{for } x, y \in X,$$

and let $f : X \to X$ denote

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < 1; \\ \frac{1}{16}, & \text{if } x \ge 1. \end{cases}$$

We next define $\psi_x : \mathbb{R}^+ \to [0, 1)$ by

$$\psi_x(t) = \begin{cases} \frac{1}{3}, & \text{if } 0 \le t \le 1; \\ \frac{t}{t+1}, & \text{if } t > 1, \end{cases}$$

and let $\varphi : \mathbb{R}^{+5} \to \mathbb{R}^+$ denote

$$\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2} \cdot \max\left\{t_1, t_2, t_3, \frac{1}{2}t_4, \frac{1}{2}t_5\right\}.$$

Then f is a generalized cyclic orbital stronger Meir-Keeler (ψ_x, φ)-contraction and 0 is the unique fixed point.

3 Fixed-point theorems (II)

In this section, we will introduce the class of generalized cyclic orbital weaker Meir-Keeler (ψ_x, ϕ)-contraction and we study the existence and uniqueness of fixed points for such mappings.

In the sequel, we denote by Φ the class of functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(ϕ_1) ϕ is lower semi-continuous, and
(ϕ_2) $\phi(0) = 0$ if and only if $t = 0$.

Definition 8 Let A and B be nonempty subsets of a metric space (X, d). Suppose $f : A \cup B \to A \cup B$ is a cyclic map such that for some $x \in A$, there exist a ψ-mapping $\psi_x : \mathbb{R}^+ \to \mathbb{R}^+$ in X and $\phi \in \Phi$ such that

$$d\bigl(f^{2n}x, fy\bigr) \le \psi_x\bigl(d\bigl(f^{2n-1}x, y\bigr)\bigr) - \phi\bigl(d\bigl(f^{2n-1}x, y\bigr)\bigr), \quad n \in \mathbb{N}, y \in A. \tag{3.1}$$

Then f is called a generalized cyclic orbital weaker Meir-Keeler (ψ_x, ϕ)-contraction.

Our second main result is the following.

Theorem 6 *Let A and B be two nonempty closed subsets of a complete metric space (X, d), and let $\psi_x : \mathbb{R}^+ \to \mathbb{R}^+$ be a ψ-mapping in X and $\phi \in \Phi$. Suppose $f : A \cup B \to A \cup B$ is a generalized cyclic orbital weaker Meir-Keeler (ψ_x, ϕ)-contraction. Then $A \cap B$ is nonempty and f has a unique fixed point in $A \cap B$.*

Proof Since $f : A \cup B \to A \cup B$ is a generalized cyclic orbital weaker Meir-Keeler (ψ_x, ϕ)-contraction and for $x \in X$, there exist a ψ-mapping $\psi_x : \mathbb{R}^+ \to \mathbb{R}^+$ in X and $\phi \in \Phi$ such that (3.1) is satisfied. Put $y = f^{2n}x$ for all $n \in \mathbb{N}$. Then we have that for each $n \in \mathbb{N}$

$$\begin{aligned} d\bigl(f^{2n}x, f^{2n+1}x\bigr) &\le \psi_x\bigl(d\bigl(f^{2n-1}x, f^{2n}x\bigr)\bigr) - \phi\bigl(d\bigl(f^{2n-1}x, f^{2n}x\bigr)\bigr) \\ &\le \psi_x\bigl(d\bigl(f^{2n-1}x, f^{2n}x\bigr)\bigr), \end{aligned}$$

and

$$\begin{aligned} d\bigl(f^{2n+1}x, f^{2n+2}x\bigr) &= d\bigl(f^{2n+2}x, f^{2n+1}x\bigr) \\ &\le \psi_x\bigl(d\bigl(f^{2n+1}x, f^{2n}x\bigr)\bigr) - \phi\bigl(d\bigl(f^{2n+1}x, f^{2n}x\bigr)\bigr) \\ &\le \psi_x\bigl(d\bigl(f^{2n+1}x, f^{2n}x\bigr)\bigr). \end{aligned}$$

Generally, we have that for each $n \in \mathbb{N}$

$$d\bigl(f^n x, f^{n+1} x\bigr) \le \psi_x\bigl(d\bigl(f^{n-1} x, f^n x\bigr)\bigr),$$

and so we conclude that for each $n \in \mathbb{N}$

$$\begin{aligned} d\bigl(f^n x, f^{n+1} x\bigr) &\le \psi_x\bigl(d\bigl(f^{n-1} x, f^n x\bigr)\bigr) \\ &\le \psi_x^2\bigl(d\bigl(f^{n-2} x, f^{n-1} x\bigr)\bigr) \\ &\le \cdots \\ &\le \psi_x^n\bigl(d(x, fx)\bigr). \end{aligned}$$

Since $\{\psi_x^n(d(x,fx))\}_{n\in\mathbb{N}}$ is decreasing, it must converge to some $\eta \ge 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of weaker Meir-Keeler type mapping ψ_x in X, there exists $\delta > 0$ such that for $x, y \in X$ with $\eta \le d(x,y) < \delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\psi_x^{n_0}(d(x,y)) < \eta$. Since $\lim_{n\to\infty} \psi_x^n(d(x,fx)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \le \psi_x^m(d(x,fx)) < \delta + \eta$, for all $m \ge m_0$. Thus, we conclude that $\psi_x^{m_0+n_0}(d(x_0,x_1)) < \eta$, and we get a contradiction. So, $\lim_{n\to\infty} \psi_x^n(d(x,fx)) = 0$, that is,

$$\lim_{n\to\infty} d\bigl(f^n x, f^{n+1} x\bigr) = 0. \tag{3.2}$$

We now claim that $\{f^n x\}$ is a Cauchy sequence. It is sufficient to show that $\{f^{2n} x\}$ is a Cauchy sequence. Suppose $\{f^{2n} x\}$ is not Cauchy. Then there exists $\varepsilon > 0$ such that for all $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \ge k$ satisfying:

(i) $d(f^{2m_k} x, f^{2n_k}) \ge \varepsilon$, and
(ii) m_k is the smallest number greater than n_k such that the condition (i) holds.

Using (3.2), we have

$$\begin{aligned} \varepsilon &\le d\bigl(f^{2m_k} x, f^{2n_k}\bigr) \le d\bigl(f^{2m_k} x, f^{2m_k-1}\bigr) + d\bigl(f^{2m_k-1} x, f^{2m_k-2}\bigr) + d\bigl(f^{2m_k-2} x, f^{2n_k}\bigr) \\ &\le d\bigl(f^{2m_k} x, f^{2m_k-1}\bigr) + d\bigl(f^{2m_k-1} x, f^{2m_k-2}\bigr) + \varepsilon. \end{aligned}$$

Let $k \to \infty$, we get

$$\lim_{n\to\infty} d\bigl(f^{2m_k} x, f^{2n_k}\bigr) = \varepsilon. \tag{3.3}$$

On the other hand, applying (3.1) with $y = f^{2n_k} x$ for all $k \in \mathbb{N}$, we get

$$d\bigl(f^{2m_k} x, f^{2n_k+1}\bigr) \le \psi_x\bigl(d\bigl(f^{2m_k-1} x, f^{2n_k}\bigr)\bigr) - \phi\bigl(d\bigl(f^{2m_k-1} x, f^{2n_k}\bigr)\bigr). \tag{3.4}$$

Since for each $k \in \mathbb{N}$

$$d\bigl(f^{2m_k} x, f^{2n_k+1}\bigr) \le d\bigl(f^{2m_k} x, f^{2n_k}\bigr) + d\bigl(f^{2n_k} x, f^{2n_k+1}\bigr), \tag{3.5}$$

and

$$d\bigl(f^{2m_k-1} x, f^{2n_k}\bigr) \le d\bigl(f^{2m_k-1} x, f^{2m_k}\bigr) + d\bigl(f^{2m_k} x, f^{2n_k}\bigr), \tag{3.6}$$

taking $k \to \infty$ and using the inequalities (3.3), (3.5) and (3.6), we get

$$\lim_{n\to\infty} d\bigl(f^{2m_k}x, f^{2n_k+1}\bigr) = \varepsilon, \tag{3.7}$$

and

$$\lim_{n\to\infty} d\bigl(f^{2m_k-1}x, f^{2n_k}\bigr) = \varepsilon. \tag{3.8}$$

Taking into account the inequalities (3.4), (3.7) and (3.8), and by the definitions of the functions ϕ and ψ_x , we get

$$\begin{aligned}\varepsilon &= \lim_{n\to\infty} d\bigl(f^{2m_k}x, f^{2n_k+1}\bigr)\\ &\leq \lim_{n\to\infty} \psi_x\bigl(d\bigl(f^{2m_k-1}x, f^{2n_k}\bigr)\bigr) - \lim_{n\to\infty} \phi\bigl(d\bigl(f^{2m_k-1}x, f^{2n_k}\bigr)\bigr)\\ &\leq \varepsilon - \phi(\varepsilon),\end{aligned}$$

which implies that $\varepsilon = 0$. Thus, $\{f^n x\}$ is a Cauchy sequence.

Since (X, d) is a complete metric space, A and B are closed, $\{f^n x\} \subset A \cup B$, there exists $\nu \in A \cup B$ such that $\lim_{n\to\infty} f^n x = \nu$. Now $\{f^{2n}x\}$ is a sequence in A and $\{f^{2n+1}x\}$ is a sequence in B, and also both converge to ν. Since A and B are closed, $\nu \in A \cap B$, and so $A \cap B$ is nonempty. On the other hand, since $\lim_{n\to\infty} d(f^{2n-1}x, \nu) = 0$ and

$$d\bigl(f^{2n}x, f\nu\bigr) \leq \psi_x\bigl(d\bigl(f^{2n-1}x, \nu\bigr)\bigr) - \phi\bigl(d\bigl(f^{2n-1}x, \nu\bigr)\bigr),$$

taking $n \to \infty$,we obtain that

$$d(\nu, f\nu) \leq 0 - \phi\bigl(d(\nu, \nu)\bigr) = 0,$$

and hence $d(\nu, f\nu) = 0$, that is, ν is a fixed point of f.

Finally, we want to show the uniqueness of the fixed point. Let μ be another fixed point of f. By the cyclic character of f, we have $\nu, \mu \in A \cap B$. Since f is a generalized cyclic orbital weaker Meir-Keeler (ψ_x, ϕ)-contraction, we have

$$d\bigl(f^{2n}x, f\mu\bigr) \leq \psi_x\bigl(d\bigl(f^{2n-1}x, \mu\bigr)\bigr) - \phi\bigl(d\bigl(f^{2n-1}x, \mu\bigr)\bigr).$$

Letting $n \to \infty$, and by the definitions of the functions ϕ and ψ_x, we obtain that

$$d(\nu, \mu) = d(\nu, f\mu) = \lim_{n\to\infty} d\bigl(f^{2n}x, f\mu\bigr) \leq d(\nu, \mu) - \phi\bigl(d(\nu, \mu)\bigr),$$

which implies that $d(\nu, \mu) = 0$. Therefore, $\nu = \mu$, and so ν is the unique fixed point of f. □

We give the following example to illustrate Theorem 6.

Example 3 Let $A = B = X = \mathbb{R}^+$ and we define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = |x - y|, \quad \text{for } x, y \in X.$$

Define $f : X \to X$ by

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1; \\ \frac{1}{16}, & \text{if } x \geq 1 \end{cases}$$

and define $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$\psi_x(t) = \frac{1}{3}t \quad \text{and} \quad \phi(t) = \frac{1}{6}t \quad \text{for } t \in \mathbb{R}^+.$$

Then f is a generalized cyclic orbital weaker Meir-Keeler (ψ_x, ϕ)-contraction and 0 is the unique fixed point.

Competing interests
The author declares that they have no competing interests.

Acknowledgements
This research was supported by the National Science Council of the Republic of China. The authors would like to thank referee(s) for many useful comments and suggestions for the improvement of the paper.

Received: 30 October 2012 Accepted: 24 March 2013 Published: 10 April 2013

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doi:10.1186/1687-1812-2013-91
Cite this article as: Chen: **Fixed point theorems of generalized cyclic orbital Meir-Keeler contractions.** *Fixed Point Theory and Applications* 2013 **2013**:91.